We consider the question of exchange interaction in Wannier-Mott excitons, for an arbitrary band structure, both in the effective-mass approximation and with allowance for the corrections, considered first by Elliott\(^{14}\), connected with the short-range potential. In addition, we calculate the corrections to the non-exchange terms, which arise in the third approximation of the kp method, and also the corrections connected with the short-range potential. The results of the calculation show that, in agreement with the results of Rashba\(^{13}\), allowance for exchange interaction in the effective-mass approximation is equivalent to allowance for the long-range forces, and refute the results of subsequent investigations\(^{15,16}\), whose authors reached opposite conclusions. Comparison of the results of the calculations of the exchange terms in the effective-mass approximation with conclusions of the phenomenological theory\(^{18-20}\) and of the calculation of the terms connected with the short-range forces with the conclusions of the theory of invariants\(^{10,15,16}\) show that these phenomenological methods are equivalent to the microscopic approach.

EXCHANGE INTERACTION IN THE EFFECTIVE-MASS APPROXIMATION

We present first the Hamiltonian describing, in the effective-mass approximation, the interaction of two electrons, one of which is in the conduction band \(m\) near an extremum \(k_c\), and the other in the valence band \(n\) near \(k_v\), the bands having an arbitrary structure near the extrema. The initial Schrödinger equation describing these electrons has the usual form

\[
\mathcal{H}_i(x_i) + \mathcal{H}_e(x_e) + \mathcal{V}(r, -r) - E\mathcal{W}(x_i, x_e) = 0, \tag{1}
\]

where \(\mathcal{V}(r) = e^2/\kappa r\), and \(x\) includes both the coordinate and time variables \(r\) and \(\alpha\). The dielectric constant \(\kappa\), generally speaking, depends here on \(r\). In the kp method, the wave function \(\Psi(x)\) is expanded in terms of the functions

\[
\psi_{s\alpha}(x) = \phi_{s\alpha}(x)e^{i\mathbf{k} \cdot \mathbf{r}},
\]

where \(\phi_{s\alpha}\) are the eigenfunctions of the Hamiltonian \(\mathcal{H}_0\) at the point \(k_0\) of band \(s\):

\[
\psi_{s\alpha}(x) = e^{i\mathbf{k} \cdot \mathbf{r}} \phi_{s\alpha}(x).
\]

Let us consider first a direct exciton, when \(k_c = k_v = k_0\). The function \(\Psi(x_1, x_2)\) will be written in the form

\[
\mathcal{W}(x_1, x_2) = \sum_{s\alpha} F_{s\alpha}(r_1, \mathbf{r}) \phi_{s\alpha}(x_1) \phi_{s\alpha}(x_2), \tag{2}
\]

and we expand the smooth function \(F_{s\alpha}(r_1, r_2)\) in a series

\[
F_{s\alpha}(r_1, r_2) = \sum_{l \geq 0} C_{l,s\alpha} e^{i\mathbf{k}_l \cdot \mathbf{r}}, \tag{3}
\]
and then
\[ \Psi(x_1, x_2) = \sum_{\alpha\beta} C_{\alpha\beta} \phi_{\alpha}(x_1) \phi_{\beta}(x_2). \]  
(4)

From the antisymmetry condition \( \Psi(x_1, x_2) = -\Psi(x_2, x_1) \) it follows that
\[ F_{\alpha}(x_1, x_2) = -F_{\beta}(x_2, x_1), \quad C_{\alpha\beta} = -C_{\beta\alpha}. \]  
(5)

Further, substituting (4) in (1), we multiply from the left by \( \phi^{\dagger}_{\alpha}(x_1) \phi_{\beta}(x_2) \) and integrate with respect to \( x_1 \) and \( x_2 \). As a result we obtain a system of equations for the coefficients \( C_{\alpha\beta} \):
\[ \sum_{\alpha\beta} \langle \phi'_{\alpha}, \phi'_{\beta} | \mathcal{H} | \phi_{\alpha}(x_1) \phi_{\beta}(x_2) \rangle C_{\alpha\beta} = E_{\alpha\beta}(x_1, x_2). \]  
(6)

The matrix \( \langle s'k'|t'k|s_{k_1}t_{k_2} \rangle \) contains the diagonal terms
\[ \langle s'k'|t'k|s_{k_1}t_{k_2} \rangle = \left( E_{s'(k_1)} + E_{t'(k_2)} \right) \delta_{s's} \delta_{t't} \delta_{k_1} \delta_{k_2}, \]  
(6a)

the intraband non-diagonal (Coulomb) terms
\[ \langle s'k'|t'k|s_{k_1}t_{k_2} \rangle = V_{s'k_1} \delta_{s's} \delta_{t't} \delta_{k_1} \delta_{k_2}, \]  
(6b)

interband Coulomb terms, which we shall consider in detail below, and interband kp terms
\[ \langle s'k'|t'k|s_{k_1}t_{k_2} \rangle = \frac{\hbar^2}{2m} (k_{p_1} + k_{p_2}) \delta_{s's} \delta_{t't} \delta_{k_1} \delta_{k_2}. \]  
(6c)

where \( P_{sp'} \) are the matrix elements of the momentum operator on the functions \( \psi_{s} \), and
\[ V_{p} = \frac{1}{2} \int V(r) e^{-ipr} dr. \]

Further, as usual, it is necessary to obtain the equation of the effective-mass method, containing only the coefficients \( C_{mk_1n_1} \) and \( C_{nk_2m_2} \), where \( m, m', \ldots \) are the conduction bands under consideration, and \( n, n', \ldots \) the valence bands, eliminating the remaining interband matrix elements \( \gamma^0_{sp} \) by means of the transformation \( \mathcal{S} \):
\[ \mathcal{S} = e^{-\mathcal{H} \gamma^0}. \]

As a result we obtain a system of equations with dimensionality \( 2N_{C}N_{V} \), where \( N_{C} \) and \( N_{V} \) are the multiplicities of the degeneracy of the conduction and valence bands at the extremum point \( k_0 \). However, taking the condition (5) into account, we obtain from these \( 2N_{C}N_{V} \) equations a system of equations with dimensionality \( N_{C}N_{V} \) only for the coefficients \( C_{mk_1n_1} \):
\[ \sum_{mk_1} \langle m'k'|n'k|\mathcal{H}_{s} |mk_1n_1 \rangle - \langle m'k'|n'k|\mathcal{H} |mk_1n_1 \rangle \times C_{mk_1n_1} = E_{C_{mk_1n_1}}. \]  
(7)

containing exchange terms—the second term of (7).

The first nonzero contribution to the exchange terms is made by the terms of third order—second in \( \gamma^0 \) and first in \( \mathcal{S} \). A contribution of the same order is contained also in the first term of (7).

The terms of fourth order in \( \gamma^0 \), i.e., in \( k \), and fifth in \( \mathcal{S} \), have no contribution to the exchange energy, since they make no contribution to the exchange energy.

In order to change over to the \( r \)-representation it is necessary to multiply Eq. (7) by \( \exp[i(k_1 \cdot r_1 + k_2 \cdot r_2)] \) and sum over \( k_1 \) and \( k_2 \). Taking into account the smallness of \( k \) and \( k' \) it is possible to replace summation over \( k_1 \) and \( k_2 \) by separate summations

\[ \text{we shall not write them out explicitly}^{2}. \]

Terms of higher order, as can be readily shown, contain the additional small parameter \( E_2/E_1 \) or \( h \gamma^2/2mE_1 \). Here \( E_2 \) is the width of the forbidden band and \( E_1 \) is the exciton binding energy.

If we retain only the terms of first order in \( \gamma^0 \) and second in \( \mathcal{S} \), then we obtain
\[ \langle m'k'|n'k|\mathcal{H}_{r} |mk_1n_1 \rangle = \langle m'k'|\Delta \mathcal{H} |mk_1n_1 \rangle \delta_{mk_1} \delta_{n_1} + \delta_{mk_2} \delta_{n_2} \delta_{k_1} \delta_{k_2} \]  
(8)

where \( \mathcal{H}(x) \) is the single-band Hamiltonian of the effective-mass method, equal to
\[ \mathcal{H}(x)_{\alpha\beta} = \frac{\hbar^2}{2m} \delta_{\alpha\beta} + \frac{\hbar^2}{2m} \sum_{\alpha\beta} k_{\alpha} \left(p_{\alpha}^2 \delta_{\alpha\beta} / (E_{\alpha} - E_{\beta}) \right). \]  
(9)

Here \( E_{s} \) is the energy in the band \( s \) at the point \( k_0 \).

In the third-order approximation, the non-exchange terms in (7) contain the contribution
\[ \langle m'k'|\Delta \mathcal{H} |mk_1n_1 \rangle = \langle m'k'|\Delta \mathcal{H} |mk_1n_1 \rangle \delta_{mk_1} \delta_{n_1} + \delta_{mk_2} \delta_{n_2} \delta_{k_1} \delta_{k_2} \]
(10)

where
\[ G_{mn}^{\alpha} = \frac{1}{2m} \sum_{\alpha\beta} \langle \gamma_{m\alpha}^{\beta} |p_{\beta} \rangle / (E_{\alpha} - E_{\beta}) \].

(11)

\[ G_{mn}^{\alpha} = \frac{1}{2m} \sum_{\alpha\beta} \langle \gamma_{m\alpha}^{\beta} |p_{\beta} \rangle / (E_{\alpha} - E_{\beta}) \].

(12)

In (9) and (12), the curly brackets denote a symmetrized product, and \( \delta_{\alpha\beta} \) is an unit antisymmetric tensor.

The exchange term in (7) differs from zero only if the matrix elements of the operator \( \gamma_{nm}^{\alpha} \) are not equal to zero, and is equal in third approximation to
\[ \langle m'k'|n'k|\mathcal{H} |mk_1n_1 \rangle = \sum_{\alpha\beta} Q_{mn}^{\alpha\beta} \delta_{mk_1} \delta_{n_1} \delta_{k_1} \delta_{k_2}. \]  
(13)

where
\[ Q_{mn}^{\alpha\beta} = \frac{\hbar^2}{2m} \langle \gamma_{m\alpha}^{\beta} |p_{\beta} \rangle / (E_{\alpha} - E_{\beta}) \].

If all \( P_{mn} \neq 0 \), then the exchange terms appear only when account is taken of terms of higher order in \( \gamma^0 \), and contain an additional small quantity.

We note that allowance for the dispersion of the dielectric constant \( k \) in (1) can lead to corrections to the non-exchange terms of the order \( E_{s}^2/E_{r} \). As noted by Moskalenko \cite{13}, all such terms, as well as the term (31) (see below), which is connected with the short-range forces, lead to a splitting of the s and p states which are degenerate in the case of spherical bands, similar to the splitting of the s and p levels of the hydrogen atom in relativistic theory \cite{7}.

\[ \text{We note that allowance for the dispersion of the dielectric constant} \]

\[ \text{\( k \) in (1) can lead to corrections to the non-exchange terms of the order} \]

\[ \text{\( E_{s}^2/E_{r} \). As noted by Moskalenko} \text{\cite{13}, all such terms, as well as the term (31) (see below), which is connected with the short-range forces, lead to a splitting of the s and p states which are degenerate in the case of spherical bands, similar to the splitting of the s and p levels of the hydrogen atom in relativistic theory} \text{\cite{7}.} \]
over \( q = k'_{1} - k_{1} = k_{2} - k'_{2} \), and in the last term, i.e., in (13), by summation over \( q = k'_{1} - k_{1} + k_{2} - k'_{2} \). The exchange term (13) will then contain the sum

\[
\sum_{r_{1}r_{2}} G_{m'n} \exp[i(k_{1}r_{1} + k_{2}r_{2})] = F_{m'n}(r_{1}, r_{2})
\]

unlike the non-exchange terms containing \( F_{mn}(r_{1}, r_{2}) \). As a result, we obtain a system of equations for the functions \( F_{mn}(r_{1}, r_{2}) \):

\[
\sum_{m'n'} G_{m'n'} \exp[i(k_{1}r_{1} + k_{2}r_{2})] = F_{m'n}(r_{1}, r_{2})
\]

(14)

Here

\[
\mathcal{H}_{m'n}(r_{1}, r_{2}) = V(r_{1}) \delta_{m'n} + \Delta \mathcal{H}_{m'n}(r_{1}, r_{2}) = \mathcal{H}_{m'n}(r_{1}, r_{2})
\]

(15)

\[
\Delta \mathcal{H}_{m'n}(r_{1}, r_{2}) = -i \frac{e^{2}}{m} \sum_{G} \left( G_{n'n} - \frac{1}{3} \text{Sp} G_{n'n} \delta_{n'n} \right)
\]

(16)

\[
\times \frac{3}{2} \frac{e_{F}^{2}}{m} \frac{4 \pi}{3} \text{Sp} G_{n'n} \delta_{n'n} + \frac{1}{2} \left( \text{Sp} G_{n'n} r_{1} \right) \right]
\]

(16)

The last term in (14), which describes the exchange interaction, is equal to

\[
\sum_{m'n'} Q_{m'n'}^{exch} \frac{\partial V}{\partial r} (r_{1} - r_{2}) \delta (r_{1} - r_{2}) \delta (r_{1} - r_{2}).
\]

(17)

The operator (14) determines the interaction of two electrons that are in different bands \( mn \) and \( n'n' \). The Hamiltonian describing the interaction of two electrons in empty and filled bands, into a Hamiltonian that describes the interaction of an electron and a positron\(^{(1)}\). Similar rules are valid also on going from electron-electron to electron-hole interaction in semiconductors, namely, to obtain the Hamiltonian describing the interaction of an electron \( m \) and a hole \( n \) it is necessary, first, to reverse the sign of that term of \( \mathcal{H}_{n'n}(k) \) which defines the presence of the electron in the filled band, and in the operators \( \langle m' n' | \mathcal{H} | m n \rangle \) and \( \langle n' n' | \mathcal{H} | m n \rangle \) which determine the interaction of the electron and the hole; second, it is necessary to replace \( \mathcal{H} \) by \( \mathcal{H}_{exch} \) and to reverse the sequence of the indices of the valence band, since creation of an electron denotes annihilation of a hole and vice versa. According to these rules, the transition from electron-electron to electron-hole interaction is effected by the transformations

\[
\mathcal{H}_{mn}(r_{1}, r_{2}) = -\mathcal{H}_{m'n'}(r_{1}', r_{2})
\]

(18)

Here \( \mathcal{H}_{m'n'} \) is the time-inversion operation. The symbol \( \mathcal{H}_{m'n'} \) denotes that if we choose as the basis the wave functions of the holes \( \phi_{n}^{h} \), then the matrix elements must be calculated with the wave functions of the "missing" electron \( \phi_{m}^{h} \). It is necessary to bear in mind here that

\[
p_{m'n} = -p_{n'm}^{h}
\]

and therefore for the matrix \( G_{n'n}^{h} \), just for \( G_{n'n} \), we have \( \mathcal{H}_{m'n'} = \mathcal{H}_{n'n} \), \( G_{m'n'}^{h} \), and for the matrix \( g_{n'n} \), we have \( g_{n'n} = -g_{n'n}^{h} \). Here the term \( \mathcal{N}_{n'n}^{h}(k_{2}) = -\mathcal{N}_{n'n}(k_{2}) \) is the usual Hamiltonian defining the hole spectrum in the effective-mass approximation.

As a result of the indicated transformations, the electron-hole interaction operator takes the form

\[
\mathcal{H}_{exch}(r_{1}, r_{2}, r_{1}', r_{2}') = \mathcal{H}_{exch}(r_{1}, r_{2}, r_{1}', r_{2}')
\]

(19)

where

\[
\begin{align*}
&\Delta \mathcal{H}_{exch}(r_{1}, r_{2}) = -i \frac{e^{2}}{m} \sum_{G} \left( G_{n'n} - \frac{1}{3} \text{Sp} G_{n'n} \delta_{n'n} \right)
\end{align*}
\]

(20)

The first three terms in (19) give the equation derived by Dresselhaus\(^{(1)}\) for the exciton in the effective-mass approximation. The last term in (19) describes corrections to the Coulomb potential.

The first two terms in \( \Delta G^{\text{eh}} \) correspond to the effective dipole-dipole and to the contact interactions, respectively. The matrix \( G^{\text{eh}} \) defining it is similar to the matrix (9) defining the spectrum. The last term in \( \Delta G^{\text{eh}} \) describes the interaction of the orbital angular momenta of the slow and fast functions; the matrix \( g^{\text{eh}} \) defining it is similar to the matrix defining the magnetic moment of the electron, which is connected with the orbital motion\(^{(1)}\). The last term in (19) is due to the exchange term (17):

\[
\mathcal{H}_{exch}(r_{1}, r_{2}, r_{1}', r_{2}') = -\sum_{G} G_{n'n}^{h} \frac{\partial V}{\partial r} (r_{1} - r_{2}) \delta (r_{1} - r_{2}) \delta (r_{1} - r_{2}).
\]

(21)

If we introduce new variables\(^{(1)}\)

\[
r = r_{1} - r_{2}, \quad R = r_{1} + r_{2}, \quad k = k_{1} + k_{2}, \quad k' = k'_{1} + k'_{2},
\]

(22)

then in terms of the variables (22) the operator (21) will be of the form

\[
\mathcal{H}_{exch}(r_{1}, r_{2}, r_{1}', r_{2}') = -\sum_{G} Q_{n'n}^{h} \frac{\partial V}{\partial R} (R - R') \delta (r) \delta (R)
\]

(23)

where

\[
Q_{n'n}^{h} = \frac{h^{2}}{m^{2} E_{g}} R_{n'n}^{h} P_{n'n}^{h} P_{n'n}^{h}
\]

(24)

The wave function of the exciton \( \psi_{K}^{h} \) can be written in the effective-mass method in the form

\[
\psi_{K}^{h}(r, R) = \frac{1}{V} \exp[iKR] \begin{bmatrix} f_{k_{1}n_1} \cdots f_{k_{1}n_1} \cdots f_{k_{n}n_{n_1}} \cdots f_{k_{n}n_{n_1}} \end{bmatrix}
\]

(25)

\(^{(1)}\)The matrices a and b can be chosen in arbitrary fashion, but in such a way that they satisfy the condition \( a + b = 1 \), so that the Jacobian of the transformation is \( \partial (r_{1}, r_{2}) / \partial (r, R) = 1 \). In this case we have \( k_{1} + k_{2} = k_{1} + k_{2} = k + K \cdot R \).
The matrix element of the operator (23), calculated with the functions (25), is
\[ \langle f|K'|e^{\delta^a}|K \rangle = \frac{\delta \epsilon_0^a}{\kappa} \sum_{\text{rec} \neq \text{rec}} Q_{\text{rec} - \text{rec}} K_{a} \text{K}_{b} \text{X}_{\text{rec} - \text{rec}} \langle 0|K_{m} \text{K}_{m} \text{X}_{\text{rec} - \text{rec}} \langle 0|f \text{K}_{m} \text{K}_{m} \text{X}_{\text{rec} - \text{rec}} \rangle (26) \]

Formula (23) and formula (26) (for a simple band) differ from the corresponding formulas given by Rashba only in that account is taken of the dielectric constant. The derivation presented above shows that formula (23) is valid for all the exciton states at an arbitrary band structure.

We note that although the term \( \epsilon^a \) is the result of allowance for the exchange interaction of the electrons in the valence band and in the conduction band, it describes a long-range electron-hole interaction, which can be regarded as the result of virtual recombination and generation of an exciton.

The figure shows clearly that on going over from the electron-electron to the electron-hole interaction in accordance with the rules indicated above, the diagram of the exchange interaction is transformed into the diagram of the annihilation interaction.

It is seen from (26) that the quantity \( \epsilon^a \) depends significantly on the direction of the vector \( K \).4

ALLOWANCE FOR SHORT-RANGE FORCES (DIRECT EXCITONS)

The matrix element of the interaction potential in Eq. (4) is equal to
\[ \langle 0|\sum_{\text{rec}} B_{m}^{a} B_{n}^{b} V_{m+n} \delta_{m+n} \delta_{m+n} \rangle = \langle 0|\sum_{\text{rec}} B_{m}^{a} B_{n}^{b} V_{m+n} \delta_{m+n} \delta_{m+n} \rangle \]
(27)
where \( \delta_{m+n} \delta_{m+n} \rangle = \langle 0|\sum_{\text{rec}} B_{m}^{a} B_{n}^{b} V_{m+n} \delta_{m+n} \delta_{m+n} \langle 0| \)

We have therefore retained in (6b) only one term from (27), the one with \( b_{L} = 0 \). We took into account here the fact that \( B_{m}^{a} B_{n}^{b} = \delta_{m+n} \delta_{m+n} \). We now take account of (27) of the terms with \( b_{L} = 0 \) and \( b_{L} \neq 0 \). It suffices here to consider the matrix elements of the operator \( V \) between the valence bands and the conduction bands.

We consider first the non-exchange terms. At small \( k \), we can neglect the dependence of \( V_{bM} + q \) on \( q \) for \( b_{M} = 0 \), and from (27) we obtain
\[ \langle m' n' k' k' | V_{m k n k} \rangle = \langle m' n' k' k' | V_{m k n k} \rangle \]
(28)

Changing over to the \( r \) representation, i.e., multiplying \( 5V \) by \( \langle q | r + q | r \rangle \), and summing over \( q \), we obtain
\[ \delta V_{r r} = \delta (r - r') \delta (r - r') \delta (r - r') \delta (r - r') \]
(29)

We consider now the exchange term
\[ \langle m' n' k' k' | V_{m k n k} \rangle = \langle m' n' k' k' | V_{m k n k} \rangle \]
(30)

Here the term with \( b_{L} = 0 \) vanishes. In the remaining terms, as before, we can neglect the dependence of \( V_{bL} + k'_{L} k'_{L} \) on \( k'_{L} - k'_{L} \), and then we obtain an expression similar to (28):
\[ \langle m' n' k' k' | V_{m k n k} \rangle = \langle m' n' k' k' | V_{m k n k} \rangle \]
(31)

On going from the electron-electron to the electron-hole interaction, in accordance with (18), from the electron-electron interaction to the electron-hole interaction, we obtain
\[ \delta V_{r r} = \delta (r - r') \delta (r - r') \delta (r - r') \delta (r - r') \]
(32)

In the coordinate representation
\[ \delta V_{r r} = \delta (r - r') \delta (r - r') \delta (r - r') \]
(33)

We note that although the term \( \epsilon^a \) is the result of allowance for the exchange interaction of the electrons in the valence band and in the conduction band, it can be regarded as the result of virtual recombination and generation of an exciton. The reason for this difference is that for an exciton the wavelength of the light, corresponding to the exciton excitation energy, i.e., \( \epsilon \), greatly exceeds the Bohr radius of the exciton \( a_{0} \), whereas for positronium, to the contrary, \( \lambda \approx h/2mc \approx a_{0} = 2h/ma_{0}^{2} \). Therefore for the exciton we can neglect retardation, which plays the major role for the positronium [48, 49].
\[ \left\langle m\vert V\vert m'\right\rangle = \int dx dx' u^*_{m}(x) u_{m'}(x) V(x, -x') u^*_{m'}(x') \]  

where \( \delta V_{\text{exc}} \) is defined as the difference between the total interaction potential \( V_{\text{total}} \) and the sum of the interaction potentials \( V_{\text{exc}} \) and \( V_{\text{int}} \). The Hamiltonian \( H \) is given by the sum of the kinetic energy and the potential energy, and the wave function \( \psi \) satisfies the time-dependent Schrödinger equation.

\[ \left\langle m\vert V\vert m'\right\rangle = \int dx dx' u^*_{m}(x) u_{m'}(x) V_{\text{exc}}(x, -x') u^*_{m'}(x') \]  

where \( \delta V_{\text{exc}} \) is defined as the difference between the total interaction potential \( V_{\text{total}} \) and the sum of the interaction potentials \( V_{\text{exc}} \) and \( V_{\text{int}} \). The Hamiltonian \( H \) is given by the sum of the kinetic energy and the potential energy, and the wave function \( \psi \) satisfies the time-dependent Schrödinger equation.

\[ \left\langle m\vert V\vert m'\right\rangle = \int dx dx' u^*_{m}(x) u_{m'}(x) V_{\text{exc}}(x, -x') u^*_{m'}(x') \]  

where \( \delta V_{\text{exc}} \) is defined as the difference between the total interaction potential \( V_{\text{total}} \) and the sum of the interaction potentials \( V_{\text{exc}} \) and \( V_{\text{int}} \). The Hamiltonian \( H \) is given by the sum of the kinetic energy and the potential energy, and the wave function \( \psi \) satisfies the time-dependent Schrödinger equation.

\[ \left\langle m\vert V\vert m'\right\rangle = \int dx dx' u^*_{m}(x) u_{m'}(x) V_{\text{exc}}(x, -x') u^*_{m'}(x') \]  

where \( \delta V_{\text{exc}} \) is defined as the difference between the total interaction potential \( V_{\text{total}} \) and the sum of the interaction potentials \( V_{\text{exc}} \) and \( V_{\text{int}} \). The Hamiltonian \( H \) is given by the sum of the kinetic energy and the potential energy, and the wave function \( \psi \) satisfies the time-dependent Schrödinger equation.

\[ \left\langle m\vert V\vert m'\right\rangle = \int dx dx' u^*_{m}(x) u_{m'}(x) V_{\text{exc}}(x, -x') u^*_{m'}(x') \]  

where \( \delta V_{\text{exc}} \) is defined as the difference between the total interaction potential \( V_{\text{total}} \) and the sum of the interaction potentials \( V_{\text{exc}} \) and \( V_{\text{int}} \). The Hamiltonian \( H \) is given by the sum of the kinetic energy and the potential energy, and the wave function \( \psi \) satisfies the time-dependent Schrödinger equation.
then we can write the matrix $\delta V_{\text{eh}}^{\text{exch}}$ for the ground state in the form\textsuperscript{[10]}

$$\delta V_{\text{eh}}^{\text{exch}} = \Delta_0 + \Delta_1 (i\sigma) + \Delta_1 \sum_i J_i \sigma_i,$$  

(46)

where $J_i$ is the matrix of the angular momentum operator $J_1$ with $j = 3/2$. For optically inactive terms, for which $f_{mn} = 0$ ($n \neq m$), we shall now show that allowance for the annihilation interaction (26) is completely equivalent to allowance for the interaction in the phenomenological theory\textsuperscript{[17-19]}. The spectrum of the excitons with allowance for the annihilation interaction (26) is determined by the secular equation

$$[E_{\text{eh}} + \mathcal{H}_{\text{ex}} - E_{\text{D}}] = 0,$$  

(48)

where

$$A_j = \frac{2\varepsilon}{3

\begin{align*}
\sum f_{mn}^e (0) p_{j}^{mn} K^n & = 0,
\end{align*}

(51)

$$\sum_{a, \nu} \alpha_a K^a \sum_{I} \frac{f_{\nu}^{v} (0) p_{j}^{mn} K^n}{A_{I} - E} - \sum_{a, \nu} \alpha_a K^a \sum_{I} \frac{f_{\nu}^{v} (0) p_{j}^{mn} K^n}{A_{I} - E} = 0,$$  

(52)

As indicated by Born and Kunc Huang\textsuperscript{[18]}, Pekar\textsuperscript{[19]}, and Agranovich and Ginzburg\textsuperscript{[20]}, the contribution of the interaction can be determined from the phenomenological equation

$$\sum_{a, \nu} \alpha_a K^a \sum_{I} \frac{f_{\nu}^{v} (0) p_{j}^{mn} K^n}{A_{I} - E} = 0,$$  

(53)

if we include in $\kappa$ the contribution made to the dielectric constant of the crystal $\Delta \varepsilon$ connected with the excitation of the excitons. Equation (53) follows from the equation

$$\text{div} \mathbf{D} = \sum_{a} \varepsilon_{a} \mathbf{K}_{a} \mathbf{K}_{a} = 0,$$  

(54)

to which Maxwell's equation reduces if retardation is neglected.

As shown in\textsuperscript{[19]}

$$\Delta \varepsilon = \frac{4\pi}{a_{0}^{2}} \sum_{I} \frac{f_{\nu}^{v} (0) p_{j}^{mn} K^n}{A_{I} - E}.$$  

(55)

It is easy to show that for an exciton described by a wave function (25), the matrix element of the current operator is

$$\mathbf{\rho} = \mathbf{\nabla} \sum_{mn} f_{mn}^e (0) \mathbf{\rho}_{mn} = \mathbf{\nabla} \sum_{mn} f_{mn}^e (0) \mathbf{\rho}_{mn}.$$  

(56)

Substituting (55) and (54) in (53), we obtain an equation that coincides with (55) (recognizing that in the approximation under consideration $\mathbf{D} = \mathbf{E}$). This shows that formula (25) takes full account of the long-range contribution in the dipole approximation, i.e., without allowance for the dependence of the matrix elements $\mathbf{P}_{mn}$ on $\mathbf{K}$. If it is necessary to take this dependence into account, then it is necessary to include in (13) terms of higher order in $\mathbf{\kappa}$. In conclusion, the authors consider it their pleasant duty to thank E. I. Rashba for an active discussion and useful remarks.

14 L. D. Landau and V. B. Berestetski, ibid. 19, 673 (1949).

Translated by J. G. Adashko