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## *PERTURBATIONS IN AN ANISOTROPIC HOMOGENEOUS UNIVERSE*

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In an anisotropic homogeneous cosmological model, the growth of the matter-density perturbations is a kinetic effect of the motion of matter in an external field, in accord with the concepts of Lifshitz and Khalatnikov[1] concerning the vacuum stage. Using this result, we find the law governing the growth of small perturbations and the general solution for finite perturbations of matter without pressure, $P = 0$. We show that in anisotropic models the perturbations increase 3–5 times faster than in the isotropic model (see formula (5)). Long-wave perturbations in a medium with $P = \epsilon/3$ increase 2–3 times faster (see formula (9)). The law governing the variation of the amplitude of acoustic and gravitational waves is also explained.

E. LIFSHITZ and I. Khalatnikov[1] have shown in very general form that the influence of matter on the space-time metric near the singularity vanishes in a certain sense for anisotropic solutions of Einstein's equations near the singularity. It is shown in their paper that during the collapse the matter moves with relativistic velocity relative to the synchronous reference frame.

In the present article we consider the cosmological problem of the growth of density perturbations in expanding matter that is at rest, in the mean, relative to the synchronous reference frame, and also the change of the amplitude of the gravitational and acoustic waves. The purpose of this article is to show that the growth of the perturbations of the matter density in an anisotropic expanding universe is a kinetic effect due to the motion of matter in a gravitational field described by the solution of the gravitation equations for empty space, and to find the laws governing the growth of the perturbations of matter density. A clear understanding of the process also makes it possible in some particular cases of the problem to advance in the analysis of a density inhomogeneity that is finite and not small.

The analysis of inhomogeneous perturbations (i.e., perturbations that depend on the coordinates) in an anisotropic homogeneous universe is of great interest. Such an analysis can be regarded as the first approximation to the solution of the problem of a universe that is not isotropic or homogeneous. The nonlinearity of the equations of general relativity theory, the complexity of the physical processes, and the mathematical difficulties make direct solution of the general problem impossible at present. It becomes necessary to acquire information concerning the character of the solution by considering particular cases, among which a major role is played by exact solutions. As a rule these solutions are degenerate; for example, they have invariance of the spherical-symmetry (i.e., rotation-group) type or of the spatial homogeneity (i.e., translation-group) type.

Weakly perturbed exact solutions form a set of much larger cardinality (coinciding with the cardinality of the general solution), since the perturbations lift the degeneracy and do not have invariance of the exact solutions. At the same time, so long as the perturbations are small, they satisfy linear equations, and their Fourier expansions lead to ordinary differential equations for the Fourier amplitudes. Therefore problems involving small perturbations combine mathematical simplicity and lucidity of the solutions with great generality of the initial conditions.

We shall consider here small perturbations of the density of matter against the background of an expanding flat anisotropic model of the universe. Departing from[1], we shall consider nonrelativistic motion of matter. A number of analytic solutions were previously obtained by one of the authors[2]. The physical interpretation of the results, showing that the growth of the perturbations is a kinematic effect, may be useful in the analysis of more complicated cases, particularly the problem of evolution of perturbations during the stage of a finite (not small) inhomogeneity of the density and finite perturbations. The analysis will be carried out for the simplest case of anisotropic solution, although a number of the results are valid also in general homogeneous Bianchi spaces.

There is a well-known solution with a flat co-moving space and P = 0, belonging to Heckmann and Schucking[3], with a metric

$$ds^2 = dt - a^2(t)d\zeta^2 - b^2(t)d\eta^2 - c^2(t)d\xi^2. \quad (1)$$

During the early stage ($t \to 0$), the expansion follows a power law: $a \sim t^{p_1}$, $b \sim t^{p_2}$, $c \sim t^{p_3}$, $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$. This law was obtained earlier by Kasner[4] in the analysis of the problem of an empty anisotropic world. As $t \to 0$, the terms in the gravitation equations containing the density and the pressure of matter, are negligible, see[1,3]. We can thus speak of a "vacuum stage" of the anisotropic world—during this stage the gravitation of matter does not play any role.

Let us present a Newtonian description of the situation. Locally, the observer feels tidal forces, in comparison with which the gravitational interaction of the neighboring volumes is negligible. The volume element of the co-moving system of coordinates compresses along one axis and expands along the other two axes, since in the general non-degenerate case we have $-1/3 < p_1 < 0 < p_2 < 2/3 < p_3 < 1$. For convenience, we put $p_1 = -\alpha$, $0 \le \alpha \le 1/3$. Let us consider first particles at rest in co-moving coordinates, i.e., with constant $\xi, \eta, \zeta$. We take pairs of particles located on some particular axis, for example $\xi_1, 0, 0$ and $\xi_2, 0, 0$ or $0, \eta_1, 0$ and $0, \eta_2, 0$.

Their relative acceleration is

$$\frac{d^2x_{1,2}}{dt^2} = p_1(p_1-1)t^{-2}x_{1,2} = \alpha(\alpha+1)t^{-2}x_{1,2} = \frac{\ddot{a}}{a}x_{1,2},$$
$$\frac{d^2y_{1,2}}{dt^2} = -p_2(1-p_2)t^{-2}y_{1,2} = \frac{\ddot{b}}{b}y_{1,2}, \quad (2)$$
$$\frac{d^2z_{1,2}}{dt^2} = -p_3(1-p_3)t^{-2}z_{1,2} = \frac{\ddot{c}}{c}z_{1,2}.$$

Here $x = a\xi$, $y = b\eta$, $z = c\zeta$. In a Newtonian interpretation, such a relative acceleration indicates action of a gravitational potential $\varphi$, satisfying the conditions (during the Kasner stage)

$$\frac{\partial^2\varphi}{\partial x^2} = -\frac{\ddot{a}}{a}, \quad \frac{\partial^2\varphi}{\partial y^2} = -\frac{\ddot{b}}{b}, \quad \frac{\partial^2\varphi}{\partial z^2} = -\frac{\ddot{c}}{c}, \quad \Delta\varphi \equiv 0 \quad (3)$$

Since $\Delta\varphi \equiv 0$, corresponding to Newton's equation for vacuum, a local observer can assume that the gravitational forces are not connected with the matter that is present, and in this sense he will call them tidal. Of course, from the point of view of relativistic theory, the gravitational field in this model is a free gravitational field of the type of a gravitational wave of infinite length; this field does not have matter as its source.

The tidal forces push apart the particles, which are at rest in this reference frame, along the x axis (they slow down the compression) and pull together the co-moving particles along the y and z axes (they slow down the expansion). Particles moving with arbitrary velocity experience the same gravitational forces, so long as their velocity is nonrelativistic.

Let us turn now to perturbations of the Hackmann-Schucking model. We are considering a Kasner vacuum stage, since the succeeding stage goes over rapidly into the trivial Friedmann model, and the behavior of the perturbations in the latter is known[5] As already mentioned, during the vacuum stage the gravitational interaction of the matter does not play any role in the behavior of the perturbations, just as it plays no role in the unperturbed motion. An approximate estimate of the role of gravitational interaction of matter is obtained by taking the instantaneous value of the increment in accordance with the Jeans formula

$$\frac{d\ln\delta}{dt} = \sqrt{4\pi G\rho}, \quad \delta \sim \exp\int dt\sqrt{4\pi G\rho}.$$

In the case of dust we substitute $\rho = \rho_m$, and in the case of radiation $\rho = \rho_r$, in accordance with the formulas $\rho_m = At^{-1}$, $\rho_r = Bt^{-4/3}$; the constants A and B are expressed in terms of the isotropization time $t_1$ (the time when the Friedmann solution becomes valid)

$$At_1^{-1} = (6\pi Gt_1^2)^{-1}, \quad Bt_1^{-4/3} = 3(32\pi Gt_1^2)^{-1}$$

and we find that during the time from the singularity ($t = 0$) to $t_1$, the sought $\ln\delta$ increases by a finite amount

$$\Delta = \int_0^{t_1}\sqrt{4\pi G\rho}\,dt = 2\sqrt{\frac{2}{3}} \text{ for } \rho = \rho_m, \quad \Delta = \frac{3}{2}\sqrt{\frac{3}{2}} \text{ for } \rho = \rho_r.$$

In an isotropic world, such an integral diverges at the lower limit. The total growth of the perturbations, connected with the kinematic growth of the perturbations in the anisotropic world, is also infinite if the initial instant is $t = 0$. Comparing with these results the final contribution of the gravitational interaction ($2\sqrt{2/3}$ or $3/2\sqrt{3/2}$), we can conclude that it is negligible. In the absence of gravitational interaction, one cannot speak of gravitational instability of a homogeneous world. But this does not, of course, mean that all the perturbations only attenuate. In the vacuum period there are growing perturbations, and the law governing this growth is even stronger than that for the growth of the perturbations (due to the Jeans gravitational instability) in the Friedmann isotropic model.

The growth of the perturbations in the anisotropic model has an entirely different, non-Jeans nature. This growth is connected with compression along one of the axes (x). In the analysis, two perfectly equivalent formulations are possible: in a co-moving unperturbed reference frame ($\xi, \eta, \zeta$), and in Euler coordinates (x, y, z). In the co-moving nonstationary system of coordinates one can speak of kinematic growth of the velocity along the $\xi$ axis; in the Euler coordinates x, y, z the motion takes place under the influence of tidal gravitational forces. As already mentioned, E. Lifshitz and Khalatnikov[1] noted that during the collapse (during the approach to the singularity) the velocity increases and reaches asymptotically a relativistic value ($v/c \to 1$) along the axis of the fastest contraction.

Let us return to cosmology and to nonrelativistic velocities. In the first formulation, in the co-moving system, the peculiar velocity[1)] for dust, is $v = a d\xi/dt \sim t^{\alpha}$. In the second formulation, we seek a general solution of the equation $d^2x/dt^2 = \alpha(\alpha+1)t^{-2}x$. This solution is

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1)Corresponding to $\xi = v_1 + v_2 t^{1+2\alpha}$, $d\xi/dt = (1+2\alpha)v_2 t^{2\alpha}$, $v = (1+2\alpha)v_2 t^{\alpha}$.

$$x = v_1 t^{-\alpha} + v_2 t^{1+\alpha}, \quad u = dx/dt = -\alpha v_1 t^{-\alpha-1} + (1+\alpha) v_2 t^{\alpha}, \quad (4)$$

and the first term describes the unperturbed motion, while the second plays the role of the perturbation. Accordingly, in the first method of analysis we can state that the perturbations increase for kinematic reasons.

Assume that at the initial instant $t_0$ we specify a density perturbation and a peculiar velocity ($\delta = \delta_0(\mathbf{r})$, $\mathbf{u} = \mathbf{u}_0(\mathbf{r})$). In this case, the independent modes are: 1) perturbation of density without perturbation of the velocity, independent of the time; 2) decreasing velocities along the second and third axes, $u_y \sim t^{-p_2}$, $u \sim t^{-p_3}$, and the ensuing density perturbations also decrease; 3) growing velocity along the first axis, $u_x \sim t^{\alpha}$, and an ensuing increasing density perturbation

$$\delta \approx u_x t k_x \sim t^{1+2\alpha} \sim \rho^{-(1+2\alpha)}. \quad (5)$$

We note that the initial amplitude of the growing component at a given velocity amplitude decreases with increasing wavelength, $k_x = \lambda^{-1} \cos(\mathbf{kx})$. Finally, neglecting the contribution of the decreasing velocities and the phase shifts, we have

$$\delta(t) = \delta_0 - u_{x_0} t_0 k_{x_0} + u_{x_0} k_{x_0} t_0 (t/t_0)^{1+2\alpha}.$$

Bearing in mind that $0 < \alpha \leq 1/3$, we see that when the density decreases by a factor of n the perturbations increase by a factor $n^{1+2\alpha}$, i.e., by $n^1 - n^{5/3}$ times. In an isotropic model with dust, $\delta \sim t^{2/3}$, $\rho \sim t^{-2}$, $\delta \sim \rho^{-1/3}$, the growth is slower by a factor $n^{1/3}$, i.e., by a factor of 3–5 (the exponent, $|d \ln \delta / d \ln \rho|$, is smaller by the same factor).

The second method of analysis (in Euler coordinates with allowance for tidal forces) is convenient for an exact solution of the problem with arbitrary initial distribution of the density and velocity of matter

$$t = t_0, \quad u = u_0(x) = -dx/dt_0 + v_0(x),$$

$$x = \left(w - \frac{v_0 t_0}{1+\alpha}\right)\left(\frac{t}{t_0}\right)^{-\alpha} + \frac{v_0 t_0}{1+\alpha}\left(\frac{t}{t_0}\right)^{1+\alpha}.$$

We introduce a Lagrangian coordinate w such that $x = w$ at $t = t_0$. Then the trajectories are given by the expression

$$x = \left(w - \frac{v_0 t_0}{1+\alpha}\right)\left(\frac{t}{t_0}\right)^{-\alpha} + \frac{v_0 t_0}{1+\alpha}\left(\frac{t}{t_0}\right)^{1+\alpha}, \quad (6)$$

where $v_0$ is a known function of w.

This instantaneous distribution of the dust density is conveniently determined as a function of t and w:

$$\rho(t;\, w) = \rho_0(w)\,(t/t_0)^{-1-\alpha}(dx/dw)^{-1}, \quad (7)$$

where x is defined by the preceding formula. It follows from this expression that the density becomes infinite at $dx/dw = 0$, which occurs when $dv_0/dw < 0$, and at the instant

$$t \approx t_0 \left[1 + \frac{1+\alpha}{t_0}\left[\frac{dv_0}{dw}\right]^{-1}\right]^{1/(1+2\alpha)}. \quad (8)$$

$\rho$ becomes infinite as a result of the intersection of the trajectories, followed by occurrence of a shock wave (see[6]).

When short waves in an elastic medium are considered (for example, in a relativistic gas with $P = \epsilon/3$), or when one considers short gravitational waves, the change in their amplitude is connected, by virtue of adiabatic invariance, with the change of the wavelength and of the frequency. The wavelength decreases and the frequency increases if the propagation is mainly along the x axis. We see that propagation along x, just like motion of dust along x in the preceding example, is not an exceptional case. By virtue of purely geometric factors, at an arbitrary initial wave vector (with components $k_{0x} \approx k_{0y} \approx k_{0z}$ of the same order at the instant $t = t_0$), $k_x$ increases and $k_y$ and $k_z$ decrease during the course of time. The direction of propagation of any wave approaches the x axis. The entire picture is similar to that considered by us earlier, the behavior of weakly-interacting particles in an anisotropic model[7] [2)].

Thus, asymptotically the wave vector is directed along x and increases with time in proportion to $t^{\alpha}$. The velocity of the gravitational waves is constant and equal to c. The velocity of the elastic waves in a high-temperature plasma (in a gas with $P = \epsilon/3$) is also constant and equal to $c/\sqrt{3}$. Consequently, in both cases the frequency increases in proportion to $k_x \sim t^{\alpha}$; it follows from adiabatic invariance that the energy of the gravitational and elastic waves contained in the given co-moving volume also increases like $\sim t^{\alpha}$. The co-moving volume increases like $i^1$ in the unperturbed Kasner solution.

The energy density of the gravitational waves is $\sim (dh/dx)^2$, where h is the dimensionless perturbation of the metric $dx^2 \to (1 + h)dx^2$. We finally obtain $h \sim t^{-(1+\alpha)/2}$; the ratio of the energy density of the gravitational waves to the energy density of the plasma increases like $\sim t^{1/3+\alpha}$, and in an isotropic world this ratio remains constant[3)].

For elastic (acoustic) waves $\epsilon \approx \rho c^2 \delta^2$; the relative amplitude of short waves ($l \ll ct$), to which adiabatic invariance is applicable, increases with increasing time, $\delta \sim t^{\alpha/2}$; in an isotropic world, this amplitude was constant. The same condition ($l \ll ct$) also pertains to gravitational waves.

Using the same method, i.e., neglecting the gravitational influence of matter, we can also readily calculate the development of long-wave ($\lambda \gg ct$) perturbations in the medium with $P = \epsilon/3$. Solving the hydrodynamic equations in the specified metric (1) and neglecting the derivatives of the pressure with respect to $\xi$, $\eta$, and $\zeta$, we obtain for the fastest-growing perturbation mode and the density

$$\delta = \delta\varepsilon/\varepsilon \sim t^{4/3+2\alpha} \sim \varepsilon^{-(1+3/2\alpha)} \quad (9)$$

in place of $\delta \sim \epsilon$ in the Friedmann model. It is interesting to note that the behavior of the absolute perturbations of the density does not depend on the equation of state of matter (in the long-wave limit).

2) This is not surprising, since free particles can be described equally well with the aid of a wave function and classically. Sometimes, according to the expression of Paradoksov [8], "quantum" mechanics helps us understand the "classical" mechanics.

3) We recall that we are postulating a plasma with a leading role of radiation, $P = \epsilon/3$, with particle collisions sufficiently frequent, so that the isotropy of their distribution and Pascal's law are not violated.

For both P = 0 and P = $\epsilon/3$ we have

$$\rho\delta|_{P=0} = \varepsilon\delta|_{P=\varepsilon/3} = t^{2\alpha}. \quad (10)$$

It should also be noted that when we consider the motion of matter against the background of an unperturbed metric, we cannot describe certain types of perturbation, namely, perturbations of the free gravitational field of the model, i.e., the field against the background of which the motion of matter was considered. As we have already noted, this general field can be regarded as a gravitational wave of infinite length. It is clear that for gravitational perturbations with large wavelength ($\lambda \gg ct$) the method of adiabatic invariance (used above to analyze waves with $\lambda \ll ct$) does not hold. Such perturbations of the entire field were considered in[1]. They also lead to perturbations of the density and velocity (with the exception of singular cases connected with the symmetry of the problem). These modes, however, increase more slowly during expansion and are not principal ones.

We do not consider in this article other more subtle problems, for example the transformation of transverse waves into longitudinal ones, or the behavior of spatially-homogeneous perturbations, when the velocity of matter becomes relativistic (the last problem was considered by Novikov[9]). However, even the examples considered here and their comparison with the exact solutions in the theory of small perturbations[2] show that the simple physical picture developed above is correct and useful.

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