EXCITATION OF SOUND IN A JOSEPHSON JUNCTION

Yu. M. IVANCHENKO and Yu. V. MEDVEDEV

Donets Physico-technical Institute, Ukrainian Academy of Sciences

Submitted November 30, 1970

Zh. Eksp. Teor. Fiz. 60, 2274–2285 (June, 1971)

We develop a method to study the Josephson transition which is based upon the Fröhlich Hamiltonian which takes the electron-phonon interaction directly into account. We show that when the electron-phonon interaction is taken into account long-wavelength lattice vibrations with the Josephson frequency may occur. We consider the influence of resonance generation of sound on the shape of the current-voltage characteristics of the junction.

AFTER Josephson's discovery[1] of weak superconductivity in a system consisting of two superconductors separated by a thin layer of dielectric, different aspects of the theory of this effect have engaged many authors (see, e.g., the survey in[2]).

The superconducting Josephson current changes with time[1] when there is a non-vanishing potential at the junction, and this causes the appearance of a variable electromagnetic field inside the junction.[3,4] The connection between the current and the field induced by it is given by the appropriate Maxwell equation.[4] When studying this interaction one usually starts from the standard model scheme suggested by Cohen, Falicov, and Phillips[5]; Prange[6] and one of the present authors[7] showed how this model can be based upon the BCS theory. One should expect that taking the electron-phonon interaction explicitly into account, apart from modifying the effective matrix element for the interaction of two metals which was obtained in[7], would admit of processes involving the emission and absorption of phonons during the electron tunneling. Hence the generation of phonons with the Josephson frequency should be possible because of the presence of an oscillating component in the electron current. One sees easily that for typical experiments in which the radiation from superconducting tunnel structures are studied[8-10] only "long-wavelength" ($\lambda \sim 10^{-3}$ to 10^{-5} cm, λ is the wavelength) acoustic phonons are possible. For such wavelengths the sizes of the usual tunnel junctions can be resonant and when one makes a suitable choice of magnetic field and voltage one can achieve a resonance generation of sound with an appreciable amplitude of the vibrations.

TUNNEL HAMILTONIAN IN THE FRÖHLICH MODEL

Following[7] we write the creation and annihilation operators for electrons at the point **r** with spin z-component σ in the form of an expansion

$$\Psi_\sigma(\mathbf{r}) = \sum_{\mathbf{p}} [f_{\mathbf{p}}^+(\mathbf{r})\alpha_{\mathbf{p}\sigma} + f_{\mathbf{p}}^-(\mathbf{r})\beta_{\mathbf{p}\sigma}]. \quad (1)$$

Here $\alpha_{\mathbf{p}\sigma}$ and $\beta_{\mathbf{p}\sigma}$ are annihilation operators of particles in states with quasi-momentum **p** and spin σ in the right-hand and left-hand side metal, respectively; $\{f_{\mathbf{p}}^\pm\}$ is a complete orthonormal set of functions which was used in[7] to obtain a splitting of the Hamiltonian into Hamiltonians of the left-hand and right-hand side electrons and a single-particle interaction Hamiltonian. The states $f_{\mathbf{p}}^\pm$ are corrected single-particle wave functions $\chi_{\mathbf{p}}^\pm$ for electrons which are incident from the right (+) and the left (-) on a symmetric barrier[1)]

$$f_{\mathbf{p}}^\pm = \chi_{\mathbf{p}}^\pm + \sum_{\mathbf{k}} (\chi_{\mathbf{k}}^{\pm *}\chi_{\mathbf{p}}^\pm)\chi_{\mathbf{k}}^+.$$

The symbol (...) indicates a scalar product. The quantity $(\chi_{\mathbf{k}}^{+*}\chi_{\mathbf{p}}^-)$ is of the order of $\sqrt{D}$ (D is the barrier transmissivity).

In terms of the operators $\alpha_{\mathbf{p}\sigma}$, $\beta_{\mathbf{p}\sigma}$ the tunnel Hamiltonian has the usual form:[7]

$$H = \sum_{\mathbf{p},\mathbf{g},\sigma} T_{\mathbf{p}\mathbf{g}}\alpha_{\mathbf{p}\sigma}^+\beta_{\mathbf{g}\sigma} + \text{h.c.} \quad (2)$$

where $T_{\mathbf{pg}}$ is the matrix element of the effective interaction between two bulk metals ($\hbar = 1$):

$$T_{\mathbf{pg}} = \frac{1}{2m}\int d\mathbf{r}_\perp \left(\chi_{\mathbf{p}}^{+*}\frac{\partial\chi_{\mathbf{g}}^-}{\partial z} - \chi_{\mathbf{g}}^-\frac{\partial\chi_{\mathbf{p}}^{+*}}{\partial z}\right)\Big|_{z=0}, \quad (3)$$

m is the electron mass.

If the oxide film serving as an insulating layer between the superconducting metals is a good dielectric, phonons incident from the metal on one side of the film will be partially reflected and partially go into the other metal without any appreciably attenuation. If the oxide is strongly disordered the phonons may be appreciably damped in the barrier region. However, if we bear in mind that only long-wavelength phonons with $\lambda \gg d$ (d is the thickness of the oxide, $d \sim 10^{-7}$ cm) are important for the generation of sound we must assume that there will be practically no damping in the barrier region. Hence, we can not assume that the barrier is for the phonons an obstacle with a small transparency, as it is for the electrons. On the contrary, the transparency of the barrier for phonons will be of the order of unity. We shall therefore consider a single system of phonons in our transition and not split it into a left-hand side and a right-hand side part. The phonon operator $\hat{\varphi}$ can then be written as an expansion:

$$\hat{\varphi}(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\sqrt{\frac{\omega_0(\mathbf{k})}{2}}[b_{\mathbf{k}}\Psi_{\mathbf{k}}(\mathbf{r}) + b_{\mathbf{k}}^+\Psi_{\mathbf{k}}^+(\mathbf{r})]. \quad (4)$$

Here $\{\Psi_{\mathbf{k}}\}$ is a complete set of phonon functions in the transition studied; $b_{\mathbf{k}}^+$ and $b_{\mathbf{k}}$ are creation and annihila-

1) For the sake of simplicity we restrict our considerations to the case of a symmetric barrier.

tion operators for phonons with quasi-momentum **k**; $\omega_0(\mathbf{k})$ is the phonon frequency.

The lattice energy operator can in the isotropic case in the continuum approximation be written in the form[11]

$$H_{\text{ph}}=\frac{1}{2}\int d\mathbf{r}\rho(\mathbf{r})\left[\dot{\mathbf{P}}^2+s^2(z)\frac{\partial\mathbf{P}}{\partial x_i}\frac{\partial\mathbf{P}}{\partial x_i}\right], \tag{5}$$

where ρ is the density, **P** the displacement of a point of the medium, s the sound velocity ($\hat{\varphi}=\sqrt{\rho}\,s\operatorname{div}\mathbf{P}$).

We assume that the sound velocity and the lattice density in the different metals are the same and equal to s_1 and ρ_1, and in the dielectric they are equal to s_2 and ρ_2.

The mechanism of the electron-phonon interaction consists in the fact that the vibrations of the medium lead to the appearance of a lattice polarization. As a result we must add to the sum of the electron and phonon energy operators a term[11]

$$ea^2\int C(z)\Psi_\sigma^+(\mathbf{r})\Psi_\sigma(\mathbf{r})\operatorname{div}\mathbf{P}\,d\mathbf{r}. \tag{6}$$

Here C is a constant of order ZeN/V (N/V = n is the number of ions per unit volume and Ze their charge) and a the lattice constant.

If we use the expansion (1) the electron-lattice interaction Hamiltonian (6) splits into three parts. The interaction of the electrons with phonons inside each of the metals will have the operator structure $\hat{\varphi}\alpha^+\alpha$ and $\hat{\varphi}\beta^+\beta$. This interaction leads to the appearance of superconducticity and it is sufficient to consider it without taking into account the renormalization following from the tunnel junction of the two metals.[2)] Moreover, there is a term giving the contribution to the electron tunneling:

$$\sum_{\mathbf{p},\mathbf{g},\sigma}\tilde{T}_{\mathbf{pg}}\alpha_{\mathbf{p}\sigma}^+\beta_{\mathbf{g}\sigma}+\text{h.c.} \tag{7}$$

$\tilde{T}_{\mathbf{pg}}$ contains here a phonon operator:

$$\tilde{T}_{\mathbf{pg}}=g\int d\mathbf{r}\hat{\varphi}(\mathbf{r})\chi_{\mathbf{p}}^{+*}(\mathbf{r})\chi_{\mathbf{g}}^-(\mathbf{r}), \tag{8}$$

where $g=(2\pi^2\zeta/p_0m)^{1/2}$ (ζ is a dimensionless constant of order unity), p_0 is the Fermi momentum so that Eq. (7) describes the tunneling of electrons from the one metal into the other with the emission or absorption of phonons.

The resulting tunneling Hamiltonian consists of the Hamiltonian (2) describing elastic tunneling processes of the electrons, and the Hamiltonian (7).

SET OF SELF-CONSISTENT EQUATIONS DETERMINING THE JOSEPHSON PHASE AND THE INDUCED LATTICE VIBRATIONS

When we connect the superconducting tunneling junction to an external circuit there arises in the neighborhood of the barrier an electromagnetic field which most simply can be taken into account by modifying the matrix elements in the tunneling Hamiltonian.[12,13] If we introduce the Josephson phase φ which is connected with the electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ in the barrier through the equations[3,4]

$$\frac{\partial\varphi}{\partial t}=2e\int_1^2 E_z(\mathbf{r},t)dz,\qquad \nabla\varphi=\frac{2e\Lambda}{\bar{c}}[\mathbf{H}(\mathbf{r}_\perp,0,t),\mathbf{n}], \tag{9*}$$

we get instead of (3) and (8)[12]

$$T_{\mathbf{pg}}^{(\varphi)}=\frac{1}{2m}\int d\mathbf{r}_\perp\left(\chi_{\mathbf{p}}^{+*}\frac{\partial\chi_{\mathbf{g}}^-}{\partial z}-\chi_{\mathbf{g}}^-\frac{\partial\chi_{\mathbf{p}}^{+*}}{\partial z}\right)\Big|_{z=0}\exp\left[-\frac{i}{2}\varphi(\mathbf{r}_\perp,t)\right],$$

$$\tilde{T}_{\mathbf{pg}}^{(\varphi)}=g\int d\mathbf{r}_\perp\hat{\varphi}(\mathbf{r})\chi_{\mathbf{p}}^{+*}(\mathbf{r})\chi_{\mathbf{g}}^-(\mathbf{r})\exp\left[-\frac{i}{2}\varphi(\mathbf{r}_\perp,t)\right]. \tag{10}$$

In Eqs. (9) **n** is a unit vector normal to the surface of the oxide, Λ the effective thickness of the layer in which **H** is non-vanishing, $\Lambda=d+2\lambda_S$, where λ_S is the depth of the penetration of the field into the superconductor.

Using the Maxwell equations to eliminate **E** and **H** we get an equation which connects the phase with the current through the transition

$$\nabla^2\varphi-\frac{1}{\bar{c}^2}\left(\frac{\partial^2\varphi}{\partial t^2}+\gamma\frac{\partial\varphi}{\partial t}\right)=\frac{j(\mathbf{r},t)}{j_s\lambda_j^2}, \tag{11}$$

where $\bar{c}=cd/2\epsilon\lambda_S$ is the propagation velocity of retarded waves in the tunnel structure, ϵ the dielectric constant of the junction, $\lambda_j^2=c^2/16\pi e\lambda_S j_S$ is the square of the "Josephson penetration depth" (j_S is the Josephson current amplitude), and γ the effective damping inside the junction.

For the sake of simplicity we shall below consider a "linear" Josephson junction, i.e., a transition for which the difference in phase φ is a function of time and of one spatial coordinate x. The experimental situation is such that φ always depends on two spatial variables. However, as a rule the dependence on one of these variables is very slow. In view of this the theories developed to describe "linear" junctions are well corroborated by experiments.

We determine the current density in (11) by taking the variational derivative of the average value of the Hamiltonian

$$H=\sum_{\mathbf{p},\mathbf{g},\sigma}(T_{\mathbf{pg}}^{(\varphi)}+\tilde{T}_{\mathbf{pg}}^{(\varphi)})\alpha_{\mathbf{p}\sigma}^+\beta_{\mathbf{g}\sigma}+\text{h.c.}$$

with respect to $\varphi(\mathbf{r}_\perp,t)$:

$$j(\mathbf{r},t)=e\delta\langle H\rangle/\delta\varphi(\mathbf{r}_\perp,t). \tag{12}$$

Using Eq. (10), bearing in mind Eqs. (2) and (12), we get

$$j(\mathbf{r},t)=-2e\operatorname{Im}\sum_{\mathbf{p},\mathbf{g},\sigma}\langle T_{\mathbf{pg}}'(\mathbf{r},t)\alpha_{\mathbf{p}\sigma}^+(t)\beta_{\mathbf{g}\sigma}(t)\rangle, \tag{13}$$

where the angle brackets indicate averaging over a non-equilibrium ensemble,

$$T_{\mathbf{pg}}'(\mathbf{r},t)=T_{\mathbf{pg}}^{(1)}+T_{\mathbf{pg}}^{(2)}=\left[\frac{1}{2m}\left(\chi_{\mathbf{p}}^{+*}\frac{\partial\chi_{\mathbf{g}}^-}{\partial z}-\chi_{\mathbf{g}}^-\frac{\partial\chi_{\mathbf{p}}^{+*}}{\partial z}\right)\Big|_{z=0}+g\int dz\,\hat{\varphi}(\mathbf{r})\chi_{\mathbf{p}}^{+*}\chi_{\mathbf{g}}^-\right]\exp\left[-\frac{i}{2}\varphi(\mathbf{r}_\perp,t)\right] \tag{14}$$

$\alpha(t)$, $\beta(t)$ are the operators α and β in the interaction representation.

We write out the average in Eq. (13), dropping terms of order D, g^2D:

2)Taking the renormalization of the matrix elements into account in the interaction of the electrons with phonons inside the metal leads to a correction in the tunneling current $\sim D^2$ and hence lies beyond the limits of accuracy of the model scheme with a tunneling Hamiltonian (see [7]).

*$[\mathbf{H},\mathbf{n}]\equiv\mathbf{H}\times\mathbf{n}$.

$$j(\mathbf{r},t) = 2e\,\mathrm{Re}\int_{-\infty}^{\infty} dt' \sum_{\mathbf{p},\mathbf{g}} \{\langle T'_{\mathbf{pg}}(\mathbf{r},t')\,[T^{(\varphi)}_{-\mathbf{p}-\mathbf{g}}(t) + \widetilde{T}^{(\varphi)}_{-\mathbf{p}-\mathbf{g}}(t)]\rangle$$
$$\times[\hat{F}_{\mathbf{p}}(t',t)\,\check{F}_{\mathbf{g}}(t,t') - \check{F}_{\mathbf{p}}(t',t)\,\hat{F}_{\mathbf{g}}(t,t')] + \langle T'_{\mathbf{pg}}(\mathbf{r},t')\,[T^{(\varphi)*}_{\mathbf{pg}}(t) + \widetilde{T}^{(\varphi)*}_{\mathbf{pg}}(t)]\rangle$$
$$\times[\hat{G}_{\mathbf{p}}(t',t)\,\check{G}_{\mathbf{g}}(t,t') - \check{G}_{\mathbf{p}}(t',t')\hat{G}_{\mathbf{g}}(t,t')]\}. \qquad (15)$$

In this equation we also neglected a term containing the average $\langle T^{(2)}_{\mathbf{pg}}\widetilde{T}^{(\varphi)*}_{\mathbf{pg}}(t')\rangle$ as it is of order ω_D/ϵ_F where ω_D is the Debye limiting energy and ϵ_F the Fermi energy.

The equal-time Green functions are evaluated by means of the equations

$$\check{\hat{G}}_{\mathbf{p}}(t,t') = \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}(\mp i)\,A_{\mathbf{p}}(\omega) f^{\pm}(\omega),$$
$$\hat{\check{F}}_{\mathbf{p}}(t,t') = \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}(\mp i)\,B_{\mathbf{p}}(\omega) f^{\pm}(\omega),$$
$$f^{\pm}(\omega) = [1+e^{\pm\beta\omega}]^{-1}; \quad \beta = 1/2T, \qquad (16)$$

where $A_{\mathbf{p}}(\omega)$ and $B_{\mathbf{p}}(\omega)$ are spectral intensities.

Zubarev[14] has made the necessary calculations for $A_{\mathbf{p}}(\omega)$ and $B_{\mathbf{p}}(\omega)$ in the thermodynamics of the superconducting state on the basis of the Fröhlich Hamiltonian. The expressions obtained in[14] for the spectral intensities are rather complicated relations. As we are solely interested in the problem of the sound generation and not in the modification of the corresponding expression for the Josephson current,[3)] we use for our calculations the functions $A_{\mathbf{p}}(\omega)$ and $B_{\mathbf{p}}(\omega)$ from the BCS theory:

$$A_{\mathbf{p}}(\omega) = \frac{1}{2}\left[\left(1+\frac{\varepsilon_{\mathbf{p}}}{E_{\mathbf{p}}}\right)\delta(\omega - E_{\mathbf{p}}) + \left(1-\frac{\varepsilon_{\mathbf{p}}}{E_{\mathbf{p}}}\right)\delta(\omega + E_{\mathbf{p}})\right],$$
$$B_{\mathbf{p}}(\omega) = -\frac{i}{2}\frac{\Delta}{E_{\mathbf{p}}}[\delta(\omega - E_{\mathbf{p}}) - \delta(\omega + E_{\mathbf{p}})]. \qquad (17)$$

Here Δ is the gap in the elementary excitation spectrum, $E_{\mathbf{p}} = \sqrt{(\epsilon_{\mathbf{p}}^2 + \Delta^2)}$ the elementary excitation energy.

Substituting Eqs. (16) and (17) into Eq. (15) for the current we get after standard transformations[7] and averaging over the atomic spacings in the plane of the tunneling junction

$$j(\mathbf{r},t) = (1 - \xi\,\mathrm{div}\langle\mathbf{P}\rangle)\int_{-\infty}^{\infty} dt'\Big\{K_s(t-t')\sin\frac{1}{2}[\varphi(\mathbf{r}_\perp,t) + \varphi(\mathbf{r}_\perp,t')]$$
$$+ K_n(t-t')\sin{}^1/_2[\varphi(\mathbf{r}_\perp,t) - \varphi(\mathbf{r}_\perp,t')]\Big\},$$

where

$$K_s(t-t') = \frac{1}{\pi e R_N}\int_{-\epsilon_F}^{\infty}\int_{-\epsilon_F}^{\infty} d\xi_1\,d\xi_2\frac{\Delta(\xi_1)\Delta(\xi_2)}{E_1E_2}\,\mathrm{th}\frac{E_2}{2T}$$
$$\sin E_1(t-t')\cos E_2(t-t'),$$
$$K_n(t-t') = \frac{1}{\pi e R_N}\int_{-\epsilon_F}^{\infty}\int_{-\epsilon_F}^{\infty} d\xi_1\,d\xi_2\,\nu(\xi_1)\nu(\xi_2)\left(\mathrm{th}\frac{E_1}{2T} - \frac{\xi_1\xi_2}{E_1E_2}\mathrm{th}\frac{E_2}{2T}\right)$$
$$\times\sin E_1(t-t')\cos E_2(t-t'),$$
$$\xi = \sqrt{\frac{4\zeta M}{3[v(0)-\varepsilon_F]}}\,s_2 p_0 d. \qquad (18)$$

Here ν is the relative density of states ($\nu(0) = 1$), $\xi_{1,2} = p^2/2m - \epsilon_F$ is the energy reckoned from the Fermi level, R_N the resistance of the junction in the normal state, $v(0)$ the height of the energy barrier in the point $z = 0$, and M the ion mass.

One can see that (15) is a rather complicated integro-differential equation determining the phase φ. Its solution must be expressed in terms of the average value $\langle\hat{\varphi}\rangle$ of the phonon operator, the equation for which in turn includes the phase φ.

We write down the equation of motion for the operator $\boldsymbol{\pi} = \rho\,\dot{\mathbf{P}}$ which is canonically conjugate to $\mathbf{P}$:

$$\frac{d\boldsymbol{\pi}}{dt} = \rho s^2\frac{\partial^2\mathbf{P}}{\partial\mathbf{r}^2} + ea^2C\frac{\partial}{\partial\mathbf{r}}\hat{n}(\mathbf{r}). \qquad (19)$$

In (19) $\hat{n}$ is the density operator of the electron gas. Using the expansion (1) for the operator Ψ_σ we can write $\hat{n}$ in the form $\hat{n} = \hat{n}_{1,1} + \hat{n}_{2,2} + \hat{n}_{1,2}$, where $\hat{n}_{1,1}$ and $\hat{n}_{2,2}$ are the operators of the electron density in the first and second metal which can be expressed in terms of the operators $\alpha^+\alpha$ and $\beta^+\beta$, while $\hat{n}_{1,2}$ is the interference term containing the operator products $\alpha^+\beta$ and $\beta^+\alpha$. The electron densities $n_{1,1} = \langle\hat{n}_{1,1}\rangle$ and $n_{2,2} = \langle\hat{n}_{2,2}\rangle$ oscillate over atomic distances because the functions $f^{\pm}_{\mathbf{p}}$ are not plane waves, while $n_{1,2} = \langle\hat{n}_{1,2}\rangle$ both oscillates and changes in the direction at right angles to the barrier plane over distances of the order $\sim$d and in the plane of the barrier over distances of the order of the wavelength of the electromagnetic wave induced in the tunnel structure by the current.

Averaging Eq. (19) over the non-equilibrium statistical ensemble and over atomic spacings we get an equation for $\langle\mathbf{P}\rangle$ in the metals:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\langle\mathbf{P}\rangle - \frac{1}{s_1^2}\left(\frac{\partial^2}{\partial t^2} + \gamma_1\frac{\partial}{\partial t}\right)\langle\mathbf{P}\rangle = 0, \qquad (20)$$

and in the dielectric:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\langle\mathbf{P}\rangle - \frac{1}{s_2^2}\left(\frac{\partial^2}{\partial t^2} + \gamma_2\frac{\partial}{\partial t}\right)\langle\mathbf{P}\rangle = -\frac{g}{s_2\sqrt{\rho_2}}\nabla\langle\hat{n}_{1,2}\rangle. \qquad (21)$$

In (20) and (21) we have introduced terms with first-order time derivatives which take the damping of the phonons into account ($\gamma_{1,2}$ are the damping coefficients of the sound vibrations).

If the transition region the induced force acts in the direction of the x-axis. It is determined by the average value of the density of the electron gas in that region:

$$\langle\hat{n}_{1,2}\rangle = 2\,\mathrm{Re}\int_{-\infty}^{\infty} dt'\Big\{\sum_{\mathbf{p},\mathbf{g}}\exp\left[-\frac{i}{2}\varphi(\mathbf{r}_\perp,t)\right]$$
$$\times\chi_{\mathbf{p}}^{+*}\chi_{\mathbf{g}}^{-}\langle[\alpha_{\mathbf{p}}^{+}(t)\beta_{\mathbf{g}}(t), H(t')]_{-}\rangle + \sum_{\mathbf{p},\mathbf{g}}\exp\left[\frac{i}{2}\varphi(\mathbf{r}_\perp,t)\right]$$
$$\times\chi_{\mathbf{p}}^{+*}\chi_{\mathbf{g}}^{-}\langle[\beta_{\mathbf{g}}^{+}(t)\alpha_{\mathbf{p}}(t), H(t')]_{-}\rangle\Big\}. \qquad (22)$$

It is necessary to complement Eqs. (20) and (21) by boundary conditions (see[17]): 1) the tangential components of the deformations and of the stresses must change continuously at the boundary of the partition of two solids when we go through the boundary plane; 2) the normal component of the stresses at the free end must vanish. This condition corresponds to total reflection of the sound wave from the boundary of the transition as a result of which the displacement $\mathbf{P}$ has a node near the boundary.

The set of Eqs. (11), (20), (21) together with the

---

3) These calculations were done in [16] on the basis Éliashberg's theory. [15]

boundary conditions can be applied to describe the interaction of the variable Josephson current with the sound created by it. To do this it is necessary to express Eq. (22) in terms of the kernels K_S and K_n which were introduced earlier:

$$\langle \hat{n}_{1,2}\rangle = \frac{1}{e\sqrt{2}[v(0)-\varepsilon_F]^{1/2}}\int_{-\infty}^{\infty} dt'\{K_s(t-t')\cos{}^1/_2[\varphi(\mathbf{r}_\perp,t)+\varphi(\mathbf{r}_\perp,t')] + K_n(t-t')\cos{}^1/_2[\varphi(\mathbf{r}_\perp,t)-\varphi(\mathbf{r}_\perp,t')]\}. \quad (23)$$

SOUND IN A JOSEPHSON JUNCTION

In the Josephson case, i.e., the case of low temperatures ($T \ll \Delta$) and small voltages ($eV \ll 2\Delta$) the terms in (18) and (23) which contain K_n give, as usually[7], a small contribution $\sim \exp(-\Delta/T)$ while terms with K_S can be transformed up to terms of order $\sim eV/2\Delta$ to quantities $\sim \sin\varphi$, $\cos\varphi$ so that we get

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{\bar{c}^2}\left(\frac{\partial^2}{\partial t^2}+\gamma\frac{\partial}{\partial t}\right)\varphi = \frac{1-\xi\operatorname{div}\langle\mathbf{P}\rangle}{\lambda_j^2}\sin\varphi, \quad (24)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{P} - \frac{1}{s_2^2}\left(\frac{\partial^2}{\partial t^2}+\gamma_2\frac{\partial}{\partial t}\right)\mathbf{P} = E\sin\varphi\nabla\varphi,$$
$$E = \xi j_s/2ed\rho_2 s_2^2. \quad (25)$$

By $\mathbf{P}$ in Eq. (25) we understand $\langle\mathbf{P}\rangle$. To simplify the notation here and henceforth we drop the brackets $\langle\,\rangle$.

We note that Eqs. (24), (25) can be obtained from the following phenomenoligical Hamiltonian:

$$H = \frac{1}{2}\int_\Omega d\mathbf{r}\left\{\frac{\pi^2}{\rho_2}+\rho_2 s_2^2\frac{\partial\mathbf{P}}{\partial x_i}\frac{\partial\mathbf{P}}{\partial x_i} + \frac{1}{d}\left[\frac{\Theta^2}{\mu}+\mu\bar{c}^2(\nabla\varphi)^2-\frac{j_s}{e}(1-\xi\operatorname{div}\langle\mathbf{P}\rangle)\cos\varphi\right]\right\}, \quad (26)$$

if there are no dissipative processes in the system ($\gamma = \gamma_{1,2} = 0$); here Θ is the momentum which is canonically conjugate to φ, $\mu = \epsilon/16\pi d e^2$.

The integration in Eq. (26) is over the region Ω of the dielectric. The first two terms in the braces are the lattice energy density. The square brackets contain the energy density of the Josephson junction taking into account the interaction with the elastic and electromagnetic fields. If $\zeta = 0$, this will be the normal expression for the Josephson energy density (see, e.g.,[2,12]). One can easily obtain the term containing the interaction from the following simple arguments. The amplitude of the Josephson current is

$$j_s \sim \exp\left\{-2\int_{-d/2}^{d/2} dz\sqrt{2m[v(z)-\varepsilon_F]}\right\} \approx \exp\{-2d\sqrt{2m[v(0)-\varepsilon_F]}\}$$

(we assume that the shape of the potential barrier is nearly rectangular). The quantity $v(0)$ has the physical meaning of the bottom edge of the conduction band of the dielectric layer. In the approximation of the deformation potential theory[18] the bottom of the band is distorted when the elastic deformation of the crystal is taken into account and shifted by an amount $(2\pi^2\zeta\rho_2/p_0 m)^{1/2}s_2\operatorname{div}\mathbf{P}$. Expanding the exponent in j_S in terms of the small magnitude of the band edge displacement we obtain Eq. (26). We note that the quantity ξ may be rather larger, but that $|\xi\operatorname{div}\mathbf{P}| \ll 1$.

To solve the set of Eqs. (24), (25) which we have obtained we shall use the method of successive approximations developing the perturbation theory in terms of the non-linear term in (24). In zeroth approximation we have for φ

$$\varphi_0 = kx+\omega t, \quad \omega = 2eV, \quad k = 2e\Lambda H/\bar{c}. \quad (27)$$

Using (27) we can rewrite Eqs. (20), (21) which determine the displacements of the lattice in the metals and the dielectric as follows

$$(P_y = P_z = 0, \ P_x = P):$$
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)P - \frac{1}{s_1^2}\left(\frac{\partial^2}{\partial t^2}+\gamma_1\frac{\partial}{\partial t}\right)P = 0,$$
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)P - \frac{1}{s_2^2}\left(\frac{\partial^2}{\partial t^2}+\gamma_2\frac{\partial}{\partial t}\right)P = kE\sin(kx+\omega t). \quad (28)$$

We write the boundary conditions in the form

$$\mu_1\frac{\partial P^{(1)}}{\partial z}\bigg|_{z=\pm d/2} = \mu_2\frac{\partial P^{(2)}}{\partial z}\bigg|_{z=\pm d/2},$$
$$P^{(1)}\bigg|_{z=\pm d/2} = P^{(2)}\bigg|_{z=\pm d/2},$$
$$\frac{\partial P}{\partial z}\bigg|_{z=\pm(L+d/2)} = 0, \quad \frac{\partial P}{\partial x}\bigg|_{x=\pm l/2} = 0. \quad (29)$$

where L is the thickness of the superconductor, l the dimension of the transition in the x-direction; μ_1 and μ_2 are the shear moduli in the metal and the dielectric. The shear moduli occur in (29) because the direction of the lattice vibrations is parallel to the plane of the junction and on the boundary between two media the displacement is a pure shear.

Equations (28) together with the boundary conditions are solved in the usual way. For regions 1 we get

$$P = \sum_n\left\{\operatorname{Im}\left[\frac{\mu_2 kE}{g_2}\frac{L_n e^{i\omega t}\sin{}^1/_2 g_2 d\cos g_1(L+d/2+z)}{A}\right]\right\}\cos\frac{\pi n}{l}x,$$
$$A = \mu_2 g_2\sin{}^1/_2 g_2 d\cos g_1 L + \mu_1 g_1\cos{}^1/_2 g_2 d\sin g_1 L, \quad (30)$$

for the transition region 2

$$P = \sum_n\left\{\operatorname{Im}\left[\frac{kEL_n e^{i\omega t}}{g_2}\left(1-\frac{\mu_1 g_1\sin g_1 L\cos g_2 z}{A}\right)\right]\right\}\cos\frac{\pi n}{l}x, \quad (31)$$

where

$$L_n = \frac{2}{l}\int_0^l e^{ikx}\cos\frac{\pi n}{l}x\,dx; \quad n = 1, 2, 3, \ldots;$$
$$g_{1,2} = \left[\frac{\omega^2}{s_{1,2}^2}-\left(\frac{\pi n}{l}\right)^2 - i\frac{\omega\gamma_{1,2}}{s_{1,2}^2}\right]^{1/2}$$

From the solutions we have found for the lattice displacement vector $\mathbf{P}$ we can find that the ω-dependence of the intensity of the n-th harmonic of the sound wave consists of a number of resonance maxima. The values of ω for which these maxima occur correspond to the following voltages at the transition:

$$V \approx \frac{1}{2e}\left[\frac{\pi m}{s_1}\left(\frac{L}{s_1^2}+\frac{\mu_2 d}{2\mu_1 s_2^2}\right)^{-1}+\frac{Ls_1}{2\pi m}\left(\frac{\pi n}{l}\right)^2\right], \quad m = 1,2,3,\ldots \quad (32)$$

According to (30) and (31) the amplitude of the induced vibrations of each harmonic of the standing wave depends then on the magnitude of the applied external magnetic field $\mathbf{H}$ reaching maximum values in the field $H = \pi cn/e\Lambda l$.

Physically the result can be understood as follows: for a finite potential V at the transition there will be a force, oscillating in time with the Josephson frequency, acting upon the dielectric layer which separates the two superconductors in the whole volume of the layer, owing to the interaction of the variable component of the electron current with the lattice vibrations in the dielectric. If the magnetic field H is applied along the y-axis the force vector will be parallel to the plane of the transition and be in the same direction as the x-axis. Moreover, along that direction the magnitude of the volume density of the induced force will be periodically changing. In the superconductors which can be considered to be resonators sound vibrations are then induced. There will be resonance in such a system if the Josephson frequency is the same as one of the eigenfrequencies of the system while the magnitude of the field H is a multiple of the wavelength of the induced sound.

The additional resonance occurring in the oxide layer corresponds to a very low voltage at the layer $V = \pi n s_2/2el$ ($l \sim 10^{-2}$ cm, $s \sim 10^5$ cm-s^{-1}). The resonance in the superconducting metals is also determined by the sound velocity s. However, if we compare Eq. (32) with Kulik's results[3] for the resonance ($V = \pi n\bar{c}/2el$), obtained by considering the interaction of the variable component of the Josephson current with the field of the electromagnetic vibrations generated by it, we can check that by changing the geometry of the sample we can easily arrange the ratio L/l to be of the order of $s/\bar{c}$. The resonance value of V is thereby lifted to a region accessible to experiment.

Near the m-th resonance the n-th harmonic of the standing wave in the superconductors has the form

$$P = (-1)^n \frac{\mu_2}{2\mu_1}\frac{kEds_1}{\pi m}\frac{\operatorname{Im}[e^{i\omega t}L_n(\Delta\omega + i\gamma_1/2)]}{(\Delta\omega)^2 + \gamma_1^2/4} \times \cos\frac{\pi n}{l}x\cos\frac{\pi m}{L}\left(\frac{d}{2}+z\right), \tag{33}$$

and its intensity

$$I = \left(\frac{\mu_2}{4\mu_1}\frac{kEds_1}{\pi m}\right)^2\left[(\Delta\omega)^2 + \frac{\gamma_1^2}{4}\right], \quad k = \frac{\pi n}{l}. \tag{34}$$

Josephson frequencies are of order of magnitude $\sim 10^9$ to 10^{10} s^{-1} and the relaxation times for electron excitations $\tau \sim 10^{-9}$ to 10^{-10} s. The interaction of a sound wave with electrons can then be considered to be the emission and absorption of sound quanta. In the framework of the isotropic BCS model the ratio of the electron sound absorption coefficients in the superconducting and normal states is equal to ($\omega < 2\Delta$)[19]

$$\gamma_s/\gamma_n = 2[1 + e^{\Delta/T}]^{-1}. \tag{35}$$

When $\omega\tau \gg s v_F$ (v_F is the electron velocity at the Fermi surface) the absorption coefficient for sound vibrations in the normal metal is proportional to the first power of the frequency and independent of the temperature and the purity of the sample.[20] As to order of magnitude $\gamma_n \sim 10^6$ to 10^7 s^{-1}.

The scattering of the energy of a sound wave by thermal phonons is not changed in the superconducting transition as the phonon spectrum of a superconductor has no peculiar features whatever as compared to the normal state. Simons' calculations[21] show that for $T < \theta_D/15$ the damping caused by the interaction between the high-frequency sound and the lattice vibrations is about an order of magnitude below what can be detected experimentally. Under reasonable assumptions we can thus take it that in a superconductor the damping is mainly determined by the interaction between the sound wave quanta and the conduction electrons.

We obtained Eqs. (33) and (34) assuming that $\gamma_1/2s_1 \ll \pi/l$. If we had not made this assumption but had assumed that the sound wave in the metal is damped over distances less than l we must, when solving the set of Eqs. (28), look for the wave in the x-direction in the form of a traveling wave. In that case we get

$$P = \operatorname{Im}\left[\frac{\mu_2 kE}{g_2}\frac{\sin{}^1/_2 g_2 d\cos g_1(L + d/2 + z)}{A}\exp i(\omega t + kx)\right],$$

$$P = \operatorname{Im}\left[\frac{kE}{g_2^2}\left(1 - \frac{\mu_1 g_1 \sin g_1 L\cos g_2 z}{A}\right)\exp i(\omega t + kx)\right],$$

$$g_{1,2} = \left[\frac{\omega^2}{s_{1,2}^2} - k^2 - i\frac{\omega\gamma_{1,2}}{s_{1,2}^2}\right]^{1/2}$$

For the sake of simplicity we assume that the elastic constants and damping coefficients of the materials making up the Josephson transition are the same. The intensity of the sound wave in the points of the maximum

$$\omega = \pi m s\left(L + \frac{d}{2}\right)^{-1} + \frac{k^2 s}{2\pi m}\left(L + \frac{d}{2}\right)$$

will be given by the formula ($\omega^2 \gg k^2 s^2$)

$$I = \left(\frac{kEds}{4\omega L}\right)^2\left[(\Delta\omega)^2 + \frac{\gamma_1^2}{4}\right]. \tag{36}$$

The change in the current-voltage characteristic arising due to the resonance excitation of sound in a Josephson junction can be obtained by writing down the equation for the constant component of the Josephson current:

$$\bar{j} = \frac{1}{l}\int \overline{j(x,t)}\,dx = \frac{j_s}{l}\left[\int \overline{\cos(kx + \omega t)\varphi_1(x,t)}\,dx - \xi\int \overline{\sin(kx + \omega t)\operatorname{div}\mathbf{P}}\,dx\right]. \tag{37}$$

Here φ_1 is a correction to the phase φ in the first order of perturbation theory and the bar indicates time-averaging. The first term in (37) leads to the usual structure on the current-voltage characteristic, which is connected with the excitation of electromagnetic waves in the tunneling layer.[3,4]

Evaluation of the second term in Eq. (37) leads to the following correction for the current ($\gamma_1/2s_1 \ll \pi/l$) in the m-th maximum:

$$\bar{j}^{max} = \sum_n \frac{\mu_2}{\mu_1}\frac{dL}{l^2}\xi E\frac{Q_m}{m^2}n^2 F_n^2\left(\frac{\Phi}{\Phi_0}\right), \tag{38}$$

where $Q_m = \omega_m/\gamma_1(m)$ is the quality factor of the m-th resonance, Φ the magnetic field flux passing through the transition, and Φ_0 a magnetic flux quantum. The functions $F_n(x)$ in (36) are defined as usually:[2]

$$F_n(x) = \frac{2}{\pi}\frac{x}{|x^2 - (n/2)^2|}\begin{cases}|\cos\pi x|, & n = 1,3,5,\ldots \\ |\sin\pi x|, & n = 2,4,6,\ldots\end{cases}.$$

The parameters in the theory, ξ and E depend in an essential way on the energy characteristics of the oxide layer. This dependence enters into them as $(v(0) - \epsilon_F)^{-1/2}$. To estimate ξ and E it is thus necessary to assign some height to the barrier at the point $z = 0$. The dependence on the height of the barrier for a given transparency for electrons occurs because in the matrix element (8) we have the product of the functions $\chi_p^\pm$ and not their derivatives as in the matrix element T_{pg}. Because of this, the smaller the gradient in the z-direction of the functions $\chi_p^\pm$ the larger the relative contribution from inelastic processes. In practice the quantity $v(0) - \epsilon_F$ can be made rather small compared to the Fermi energy. It seems that in this respect S-N-S junctions are very successful. However, for such transitions the relative weight of the Josephson component of the current drops[22] so that the most suitable layer from the point of view of a maximum generation of sound for a given transparency of the junction will be conductors or oxides with a small magnitude of the forbidden band.

We now estimate the experimental possibilities to observe the generation of sound in Josephson junctions. For typical experiments (see the review[2]) the dimensions of the transitions are the following: $d \sim 10^{-7}$ cm, $L \sim 10^{-5}$ cm, $l \sim 10^{-2}$ cm. If we take $v(0) - \epsilon_F \sim 10^{-2}$ ϵ_F, and $\lambda_j \sim l$, we get for ξ and E the values $\sim 10^{-2}$ and 10^{-9}. The power of the sound vibrations generated by the junction will then for $\gamma_1 \sim 10^4\ s^{-1}$ and $\omega \sim s^{-1}$ be $\sim 10^{-10}$ W. We note that experimentally electromagnetic radiation with a power $\sim 10^{-14}$ W has been registered. It is clear from Eq. (35) that the damping coefficient γ_1 can be appreciably lowered by decreasing the temperature. This is important for an indirect observation of the sound resonances on the current-voltage characteristic. One sees easily that for $T \sim 1^\circ K$, $\gamma_1 \sim 1$. Then $\Delta j_e/\Delta j_s \sim 10$, where Δj_e is the increase in the constant component of the current due to the resonance excitation of electromagnetic waves and Δj_s the corresponding increase due to sound. This estimate is made for a magnetic field in which the maximum sound intensity is reached with $n = 10$. It can be seen from Eqs. (34), (36), and (38) that the whole effect increases with increasing field $\sim H^2$. However, in strong fields when $\sin \pi x/x \sim 2e\Theta_{eff}/j_s$ thermal fluctuations will appear which arise in the junction and are produced by the external field (Θ_{eff} is the effective temperature of the fluctuations). In the fluctuation region the effect will be appreciably reduced.

[1] B. D. Josephson, Phys. Lett. 1, 251 (1962); Rev. Mod. Phys. 36, 221 (1964).

[2] I. O. Kulik and I. K. Yanson, Effekt Dzhosefsona v sverkhprovodyashchikh tunnel'nykh strukturakh (The Josephson Effect in Superconducting Tunneling Structures), Nauka, 1970.

[3] I. O. Kulik, ZhETF Pis. Red. 2, 124 (1965) [JETP Lett. 2, 84 (1965)].

[4] Yu. M. Ivanchenko, A. V. Svidsinskiĭ, and V. A. Slyusarev, Zh. Eksp. Teor. Fiz. 51, 194 (1966) [Sov. Phys.-JETP 24, 131 (1967)].

[5] M. H. Cohen, L. M. Falicov, and J. C. Phillips, Phys. Rev. Lett. 8, 316 (1962).

[6] R. E. Prange, Phys. Rev. 131, 1083 (1963).

[7] Yu. M. Ivanchenko, K teorii mnogochastichnogo tunnelirovaniya (On the Theory of Many-body Tunneling) Preprint Donets Phys.-Tech. Inst., Acad. Sc., Ukr.S.S.R., No. 76, 1965.

[8] I. K. Yanson, V. M. Svistunov, and I. M. Dmitrenko, Zh. Eksp. Teor. Fiz. 48, 976 (1965) [Sov. Phys.-JETP 21, 650 (1965)].

[9] D. D. Coon and M. D. Fiske, Phys. Rev. 138, A744 (1965).

[10] A. A. Galkin and V. M. Svistunov, ZhETF Pis. Red. 5, 396 (1967) [JETP Lett. 5, 323 (1967)].

[11] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Quantum Theory Field Methods in Statistical Physics) Fizmatgiz, 1962 [Pergamon Press, Oxford, 1965].

[12] Yu. M. Ivanchenko, Zh. Eksp. Teor. Fiz. 52, 1320 (1967) [Sov. Phys.-JETP 25, 878 (1967)].

[13] M. J. Stephen, Phys. Rev. 182, 531 (1969).

[14] D. N. Zubarev, Dokl. Akad. Nauk SSSR 132, 1055 (1960) [Sov. Phys.-Doklady 5, 570 (1960).

[15] G. M. Éliashberg, Zh. Eksp. Teor. Fiz. 38, 966 (1960) [Sov. Phys.-JETP 11, 696 (1960)].

[16] T. Fulton and D. McCumber, Phys. Rev. 175, 585 (1968).

[17] L. D. Landau and E. M. Lifshitz, Teoriya uprugosti (Theory of Elasticity), Nauka, 1965 [Pergamon Press, Oxford, 1970].

[18] A. I. Ansel'm, Vvedenie v teoriyu polyprovodnikov (Introduction to the Theory of Semiconductors), Gostekhizdat, 1962.

[19] J. Bardeen, L. Cooper, and J. Schrieffer, Phys. Rev. 108, 1175 (1957).

[20] A. I. Akhiezer, M. N. Kaganov, and G. Ya. Lyubarskiĭ, Zh. Eksp. Teor. Fiz. 32, 837 (1957) [Sov. Phys.-JETP 5, 685 (1957)].

[21] S. Simons, Proc. Camb. Phil. Soc. 53, 702 (1957).

[22] L. G. Aslamazov, A. I. Larkin, and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 55, 323 (1968) [Sov. Phys.-JETP 28, 171 (1969)].

Translated by D. ter Haar
246