SOVIET PHYSICS JETP VOLUME 34, NUMBER 5 MAY, 1972

STATISTICAL PROPERTIES OF GAS LASER EMISSION IN A LONGITUDINAL MAGNETIC FIELD

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Submitted May 19, 1971

Zh. Eksp. Teor. Fiz. **61**, 1801–1807 (November, 1971)

Fluctuations of total emission energy, polarized components, and their line widths as functions of the longitudinal magnetic field and the type of atomic transition are determined in the small emission energy approximation. The emission depolarization coefficient is determined taking the resonator anisotropy into account.

1. In the present work we consider the effect of longitudinal magnetic field on the fluctuation characteristics of a gas laser; this effect is not taken into account in the semi-classical approach (see[1] for example). The interest in quantum theory was expressed in the recent work of H. Hübner[2] who derived the Fokker-Planck equation for the photon distribution function in exact resonance and the $j = 0 \to j = 1$ transition, taking into account magnetic field splitting of the $j = 1$ level, and found a stationary solution for the case of an isotropic resonator.

A solution of the problem for the case of anisotropic resonator Q, off-resonance, and any type of atomic transition is difficult because of the non-potentiality of the probability current for the polarized field components. The Q anisotropy can be taken into account in the case of exact resonance for transitions between levels with total momenta $j \to j + 1$ and $\frac{1}{2} \to \frac{1}{2}$ (Sec. 3). We find the off-resonance solution of the distribution function by neglecting the anisotropy (Secs. 4 and 5).

2. We write the equation for photon distribution function P in the sudarshan-Glauber coherent-state representation. The equation is derived by a method presented in[3,4] ($\hbar = 1$):

$$\frac{2Q}{\omega}\frac{\partial P}{\partial t} + \sum_{q=\pm}\left\{\frac{\partial}{\partial z_q}\left[(\xi - i\Omega_0)z_q - \alpha z_{-q} - 2\beta z_q(B_q|z_q|^2) + AC_q|z_{-q}|^2 - \frac{\partial}{\partial z_q^*}\right]P + \text{c.c.}\right\} = 0,$$

$$\Omega_0 = \frac{2}{\sqrt{\pi}}\frac{\mu_0 gH}{ku}, \quad B_\pm = 1 + [1 - i(\delta \mp \Omega)]^{-1}, \tag{1}$$

$$C_\pm = \frac{1}{2}\left[\frac{1}{1 - i\delta} + \frac{1}{1 \pm i\Omega} + \frac{1}{1 \pm 2i\Omega'}\left(\frac{1}{1 - i(\delta \mp \Omega)} + \frac{1}{1 \pm l\Omega}\right)\right]$$

$$\Omega' = \Omega\frac{\gamma_s}{\nu}.$$

Here $z_\pm$ are the eigenvalues of the annihilation operators for photons with different circular polarization, and ξ is the relative pumping excess over threshold ($\xi \ll 1$), A is nonlinear coupling coefficient of circular field components at Zeeman sublevels[4]

$$\beta = 2\pi\omega \sum_{l,\lambda}|d_{l\lambda}^+|^4 \Big/ \sum_{l,\lambda}|d_{l\lambda}^+|^2 V\gamma\gamma_s$$

is the saturation parameter (l and λ are upper and lower state sublevel numbers), V is the volume of the system, $d_{l\lambda}^\pm$ are the circular components of the dipole moment, γ and γ_S are the widths of the levels and of the luminescence lines, $\delta = (\omega - \omega_0)/\gamma_S$ is the dimensionless mismatch of the resonator frequency $\omega = kc$ relative to the amplification line center ω_0, the quantity $\Omega = \mu_S gH/\gamma_S$ characterizes Zeeman splitting in magnetic field H for equal g factors, Q is the average Q of the resonator, and α is the relative difference of the Q factors ($Q_y > Q_x$, $0 < \alpha \ll 1$).

We obtained (1) making the usual assumptions of small characteristic frequencies relative to the Doppler width ku($\gamma_S|\delta|$, $\gamma_S\Omega$, $\gamma_S \ll$ ku); the expression in brackets in the equation without the gradient operator corresponds to the classical current[5]. It is convenient to use spherical coordinates in (1), since the equation for the stationary distribution function admits of a solution with a well determined phase shift φ between the circular field components and with a total number of photons $n = n_+ + n_-$:

$$z_\pm = \left[\frac{n}{2}(1 \mp \cos\theta)\right]^{1/2}\exp\left[\frac{i}{2}(\psi \pm \varphi)\right].$$

Here

$$\frac{Q}{\omega}\frac{\partial P}{\partial t} + \frac{1}{n}\frac{\partial}{\partial n}n^2\left\{P[\xi - \alpha\cos\varphi\sin\theta - n\beta(D_1 + D_2\cos^2\theta + D_3\cos\theta)] - \frac{\partial P}{\partial n}\right\} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left\{P\cos\theta[-\alpha\cos\varphi + n\beta(D_2\sin\theta - D_4\operatorname{tg}\theta)] - \frac{1}{n}\frac{\partial P}{\partial\theta}\right\} + \frac{\partial}{\partial\varphi}\left\{P\left[\frac{\alpha\sin\varphi}{\sin\theta} + \beta n(F_1\cos\theta + F_3) - \Omega_0\right] - \frac{1}{n\sin^2\theta}\left(\frac{\partial P}{\partial\varphi} + \cos\theta\frac{\partial P}{\partial\psi}\right)\right\} + \frac{\partial}{\partial\psi}\left\{P[\alpha\sin\varphi\operatorname{ctg}\theta - \beta n(F_2 + F_4\cos\theta)] - \frac{1}{n\sin^2\theta}\left(\frac{\partial P}{\partial\psi} + \cos\theta\frac{\partial P}{\partial\varphi}\right)\right\} = 0, \tag{2}$$

where $2D_{1,2} = \operatorname{Re}[B_+ + B_- \pm A(C_+ + C_-)]$, $D_3 = \operatorname{Re}(B_+ - B_-)$, $2D_4 = \operatorname{Re}[B_+ - B_- + A(C_+ - C_-)]$, $2F_{1,2} = \operatorname{Im}[B_+ + B_- \mp A(C_+ + C_-)]$, $2F_{3,4} + \operatorname{Im}[B_- - B_+ \pm A(C_- - C_+)]$. We note that $D_{3,4} = F_{1,2} = 0$ for $\delta = 0$ and $D_{3,4} = F_{3,4} = \Omega_0 = 0$ for $H = 0$.

Further analysis is based on the assumption that $\xi - \alpha \gg \sqrt{\beta}$, which means that the system is not in thermodynamic equilibrium (regeneration regime).

3. We consider a stationary solution of (2) for $A \le 1$ (transitions $j \longleftrightarrow j + 1$ and $\frac{1}{2} \to \frac{1}{2}$) in exact resonance $\delta = 0$. The distribution function does not depend on the angle φ in this case. Its solution is sought near

the maximum points that are related by the classical equations

$$\sin\theta_0 = 1,\ \alpha\cos\varphi_0 = \xi - n_0\beta D_1,\ \alpha\sin\varphi_0 = \Omega_0 - n_0\beta F_3 \qquad (3)$$

and that determine linearly polarized emission with the polarization plane rotated through the angle $\varphi_0/2$ from the x axis[5]. Taking the non-potentiality of probability current into account and proceeding analogously to[6] we obtain the following expression for the distribution function near the maximum:

$$\ln\frac{P}{P_0} = -\frac{1}{1+\rho^2}\left[\frac{a_1}{2}(n-n_0)^2 - a_{12}(n-n_0)(\varphi-\varphi_0) + \frac{a_2}{2}(\varphi-\varphi_0)^2\right] - \frac{b}{2}(\theta-\theta_0)^2,$$

$$\rho = (2\alpha\sin\varphi_0 - \Omega_0)\,/\,(\xi - 2\alpha\cos\varphi_0),\quad a_1 = \beta(D_1 - \rho F_3),$$

$$a_{12} = n_0\beta(F_3 + \rho D_1),$$

$$a_2 = n_0\alpha(\rho\sin\varphi_0 - \cos\varphi_0),\qquad b = n_0(n_0\beta D_2 - \alpha\cos\varphi_0). \qquad (4)$$

The conditions for the maximum of the distribution function $(a_1, a_2, (a_1a_2 - a_{12}^2), b > 0)$ coincide with the stability conditions of the classical solution (3). Since we are operating within an approximation for which the distribution function has a clearly expressed Gaussian characteristic, these coefficients are subject to the strengthened condition $(a_1a_2 - a_{12}^2)/a_1 \sim b \gg 1$.

The dispersion characteristics can be readily found after computing the normalization of P_0 as a function of the parameters a and b. The obtained results are valid for $\alpha\xi \gg \beta$: furthermore the transition $0 \to 1$ ($A = 1$) for weak magnetic fields is subject to an additional condition $\Omega'\xi \gg \sqrt{\beta}$. We write the mean values and dispersions for the total number n of photons and for photons with circular $n_\pm$ and linear $n_{x,y}$ polarizations:

$$\langle(\Delta n)^2\rangle = a_2(1+\rho^2)(a_1a_2 - a_{12}^2)^{-1},\ \langle n\rangle = n_0,$$

$$\langle(\Delta n_\pm)^2\rangle = {}^1/_4[\langle(\Delta n)^2\rangle + n_0^2/b],\ \langle n_\pm\rangle = n_0/2,$$

$$\langle n_{x,y}\rangle = \frac{1}{2}n_0(1\pm\cos\varphi_0) \mp \langle(\Delta n)^2\rangle\left(\frac{a_{12}\sin\varphi_0}{a_2} + \frac{1}{4}\frac{a_1\cos\varphi_0}{a_2}\right) \mp \frac{n_0}{4b}\cos\varphi_0,$$

$$\langle(\Delta n_{x,y})^2\rangle = \frac{1}{4}\frac{\langle(\Delta n)^2\rangle}{a_2}\Big[a_2(1\pm\cos\varphi_0)^2 \mp 2a_{12}n_0\sin\varphi_0(1\pm\cos\varphi_0) + a_1n_0^2\sin^2\varphi_0\Big]. \qquad (5)$$

All dispersion formulas have accuracy up to terms of the order of ξ. It is possible to take in them the limit as $\cos\varphi_0 \to -1$ (for $H \to 0$ for example), with the exception of the formula for $\langle(\Delta n_x)^2\rangle$ which is cumbersome and is therefore written for those values of the magnetic field H for which $\sin^2\varphi_0 \gg \sqrt{\beta}/\alpha$.

The depolarization coefficient d[7] and the emission line width ν are also readily computed:

$$d = \frac{1}{4}\left(\frac{1}{b} + \frac{a_1(1+\rho^2)}{a_1a_2 - a_{12}^2}\right), \qquad (6)$$

$$\nu = \omega\,/\,4Qn_0. \qquad (7)$$

In the limit $H = 0$, Eqs. (5)--(7) yield the results of[4] (an exact formula is used here for $\langle(\Delta n_x)^2\rangle$). From now on we assume that $\xi \gg \alpha$ ($\alpha \sim 10^{-3}$ [5]). As the magnetic field increases in the region

$$\alpha \gg \Omega\left|\xi - 2\frac{\gamma_s}{ku}\frac{1+A}{1+2A\gamma_s/\gamma}\right| > 0\quad\left(\frac{\gamma_s}{\gamma} \gg 1\right)$$

the variation of relative fluctuation of the total number of photons and of the depolarization coefficient depends on the relative pumping excess over threshold. In the range of values of ξ

$$\alpha\xi \gtrless (\xi-\xi_0)^2 \geqslant 0,\quad \xi_0 = 2\frac{\gamma_s}{ku}\frac{(1+A)}{1+2A\gamma_s/\gamma}, \qquad (8)$$

n_0 increases and $\langle(\Delta n)^2\rangle/n_0^2$ and d decrease:

$$n_0 = n_0|_{H=0}\left(1 + \Omega^2\frac{1+2A(\gamma_s/\gamma)^2}{2(1+A)}\right),$$

$$\frac{\langle(\Delta n)^2\rangle}{n_0^2} = \frac{\langle(\Delta n)^2\rangle}{n_0^2}\Big|_{H=0}\left(1 - \Omega^2\frac{1+2A(\gamma_s/\gamma)^2}{2(1+A)}\right),$$

$$d = d|_{H=0}\left[1 - \Omega^2\frac{1+4A(\gamma_s/\gamma)^2}{4(1+A)^2}\left(1 + \frac{\xi}{\alpha}\mathrm{Ant}\,A\right)\right]. \qquad (9)$$

The reverse effect is observed outside the range of energies (8)

$$n_0 = n_0|_{H=0}\left(1 - \frac{\alpha}{2\xi}\sin^2\varphi_0\right),\ \sin\varphi_0 = -\frac{\Omega}{\alpha}(\xi-\xi_0)\frac{1+2A\gamma_s/\gamma}{2(1+A)},$$

$$\frac{\langle(\Delta n)^2\rangle}{n_0^2} = \frac{\langle(\Delta n)^2\rangle}{n_0^2}\Big|_{H=0}\left(1 + \frac{2\alpha}{\xi}\sin^2\varphi_0\right),$$

$$d = d|_{H=0}\left[1 + \frac{\alpha}{2\xi}\sin^2\varphi_0\left(1 + \frac{\xi}{2\alpha}\mathrm{Ant}\,A\right)\right]. \qquad (10)$$

In the above formulas, for the case $A \neq 0$, we neglect unity as compared to $\gamma_s A/\gamma$. In the second case the increase of fluctuations and depolarization is due to the increased fluctuations of the y-th component of the emission. According to (7), as the magnetic field increases, the emission line width decreases within the energy range (8) and increases away from it. The appearance of the characteristic quantity ξ_0 can be attributed to the interaction of "Bennett holes" burned by the circular field components in the corresponding amplification contours that move apart as the magnetic field increases.

As the magnetic field increases further the angle of rotation of the polarization plane from the y axis increases. In the region of critical values of the magnetic field where $|\sin\varphi_0| \to 1$ the fluctuation of the total photon number and depolarization increase sharply:

$$\frac{\langle(\Delta n)^2\rangle}{n_0^2} = \frac{\alpha}{n_0^2\beta}[\alpha^2(D_1^2 + F_3^2) - (\xi F_3 - \Omega_0 D_1)^2]^{-1/2},$$

$$d = \frac{1}{4}\left(\frac{1}{b} + \frac{a_1 n_0\langle(\Delta n)^2\rangle}{\alpha n_0^2}\right),$$

$$\alpha \gg [\alpha^2(D_1^2 + F_3^2) - (\xi F_3 - \Omega_0 D_1)^2]^{1/2} \gg \beta/\xi. \qquad (11)$$

4. To simplify computation in the case of the isotropic resonator ($\alpha = 0$) we only consider small magnetic fields and small detunings (Ω, Ω', $\delta^2 \ll 1$) for any A, excluding $A = 1$. The stationary distribution function does not depend on the angles φ and ψ. This means that the circular polarizations are split by frequencies and that, depending on the transition type, the function has one ($A < 1$) or two ($A > 1$) maxima whose coordinates are related by

$$A < 1,\qquad \cos\theta_0 = \frac{\delta}{2}\frac{2\Omega + A(\Omega+\Omega')}{1-A},\qquad n_0 = \frac{\xi}{\beta D_1}, \qquad (12)$$

$$A > 1,\qquad \cos\theta_0 = \pm 1,\qquad n_{0\mp} = \frac{\xi}{\beta}\frac{1}{D_1 + D_2 \mp 4\delta\Omega}$$

$$D_{1,2} = (1 \pm A)(2 - \delta^2 - \Omega^2) \mp A({}^1/_2\Omega^2 + 2\Omega\Omega' + 4\Omega'^2). \qquad (13)$$

The first case corresponds in the classical theory to elliptic polarization, the ellipse rotating at the rate

$$\frac{\omega}{2Q}\left[\frac{\xi}{4(1+A)}(2\Omega+3A\Omega+4A\Omega')-\Omega_0\right]$$

and having an eccentricity $|\cos\theta_0|/2$. The second case corresponds to circular polarization (left- or right-handed).

For $A<1$, expanding the distribution function near the maximum

$$\ln\frac{P}{P_0}=-\beta D_1(n-n_0)^2+\frac{A\delta(\Omega+\Omega')\xi}{2}(n-n_0)(\theta-\theta_0)-\frac{\xi^2D_2}{\beta D_1^2}(\theta-\theta_0)^2 \qquad (14)$$

and neglecting as before the small terms of the order of ξ as compared to unity, we find the following expressions for the relative fluctuations and emission line widths $\nu_\pm$ of the circularly polarized field components:

$$\langle(\Delta n)^2\rangle/n_0^2=\beta D_1/\xi^2,\quad \langle n\rangle=n_0,$$
$$\frac{\langle(\Delta n_\pm)^2\rangle}{\langle n_\pm\rangle^2}=\beta D_1\xi^{-2}\left[1+\frac{D_1}{D_2}\pm\pm 2\delta\frac{1+A}{(1-A)^2}\left(\Omega+A\left(1-\frac{A}{2}\right)(\Omega+\Omega')\right)\right], \qquad (15)$$
$$\langle n_\pm\rangle={}^1/_2n_0(1\mp\cos\theta_0),\quad \nu_\pm=\omega/4Q\langle n_\pm\rangle.$$

In the absence of nonlinear coupling of the circular field components ($A = 0$, transitions $1/2\to1/2$) these expressions are considerably simplified:

$$\langle(\Delta n)^2\rangle/n_0^2=\beta\xi^{-2}(2-\delta^2-\Omega^2),\ n_0=\xi/\beta(2-\delta^2-\Omega^2),$$
$$\frac{\langle(\Delta n_\pm)^2\rangle}{\langle n_\pm\rangle^2}=4\beta\xi^{-2}\left[1-\frac{(\delta\mp\Omega)^2}{2}\right],\quad \langle n_\pm\rangle=\frac{\xi}{4\beta}\left[1+\frac{(\delta\mp\Omega)^2}{2}\right]. \qquad (16)$$

The relative dispersions of total energy decrease with increasing magnetic field and mismatch. We note that $\langle n_+\rangle \gtrless \langle n_-\rangle$, $\nu_+\gtrless\nu_-$, $\langle(\Delta n_+)^2\rangle/\langle n_+\rangle^2 \lessgtr \langle(\Delta n_-)^2\rangle/\langle n_-\rangle^2$ for $\delta\lessgtr 0$ and $H\neq 0$. Thus if $|\delta|\gg\Omega'$ the emission line width and relative fluctuations of the circular component with a smaller (larger) number of photons increase (decrease) proportionally to H as the magnetic field increases. On the other hand if $|\delta|\ll\Omega'$ the relative fluctuations and emission line widths of both components decrease with increasing H and the decrease is faster for the component with the larger number of photons. For $A = 0$ these characteristics assume maximum values at the points $\delta=\pm\Omega$ for components $n_\pm$ respectively.

5. We now turn to generation regime (13) (transitions $j\to j$ with $j>1$) under the previous assumptions of δ and $\Omega'\ll1$. At resonance the stationary distribution function is determined exactly:

$$P=P_0\exp\{n\xi-{}^1/_2n^2\beta(D_1+D_2\cos^2\theta)\}. \qquad (17)$$

For $A = 1$ it coincides with the expression obtained in[2]. This distribution function is symmetrical about the plane $\Theta=\pi/2$. Following[4,6] we compute the time of transition between metastable states (13)

$$\tau=16\pi\frac{Q}{\omega}\left(\frac{n_0}{\beta}\right)^{-1/2}\frac{D_1^{3/2}}{\xi|D_2|}\exp\left[\frac{\xi n_0|D_2|}{2D_1}\right]. \qquad (18)$$

It turns out to be exponentially large in the approximation adopted for the parameters ξ and β and decreases with increasing magnetic field. In the time $\nu^{-1}\ll t\ll\tau$ we have right or left polarized emission depending on initial conditions. Increasing the detuning introduces differences between the height of maxima at points (13) and the probabilities of the corresponding states. In connection with this, the transition time from one metastable level to another also changes, increasing for the transition from a more probable to a less probable state and decreasing for the opposite transition. These times can be qualitatively evaluated from (18) after substituting in it the corresponding values of (13) taking the detuning into account.

When the detuning is small the argument of the exponential does not change significantly and the transition time between maxima remains exponentially large as before. Off-resonance we can use quasi-stationary distribution functions for the above time interval:

$$\ln\frac{P_q}{P_{0q}}=-\frac{\xi(n-n_{0q})^2}{2n_{0q}}+\frac{n_{0q}^2\beta}{2}(D_2+qD_4)(\theta-\theta_{0q})^2,$$
$$D_4=\delta[2\Omega+A(\Omega+\Omega')]. \qquad (19)$$

The index $q=\pm$ shows the domain of a maximum, the approach to which depends on the initial conditions. The ratio of probabilities P_{0+}/P_{0-} of states $q=\pm$ is determined by the exponent $\delta\Omega\xi^2/\beta$; they are equally probable only when $|\delta|\Omega\gg\beta/\xi^2$.

The fluctuation characteristics and the emission line widths for circular components in these states have the form

$$\langle n\rangle\approx\langle n_q\rangle=n_{0q}=\frac{\xi}{4\beta}\left[1+\frac{(\delta-q\Omega)^2}{2}\right],\quad \langle n_{-q}\rangle=[2\beta n_{0q}|D_2+qD_4|]^{-1},$$
$$\langle(\Delta n)^2\rangle/n_{0q}^2=\langle(\Delta n_q^2)\rangle/\langle n_q\rangle^2=1/\xi n_{0q},\quad \langle(\Delta n_{-q})^2\rangle=\langle n_{-q}\rangle^2,$$
$$\nu_q=\omega/4Qn_{0q},\quad \nu_{-q}/\nu_q=n_{0q}/\langle n_{-q}\rangle\gg1. \qquad (20)$$

The relative dispersion and emission line width for the n_α component are maximum in the point $\delta=q\Omega$ that marks the most favorable conditions for amplification of the opposite polarization whose intensity n_{-q} is completely determined by fluctuations increasing with growing H for $|\delta|\ll\Omega'$.

The authors thank V. S. Smirnov for discussions of issues in the course of the work.

Note added in proof (September 24, 1971). Regime (12) has a large degree of depolarization $d=1-2|\cos\theta_0|$. The polarized portion of the emission represents a superposition of two circular modes of equal intensity and different frequency (G. I. Smirnov, B. L. Zhelnov. Paper at Second Vavilov Conference on Nonlinear Optics, Novosibirsk, June 1971).

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Translated by S. Kassel
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