SOVIET PHYSICS JETP VOLUME 34, NUMBER 6 JUNE, 1972

Vacuum Polarization and Photon Splitting in an Intense Field

V. O. PAPANYAN AND V. I. RITUS

P. N. Lebedev Physics Institute, USSR Academy of Sciences

Submitted July 2, 1971

Zh. Eksp. Teor. Fiz. **61**, 2231–2241 (December, 1972)

Owing to vacuum polarization, splitting of a photon into two photons becomes possible in an intense electromagnetic field. The general structure of the polarization operator which describes three-photon interaction in an intense crossed field is determined. Its concrete expression for real photons in terms of two invariant functions is found. These functions describe the spectrum and polarization of finite photons as functions of the initial photon energy and polarization. For an initial photon energy of 25 GeV and a field strength of 4.10^8 Oe the photon mean free path with respect to photon splitting is of the order of several centimeters.

1. INTRODUCTION

WHEN photons interact with an external field, they can split into two: $\gamma \to \gamma' + \gamma''$. This process, which is characteristic of nonlinear interaction of light with light and is forbidden in vacuum by the charge symmetry of the theory (the Furry theorem), was considered earlier by Skobov[1], Minguzzi[2], and Sannikov[3] in the lowest approximation in the external field. Recently, the splitting of a photon was considered again (also in the lowest order in the external field) by Adler et al.[4], Bialynicki-Birula[6], and Gol'tsov and Skobelev[6] in connection with the possible existence of exceedingly intense magnetic fields in pulsars. The results of these investigations contradict one another: whereas in[1-3,6] the amplitude of the process is linear in the field, in[4,5] it is proportional to the third power of the field.

In the present paper we consider the splitting of a photon in a constant crossed field ($\mathbf{E} \perp \mathbf{H}$, $\mathbf{E} = \mathbf{H}$) of arbitrary intensity. The advantages ensuing from considering the process in precisely such a field are deduced from the following considerations: the magnitude of the effect becomes decernible when the intensity of external field in the system, where the incident photon has an energy on the order of mc^2, turns out to be comparable with

$$F_0 = m^2c^3/e\hbar = 4.4\cdot 10^{13} \text{ э.} \tag{1}$$

which is the characteristic field in quantum electrodynamics[1)]. Since the fields in laboratories or even in pulsars are much weaker than F_0, noticeable effects arise for photons with energy much larger than mc^2. But then any external field in a system where the photon has an energy on the order of mc^2 will be very close to a plane-wave field. If the characteristic wavelength and the period of the field are in addition large in comparison with m/eF—the length and time of formation of the process in a field of intensity F—then such a field can be regarded as a constant crossed field.

In other words, if among the three invariants[2)]

$$\varepsilon = \frac{e}{m^2}(\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F})^{1/2}, \quad \eta = \frac{e}{m^2}(\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F})^{1/2}, \quad \varkappa = \frac{\sqrt{(eF_{\mu\nu}l_\nu)^2}}{m^3}, \tag{2}$$

on which the total probability of the splitting of the photon with momentum l in an arbitrary constant field depends, the pure field invariants ϵ and η are small compared with unity and with the dynamic invariant κ, then ϵ and η can be neglected, which is equivalent to considering a process in a constant crossed field.

At $\kappa \sim 1$ the splitting probability is of the order of

$$\alpha^3\left(\frac{mc^2}{\hbar}\right)\frac{mc^2}{l_0}n_\gamma \tag{3}$$

(l_0 and n_γ are the energy and density of the incident photons), corresponding to a reciprocal photon lifetime relative to splitting $\tau^{-1}_{\gamma\to 2\gamma} \sim 3\cdot 10^{14}(mc^2/l_0 \text{ sec}^{-1}$. The values $\kappa \sim 1$ are reached, for example, for photons with energy 25 GeV (the usual energy of modern accelerators) in a field $F \sim 4 \times 10^8$ Oe. Then $\tau^{-1}_{\gamma\to 2\gamma} \sim 6 \times 10^9 \text{ sec}^{-1}$. We note that fields $F \sim 2 \times 10^7$ Oe, i.e., smaller by only a factor of 20 than those mentioned above, have by now been reached in lasers.

The splitting probability obtained by us for arbitrary κ is proportional to κ^6 and coincides with that obtained in[4,5] at $\kappa \ll 1$, and behaves like $\kappa^{2/3}$ at $\kappa \gg 1$.

2. KINEMATIC STRUCTURE OF POLARIZATION OPERATOR OF THREE PHOTONS

In the lowest order in the radiation field, the interaction of three photons is described by the two Feynmann diagrams shown in Fig. 1. They differ from each other by permutation of two photons or by the change of the sign of the momentum of the electron loop. In vacuum, such diagrams cancel each other (the Furry theorem), whereas in an external field with which the virtual electrons of the loop interact their sum is not equal to zero. The diagrams of Fig. 1 can also be parts of more complicated diagrams, so that the mo-

[1)]We shall henceforth use a system of units in which $\hbar = c = 1$, $\alpha = e^2/4\pi = 1/137$ and the notation $l_\mu = (\mathbf{l}, il_0)$, $ll' = \mathbf{ll}' - l_0l_0'$, $\gamma_\mu^+ = \gamma_\mu$, $F^*_{\mu\nu} = (i/2)\epsilon_{\mu\nu\lambda\sigma}$, $F_{\lambda\sigma}$ is a tensor dual to $F_{\mu\nu}$.

[2)]The field invariants ϵ and η have a simple physical meaning—these are the values of the electric and magnetic field intensities in the system where they are parallel, referred to F_0; $\mathfrak{F} = 1/4F^2_{\mu\nu}$ and $\mathfrak{G} = 1/4F_{\mu\nu}F^*_{\mu\nu}$ are the ordinary field invariants.

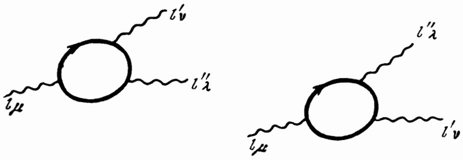

FIG. 1

menta l, l', and l'' will belong to virtual photons, and their squares will differ from zero. The sum of the diagrams of this type forms a third-rank tensor $\Pi_{\mu\nu\lambda}(l, l', l'')$, which we shall call the polarization operator of the third rank. The kinematic structure of this operator can be found by a method described in[7] and is suitable for diagrams with any number of external photon lines.

In an arbitrary constant field, the polarization operator is given by the function of operators

$$-i\partial_\mu, \quad -iF_{\mu\nu}\partial_\nu, \quad -iF_{\mu\nu}{}^*\partial_\nu, \quad -iF_{\mu\nu}F_{\nu\lambda}\partial_\lambda, \tag{4}$$

which commute with the momentum operator $-i\partial_\mu$. Therefore the polarization operator is diagonal in the momentum representation.

$$\Pi_{\mu\nu\lambda}(l, l', l'') \to (2\pi)^4\delta(l + l' + l'')\,\Pi_{\mu\nu\lambda}(ll'l'').$$

The operator $\Pi_{\mu\nu\lambda}$ should be a transverse charge- and space-even tensor which is symmetrical with respect to permutation of any pair of photons.

The polarization vector of a photon with momentum l_μ can be resolved into four independent vectors:

$$l_\mu,\quad L_\mu = F_{\mu\nu}l_\nu,\quad L_\mu{}^* = F_{\mu\nu}{}^*l_\nu,\quad G_\mu = L^{-2}l^2F_{\mu\nu}F_{\nu\lambda}l_\lambda + l_\mu, \tag{5}$$

which are mutually orthogonal if $F_{\mu\nu}F^*_{\mu\nu} = 0$. Owing to the transversality condition, the projection on the momentum l_μ will always be equal to zero: the photon can be only in the three polarization states L_μ, L^*_μ, and G_μ (a real photon, for which $l^2 = 0$, cannot be in the state G_μ, which in this case reduces to l_μ). The polarization state $\psi_{n_1n_2n_3}(l_1\mu_1, l_2\mu_2, \ldots, l_n\mu_n)$ of a system of n photons, of which n_1, n_2, and n_3 are respectively in states of the type L, L*, G($n_1 + n_2 + n_3 = n$), is a product of these single-photon states with invariant coefficient $c_{n_1n_2n_3}(l_1, l_2 \ldots l_n)$ of suitable charge and space parity, symmetrized over all possible permutations of the photons. The charge and space parities $C = P = (-1)^n$ of the state $\psi_{n_1n_2n_3}$ are determined by the parities $C = (-1)^{n_1+n_2}$ and $P = (-1)^{n_1+n_3}$ of the product of the single-photon states and by the parities $C = (-1)^{n_3}$ and $P = (-1)^{n_2}$ of the invariant coefficients. The transverse symmetrical states of three photons are listed in the table. In this table, the vectors (5) for the momentum l' and l'' are marked respectively by single and double primes, the symbol $\sum_{\text{sym}}$ denotes symmetrization over the possible permutations of the photons, while the signs + and − of the coefficients c denote their charge and space parities. The coefficients $c_{n_1n_2n_3}$ with any index $n_i \geq 2$ are symmetrical with respect to permutation of the photons of the i-th type. The symmetrization in the table is therefore carried out over photons of different types, for example

$$\psi_{120} = \frac{L_\mu L_\nu{}^{*\prime}L_\lambda{}^{*\prime\prime}}{LL^{*\prime}L^{*\prime\prime}}\,c_{120}(ll'l'') + \frac{L_\nu{}'L_\mu{}^*L_\lambda{}^{*\prime\prime}}{L'L^*L^{*\prime\prime}}\,c_{120}(l'll'') + \frac{L_\lambda{}''L_\nu{}^{*\prime}L_\mu{}^*}{L''L^{*\prime}L^*}\,c_{120}(l''l'l). \tag{6}$$

	n_1	n_2	n_3	$\psi_{n_1n_2n_3}(l\mu, l'\nu, l''\lambda)$
1	3	0	0	$c^{++}_{300}(ll'l'')\,L_\mu L'_\nu L''_\lambda/LL'L''$
2	2	1	0	$\sum_{\text{sym}} c^{+-}_{210}(ll'l'')\,L_\mu L'_\nu L^{*\prime\prime}_\lambda/LL'L^{*\prime\prime}$
3	2	0	1	$\sum_{\text{sym}} c^{-+}_{201}(ll'l'')\,L_\mu L'_\nu G''_\lambda/LL'G''$
4	1	2	0	$\sum_{\text{sym}} c^{++}_{120}(ll'l'')\,L_\mu L^{*\prime}_\nu L^{*\prime\prime}_\lambda/LL^{*\prime}L^{*\prime\prime}$
5	1	1	1	$\sum_{\text{sym}} c^{--}_{111}(ll'l'')\,L_\mu L^{*\prime}_\nu G''_\lambda/LL^{*\prime}G''$
6	1	0	2	$\sum_{\text{sym}} c^{++}_{102}(ll'l'')\,L_\mu G'_\nu G''_\lambda/LG'G''$
7	0	3	0	$c^{+-}_{030}(ll'l'')\,L^*_\mu L^{*\prime}_\nu L^{*\prime\prime}_\lambda/L^*L^{*\prime}L^{*\prime\prime}$
8	0	2	1	$\sum_{\text{sym}} c^{-+}_{021}(ll'l'')\,L^*_\mu L^{*\prime}_\nu G''_\lambda/L^*L^{*\prime}G''$
9	0	1	2	$\sum_{\text{sym}} c^{+-}_{012}(ll'l'')\,L^*_\mu G'_\nu G''_\lambda/L^*G'G''$
10	0	0	3	$c^{-+}_{003}(ll'l'')\,G_\mu G'_\nu G''_\lambda/GG'G''$

From the tensor of the constant crossed field $F_{\mu\nu}$ and the momenta l, l', and l'', which are connected by the conservation law $l + l' + l'' = 0$, we can form, besides the three squares of the momenta, also four independent invariants, namely, two charge-even scalars κ and κ' (the third is connected with them by the conservation law $\kappa + \kappa' + \kappa'' = 0$), one charge-odd scalar $\rho = F_{\alpha\beta}l'_\alpha l_\beta$, and one charge-odd pseudoscalar $\tau = F^*_{\alpha\beta}l'_\alpha l_\beta$. The last two are antisymmetrical with respect to permutation of any two photons. Their product $\rho\tau$ forms a single charge-even pseudoscalar which is symmetrical with respect to permutation of the photons. Therefore, from parity considerations, the functions c^{-+}_{201}, c^{-+}_{021}, and c^{-+}_{003} are proportional to ρ, the function c^{--}_{111} is proportional to τ, and the functions c^{+-}_{210}, c^{+-}_{030}, and c^{+-}_{012} are proportional to $\rho\tau$. The first three cannot be symmetrical with respect to permutation of a photon of one type, and therefore the states ψ_{201}, ψ_{021}, and ψ_{003} are missing. Thus, there are only eight states of the system of three photons.

For real photons $l^2 = l'^2 = l''^2 = 0$, and it follows from the conservation law that the momenta l, l', and l'' are collinear. In this case $\rho = \tau = 0$ and there remain only the states ψ_{300}, ψ_{120}, and ψ_{102}. For real photons, however, $G = l$, and therefore the state ψ_{102} is missing because of the transversality condition, and the system of real photons is described by the two states ψ_{300} and ψ_{120}.

The respresentation of the polarization operator $\Pi_{\mu\nu\lambda}$ in terms of the invariant functions $c_{n_1n_2n_3}$ arises automatically if we put $\Pi_{\mu\nu\lambda} = \delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\lambda\gamma}\Pi_{\alpha\beta\gamma}$ and expand the δ symbols in the system of vectors (5) pertaining respectively to the momenta l, l', and l'', for example,

$$\delta_{\mu\alpha} = \frac{l_\mu l_\alpha}{l^2} + \frac{L_\mu L_\alpha}{L^2} + \frac{L_\mu{}^*L_\alpha{}^*}{L^{*2}} + \frac{G_\mu G_\alpha}{G^2}. \tag{7}$$

Owing to the transversality, $l_\alpha\Pi_{\alpha\beta\gamma} = 0$ etc., only 27 terms remain in the product of the three δ symbols, and form ten groups with different sets of single-particle states (see the table). The coefficients $c_{n_1n_2n_3}$ arise as contractions of the tensor $\Pi_{\alpha\beta\gamma}$ with the vectors L, L*, and G, for example $c_{210}(ll'l'') = \Pi_{\alpha\beta\gamma}L_\alpha L'_\beta L^{*\prime\prime}_\gamma/LL'L^{*\prime\prime}$. Three of them, c_{201}, c_{021}, and c_{003}, are equal to zero owing to the absence of charge-odd scalars that are symmetrical under permutation of at least one pair of photons.

3. POLARIZATION OF VACUUM AND SPLITTING OF PHOTONS

If we describe the motion of an electron in a constant crossed field by the Green's function $G(x_2, x_1)$ in the proper-time representation[8]

$$G(x_2,x_1)=\exp\left(ie\int_{x_1}^{x_2} dx'A(x')\right)S(x_2-x_1),$$

$$S(x)=\frac{-i}{(4\pi)^2}\int_0^\infty\frac{ds}{s^2}\exp\left[i\frac{x^2}{4s}-is\left(m^2+\frac{(eFx)^2}{12}\right)\right]\left[m-\frac{i\gamma x}{2s}+\frac{ise^2\gamma FFx}{3}+\frac{imse\sigma F}{2}+\frac{e\gamma_5\gamma F^*x}{2}\right], \quad (8)$$

then the matrix element of the first diagram of Fig. 1 can be represented in the form

$$M_{\mu\nu\lambda}(ll'l''F)=(2\pi)^4\delta(l+l'+l'')I_{\mu\nu\lambda}(ll'l''F),$$
$$I_{\mu\nu\lambda}(ll'l''F)=-e^3\int d^4x\,d^4x''\,\mathrm{Sp}[S(x'')\gamma_\lambda S(x')\gamma_\nu S(x)\gamma_\mu]\times\exp\left(\frac{ie}{2}x''Fx-il'x+il''x''\right), \quad (9)$$

where $x + x' + x'' = 0$. Owing to charge symmetry, the matrix element of the second diagram in Fig. 1 differs from the given one in that its sign is reversed and the substitution $F \to -F$ is made. Thus, the polarization operator is determined by the formula

$$\Pi_{\mu\nu\lambda}(ll'l''F)=I_{\mu\nu\lambda}(ll'l''F)-I_{\mu\nu\lambda}(ll'l''-F), \quad (10)$$

which shows explicitly its odd dependence on the field F.

After integrating with respect to x and x″, the integral I takes the form

$$I_{\mu\nu\lambda}=\frac{ie^3}{(2\pi)^2}\iiint_0\frac{ds\,ds'\,ds''}{(s+s'+s'')^2}e^{-i\varphi}Q_{\mu\nu\lambda}, \quad (11)$$

in which the phase of the integrand is

$$\varphi=m^2(s+s'+s'')+\beta\left(\frac{l^2}{s'}+\frac{l'^2}{s''}+\frac{l''^2}{s}\right)+2\beta el''Fl'+\frac{\beta}{3}[(eFl)^2\gamma+(eFl')^2\gamma'+(eFl'')^2\gamma''], \quad (12)$$

where $\beta = ss's''/(s + s' + s'')$, $\gamma = s - s' + s'' + ss''/s'$, and γ' and γ'' are obtained from γ and γ' respectively by cyclic permutation of the proper times: $s \to s' \to s'' \to s$. The tensor $Q_{\mu\nu\lambda}$ can be represented as a result of the action of the operator $\hat{Q}_{\mu\nu\lambda}$, which is differential in the momenta, on $e^{i\varphi}$, namely $\hat{Q}_{\mu\nu\lambda}e^{-i\varphi} = e^{-i\varphi}Q_{\mu\nu\lambda}$. This operator is determined by the trace

$$\hat{Q}_{\mu\nu\lambda}={}^1\!/_4\mathrm{Sp}[(m+i\gamma V''+i\sigma T''+\gamma_5\gamma A'')\gamma_\lambda(m+i\gamma V'+i\sigma T'+\gamma_5\gamma A')\gamma_\nu(m+i\gamma V+i\sigma T+\gamma_5\gamma A)\gamma_\mu], \quad (13)$$

in which

$$V_\alpha=-\hat{x}_\alpha/2s+se^2F_{\alpha\beta}F_{\beta\gamma}\hat{x}_\gamma/3,\quad T_{\alpha\beta}=mseF_{\alpha\beta}/2,\quad A_\alpha=eF_{\alpha\beta}{}^*\hat{x}_\beta/2, \quad (14)$$

$V'_\alpha = V_\alpha(\hat{x}', s')$ etc., and the operators $\hat{x}_\alpha$, $\hat{x}'_\alpha$, and $\hat{x}''_\alpha$ are defined by the relations

$$\hat{x}_\alpha=i\frac{\partial}{\partial l_\alpha'},\quad \hat{x}_\alpha''=-i\frac{\partial}{\partial l_\alpha''},\quad \hat{x}+\hat{x}'+\hat{x}''=0. \quad (15)$$

The explicit expression for $\hat{Q}_{\mu\nu\lambda}$ contains terms that are linear and cubic in the operators x, and is given in the appendix. The tensor $Q_{\mu\nu\lambda}$ differs from $\hat{Q}_{\mu\nu\lambda}$ in that the operators of the coordinates are replaced by their "eigenvalues," and in the linear terms

$$\hat{x}_\alpha\to x_\alpha=2\beta\left[\frac{l'}{s''}-\frac{l}{s'}-eFl''-\frac{e^2}{3}(FFl'\cdot\gamma'-FFl\cdot\gamma)\right]_\alpha, \quad (16)$$

while all the others are obtained by cyclic permutation of the primes of the momenta and of the proper times, and in the cubic terms we have

$$\hat{x}_\alpha\hat{x}_\beta'\hat{x}_\gamma''\to x_\alpha x_\beta' x_\gamma''-2i\beta\left\{x_\alpha\left[\frac{\delta_{\beta\gamma}}{s}+eF_{\beta\gamma}-\frac{e^2}{3}(FF)_{\beta\gamma}\gamma''\right]+x_\beta'\left[\frac{\delta_{\gamma\alpha}}{s'}+eF_{\gamma\alpha}-\frac{e^2}{3}(FF)_{\gamma\alpha}\gamma\right]+x_\gamma''\left[\frac{\delta_{\alpha\beta}}{s''}+eF_{\alpha\beta}-\frac{e^2}{3}(FF)_{\alpha\beta}\gamma'\right]\right\}. \quad (17)$$

Thus, the tensor $Q_{\mu\nu\lambda}$ is a complicated function of the momenta, of the field, and of the proper times.

The proper-time representation is convenient because of its symmetry. However, for practical calculations and for an examination of particular cases, it is more convenient to use the u, v, η representation, which is obtained from (11) by changing over to the dimensionless variables

$$\eta=m^2(s+s'+s''),\quad u=(s+s'')/(s+s'+s''),\quad v=s''/(s+s''). \quad (18)$$

In this representation

$$I_{\mu\nu\lambda}=\frac{ie^3}{(2\pi)^2m^2}\int_0^1du\,u\int_0^1dv\int_0^\infty d\eta\,e^{-i\varphi}Q_{\mu\nu\lambda}, \quad (19)$$

where the phase φ and the tensor $Q_{\mu\nu\lambda}$ turn out to be polynomials in u, v, and η:

$$\varphi=\Lambda\eta+K\eta^2+\frac{\Omega\eta^3}{3},$$
$$\Lambda=1+\frac{l^2}{m^2}u^2v(1-v)+\frac{l'^2}{m^2}u(1-u)(1-v)+\frac{l''^2}{m^2}uv(1-u),$$
$$K=2u^2(1-u)v(1-v)\frac{el''Fl'}{m^4},$$
$$\Omega=\varkappa^2u^2\{[v(1-uv)-\theta(1-u)]^2+4\theta u(1-u)v(1-v)^2\},$$
$$\theta=-\varkappa'/\varkappa. \quad (20)$$

We note that Ω is positive for all real θ and κ.

We present in this paper the form of the polarization operator on the mass shell of the photons, i.e., at $l^2 = l'^2 = l''^2 = 0$. In this case it is characterized by two invariant functions

$$\Pi_{\mu\nu\lambda}(ll'l''F)=\frac{L_\mu L_\nu' L_\lambda''}{LL'L''}c_{300}(ll'l'')+\sum_{\mathrm{sym}}\frac{L_\mu L_\nu^{*\prime} L_\lambda^{*\prime\prime}}{LL'L''}c_{120}(ll'l''), \quad (21)$$

which depend on two independent invariants, chosen to be κ and $\theta = -\kappa'/\kappa$. The coefficients c_{n_1,n_2,n_3} are determined by the integrals

$$c_{n_1n_2n_3}=\frac{e^3m\varkappa}{2\pi^2}\int_0^1du\,u\int_0^1dv\,z\left[pf(z)-qzf'(z)+\frac{\varkappa^2z^3r}{3}f'''(z)\right]_{n_1n_2n_3}, \quad (22)$$

which contain a special function f(z), which arises as a result of integration with respect to η and is characteristic of processes in a constant field (see[7 9]):

$$f(z)=i\int_0^\infty dt\,e^{-i(tz+t^3/3)}.$$

Its real and imaginary parts are tabulated, the imaginary part being the well known Airy function; $z = \Omega^{-1/3}$, see (20). Thus, the coefficients c_{300} and c_{120} differ only in the polynomials p, q, and r, to which the index $n_1n_2n_3$ on the right in (22) pertains.

For p, q, and r we have obtained the following expressions:

$$
\begin{aligned}
p_{300} = q_{300} &= -1 + 2u - 2uv + 2u^2v - u^2v^2 - \theta u(2-u)(1-2v),\\
r_{300} &= u^2v^2(1 - 2u + 2u^2v - u^2v^2)(3 - 10uv + 11u^2v^2)\\
&+ \theta u^2 v[-2 - 8v + 2u(1 + 15v + 10v^2) + u^2v(-15 - 100v + 8v^2)\\
&+ 2u^3v(-8 + 50v + 17v^2 - 10v^3) + u^4v(8 - 16v - 47v^2 + 22v^3)]\\
&+ \theta^2u^2(1-u)[-1 + 4v + 8v^2 + u(3 + 4v - 52v^2)\\
&+ u^2(-2 - 28v + 89v^2 + 16v^3 - 8v^4) + u^3v(22 - 47v - 16v^2 + 8v^3)]\\
&+ \theta^3u^2(1-u)^2(4 - 12u + 11u^2)(1-2v);
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
p_{120} &= -uv(2-uv) - \theta u(2-u)(1-2v), \quad q_{120} = (1-uv)^2 - \theta u(2-u)(1-2v),\\
r_{120} &= u^2v^2(1-uv)^3(1+uv) + \theta u^2v(1-uv)[2 - 8v + 2u(-3 + 8v - 2v^2)\\
&+ u^2(4 - 5v + 4v^3) + u^3v(-4 + 7v - 6v^2)] + \theta^2u^2(1-u)\\
&\times [-3 + 4v + 8v^2 + 3u(1 - 4v^2) + u^2v(2 - 13v + 16v^2 - 8v^3)\\
&+ u^4v(-4 + 15v - 16v^2 + 8v^3)] + \theta^3u^2(1-u)^2(4-u^2)(1-2v).
\end{aligned} \tag{24}
$$

We define the function $c(\kappa, \theta)$ for real κ by the representation (22), in which $\kappa^{1/3}$ stands for the real value of the root having the same sign as κ. In general, on the other hand, the representation (22) defines the analytic function $c(\kappa, \theta)$ in the regions $-\pi/2 < \arg\kappa < 3\pi/2$, $5\pi/2 < \arg\kappa < 9\pi/2$ etc., with an essential singularity and a branch point at $\kappa = 0$. Its values at the real points κ and $-\kappa$ of the same region of analyticity are connected by the relation $c(-\kappa, \theta) = -c^*(\kappa, \theta)$, and at the same points but belonging to neighboring regions (as is the case in our, causal, definition), they are connected by the relation $c(-\kappa, \theta) = -c(\kappa, \theta)$. These relations reflect the charge symmetry of the theory (cf. the mass and polarization operators in[7]).

The symmetry of the function $c_{300}(\kappa, \theta)$ relative to the substitutions $l \rightleftarrows l'$, $l' \rightleftarrows l''$, and $l'' \rightleftarrows l$ is expressed by the relations

$$
\begin{aligned}
c_{300}(\varkappa, \theta) &= -c_{300}\left(\varkappa\theta, \frac{1}{\theta}\right) = c_{300}(\varkappa, 1-\theta)\\
&= -c_{300}\left(\varkappa(1-\theta), -\frac{\theta}{1-\theta}\right),
\end{aligned} \tag{25}
$$

whereas $c_{120}(\kappa, \theta)$ is symmetrical only with respect to the substitution $l' \rightleftarrows l''$

$$
c_{120}(\varkappa, \theta) = c_{120}(\varkappa, 1-\theta). \tag{26}
$$

We have used in (25) the fact that the functions $c(\kappa, \theta)$ are odd in κ. The proof of these relations is easiest to obtain from the representation of the proper-time, in which the obtained functions κp, κq, $\kappa^3 r$, and z go over into themselves following the indicated substitutions if the integration variables are also suitably redefined; for example, the substitution $l \rightleftarrows l'$ calls for the substitution $s' \rightleftarrows s''$, which corresponds in the representation ((22) to the substitutions $u \to u' = 1 - uv$ and $v \to v' = (1-u)(1-uv)^{-1}$.

At small values of κ, the effective values in the integral (22) are $u, v \sim 1$ and $z \sim \kappa^{-2/3} \gg 1$. In this case $f(z) \approx z^{-1} + 2z^{-4}$, if we neglect the terms $\sim z^{-7}$ for the real part and the exponentially small terms for the imaginary part. We then obtain

$$
c_{300}(\varkappa, \theta) = \frac{8e^3m}{105\pi^2}\varkappa^3\theta(1-\theta), \quad c_{120}(\varkappa, \theta) = \frac{13e^3m}{315\pi^2}\varkappa^3\theta(1-\theta). \tag{27}
$$

These expressions lead to the photon-splitting amplitude obtained by Adler et al.[4] and by the Bialynicki-Birulas[5] in the lowest approximation in an external field.

At larger values of κ the values effective in the integral (22) are $u, v \sim 1$ and $z \sim \kappa^{-2/3} \ll 1$. In this case, replacing the function f and its derivatives by their values at zero, we obtain

$$
c_{n_1n_2n_3}(\varkappa, \theta) = \frac{e^3m(3\varkappa)^{1/3}}{12\pi^2}\Gamma\left(\frac{1}{3}\right)(1 + i\sqrt{3})\,g_{n_1n_2n_3}(\theta), \tag{28}
$$

$$
g_{n_1n_2n_3}(\theta) = \int_0^1 du\,u\int_0^1 dv\,\frac{[p + r/3\omega]_{n_1n_2n_3}}{\omega^{1/3}}, \quad \omega = \varkappa^{-2}\Omega.
$$

The functions $g_{300}(\theta)$ and $g_{120}(\theta)$ were obtained numerically and are shown in Fig. 2.

The amplitude $T_{ll'l''}$ of the splitting of a photon with momentum l and polarization e into two photons with momenta l' and l'' and polarizations e' and e'' is determined in terms of the polarization operator on the mass shell by the relation

$$
iT_{ll'l''} = (2\pi)^4\delta(l - l' - l'')\frac{\Pi_{\mu\nu\lambda}(-ll'l''F)\,e_\mu e_\nu' e_\lambda''}{\sqrt{8l_0l_0'l_0''}}. \tag{29}
$$

It is obvious that the functions c are the invariant splitting amplitudes with definite momenta and polarizations of the initial and final photons. Thus, $c_{120}(-ll'l'') = -c_{120}(\kappa, \theta)$ is the amplitude for the splitting of a photon with momentum l and polarization along L into photons with respective momenta l' and l'' and polarizations along $L^{*\prime}$ and $L^{*\prime\prime}$, while $c_{120}(l' - ll'') = c_{120}(\kappa\theta, 1/\theta)$ is the amplitude for the splitting of a photon with momentum l and polarization L^* into photons with momenta l' and l'' and with polarizations along L' and $L^{*\prime\prime}$. The splitting probability per unit volume and per unit time is

$$
W = \frac{1}{2}\int\frac{|T_{ll'l''}|^2}{VT}\frac{d^3l'd^3l''}{(2\pi)^6} = \frac{1}{32\pi l_0}\int_0^1 d\theta\,|\Pi_{\mu\nu\lambda}(-ll'l'')\,e_\mu e_\nu' e_\lambda''|^2. \tag{30}
$$

The factor $1/2$, in integration over finite states, takes into account the identity of the two final photons, see[10], p. 285. For the probability of splitting of an unpolarized photon into photons with arbitrary polarization we have

$$
W = \frac{1}{64\pi l_0}\int_0^1 d\theta\left[|c_{300}|^2 + \sum_{\text{Sym}}|c_{120}|^2\right]
$$

$$
= \begin{cases}
\dfrac{361\,\alpha^3m^2\varkappa^6}{992\,250\,\pi^2 l_0}, & \varkappa \ll 1.\\[2ex]
\dfrac{\Gamma^2(1/3)\,\alpha^3m^2(3\varkappa)^{2/3}}{36\pi^2l_0}\displaystyle\int_0^1 d\theta\left[g_{300}^2(\theta) + g_{120}^2(\theta) + \theta^{2/3}g_{120}^2\left(\frac{1}{\theta}\right)\right.\\
\qquad\left. + (1-\theta)^{2/3}g_{120}^2\left(\frac{1}{1-\theta}\right)\right] & \varkappa \gg 1.
\end{cases} \tag{31}
$$

The last integral is numerically equal to 0.88. At

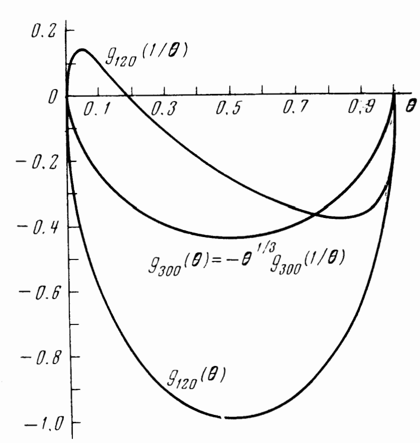

FIG. 2. The functions $g_{300}(\theta)$, $g_{120}(\theta)$, characterizing the spectrum and polarization of the final photons at $\kappa \gg 1$.

large κ, the splitting probability varies like the probability of pair production by a photon[11], but its coefficient is smaller by a factor 137^2.

When $\kappa \ll 1$, the splitting probability is proportional to the sixth power of a small parameter with a small numerical coefficient on the order of 10^{-4}. Therefore even at the exceedingly intense fields in pulsars ($F \sim 10^{12}$ Oe), the effects of splitting of optical or x-ray photons are small, owing to the low photon energy. The splitting of photons with energy of several dozen GeV is a different matter. Here, even at fields $F \sim 10^8$ Oe, the parameter κ becomes of the order of unity, and the probability becomes of the order of $\alpha^3 m^2/l_0$, see (3). The effect can be used in principle for photon polarization.

In conclusion let us make a few remarks concerning the behavior of the polarization operator (or amplitude) at small and large values of κ.

The linear dependence of the amplitude on the field[3)], obtained in error in[1-3], has led to the remarks (see[4,5]) that such a dependence is inadmissible by considerations of relativistic and gauge invariance. However, relativistic and gauge invariance, charge symmetry, and Bose statistics of photons only limit the polarization operator to the form (21) and the symmetry relations (25) and (26) for the invariant functions, and do not forbid, for example, a linear dependence of c_{120} on κ.

A linear dependence as $\kappa \to 0$ is actually excluded by the physical requirement that the polarization operator be finite at finite values of the field and momentum. In fact, if the angle ζ between the momentum **l** and the vector **E** × **H** tends to zero, then the dynamic variable κ tends to zero like ζ^2, and the third and zeroth components of the vectors[4)]

$$\frac{L_\mu}{L} = \left(1.0, \frac{l_1}{l_-}, i\frac{l_1}{l_-}\right), \quad \frac{L_\mu^*}{L} = \left(0 - 1, -\frac{l_2}{l_-}, -i\frac{l_2}{l_-}\right) \tag{32}$$

tend to infinity like ζ^{-1}. Inasmuch as in this case the polarization operator should not only not diverge, but vanish, the invariant functions c should tend to zero more strongly than ζ^3 or $\kappa^{3/2}$. The real parts of the functions $c_{n_1 n_2 n_3}$ vanish like κ^3 as $\kappa \to 0$ because they can be expanded in a perturbation-theory series in terms of the field and because they are odd. We note that the imaginary parts of the functions $c_{n_1 n_2 n_3}$ behave like $e^{-1/\kappa}$ as $\kappa \to 0$, and cannot be expanded in a perturbation-theory series. In general, the point $\kappa = 0$ is essentially singular.

In the opposite limiting case, $\kappa \gg 1$, the functions $c_{n_1 n_2 n_3}$, together with the splitting amplitude and probability, have the remarkable property that they are independent of the electron mass ($\kappa \propto m^{-3}$). This means that in a system where the electrons have an energy on the order of mc^2, the process evolves in a region that is small compared with the Compton length. A similar scaling law is characteristic of the simplest processes in an intense field[11], but is violated by radiative corrections[7].

We also call attention to the fact that when $\kappa \gg 1$ the amplitude for the splitting of the polarization channel $L^* \to L'L^{*''}$ vanishes at the point $\theta \approx 0.18$ owing to a certain interference effect, see Fig. 2.

3)The error was due to failure to take the phase factor of the Green's function[8] into account.

4)Expressions (32) are expressed in a special coordinate system with axes 1,2,3, along **E**, **H**, and **E**×**H**, while $l_- = l_0 - l_3$.

APPENDIX

We present here an explicit expression for the tensor $\hat{Q}_{\mu\nu\lambda}$, defined by the formula (13):

$$\begin{aligned} Q_{\mu\nu\lambda} = i[&P_{\mu\nu\lambda} + P'_{\nu\lambda\mu} + P''_{\lambda\mu\nu} + \delta_{\mu\nu}(-R_{\lambda\alpha\alpha} + R'_{\lambda\alpha\alpha} + R''_{\lambda\alpha\alpha}) \\ &+ \delta_{\lambda\nu}(R_{\mu\alpha\alpha} - R'_{\mu\alpha\alpha} + R''_{\mu\alpha\alpha}) + \delta_{\lambda\mu}(R_{\nu\alpha\alpha} + R'_{\nu\alpha\alpha} - R''_{\nu\alpha\alpha}) \\ &- R_{\mu\lambda\nu} - R'_{\nu\mu\lambda} - R''_{\lambda\nu\mu} - R_{\mu\nu\lambda} + R_{\lambda\mu\nu} - R_{\nu\lambda\mu}] \\ &+ {}^1/_3[\varepsilon_{\mu\nu\lambda\alpha}(S_{\alpha\beta\beta} + S'_{\alpha\beta\beta} + S''_{\alpha\beta\beta}) + (\delta_{\mu\nu}\varepsilon_{\lambda\alpha\beta\gamma} + \delta_{\nu\lambda}\varepsilon_{\mu\alpha\beta\gamma} \\ &+ \delta_{\lambda\mu}\varepsilon_{\nu\alpha\beta\gamma})S_{\alpha\beta\gamma} - 2\varepsilon_{\mu\nu\alpha\beta}(S'_{\lambda\alpha\beta} - S''_{\lambda\alpha\beta}) - 2\varepsilon_{\nu\lambda\alpha\beta}(S''_{\mu\alpha\beta} \\ &- S_{\mu\alpha\beta}) - 2\varepsilon_{\lambda\mu\alpha\beta}(S_{\nu\alpha\beta} - S'_{\nu\alpha\beta})]. \end{aligned} \tag{A.1}$$

Here

$$\begin{aligned} P_{\mu\nu\lambda} = \delta_{\mu\nu}\{&m^2(-V + V' + V'') + 2m[-T(V' - V'') + T'(V'' - V) \\ &+ T''(V - V') + T^*A - T^{*\prime}A' - T^{*\prime\prime}A''] + 8VT'T''\}_\lambda \\ + 4m(&T_{\mu\nu}'V_\lambda'' + T_{\mu\nu}''V_\lambda' + T_{\mu\nu}^{*\prime}A_\lambda'' - T_{\mu\nu}^{*\prime\prime}A_\lambda') - 2im\varepsilon_{\mu\nu\lambda\alpha}(T^*V' + T^{*\prime}V)_\alpha \\ &- 8[(TT')_{\mu\nu}V_\lambda'' + (TT'')_{\mu\nu}V'_\lambda] - 8i\varepsilon_{\mu\nu\alpha\beta}A_\alpha(T'T'')_{\beta\lambda}, \end{aligned} \tag{A.2}$$

$$R_{\mu\nu\lambda} = V_\mu(V_\nu'V_\lambda'' - A_\nu'A_\lambda'') - A_\mu(A_\nu'V_\lambda'' + V_\nu'A_\lambda''), \tag{A.3}$$

$$S_{\mu\nu\lambda} = V_\mu(A_\nu'V_\lambda'' + V_\nu'A_\lambda'') + A_\mu(V_\nu'V_\lambda'' - A_\nu'A_\lambda''), \tag{A.4}$$

and analogous primed and double-primed tensors are obtained from the presented cyclic permutations of the primes of the quantities V, A, and T. Since the tensors R and S go over into themselves following cyclic permutation of the indices and primes, we get

$$R_{\mu\nu\lambda} = R'_{\nu\lambda\mu} = R''_{\lambda\mu\nu}, \quad S_{\mu\nu\lambda} = S'_{\nu\lambda\mu} = S''_{\lambda\mu\nu}. \tag{A.5}$$

1V. G. Skobov, Zh. Eksp. Teor. Fiz. **35,** 1315 (1958) [Sov. Phys. JETP **8,** 919 (1959)].

2A. Minguzzi, Nuovo Cimento **19,** 847 (1961).

3S. S. Sannikov, Zh. Eksp. Teor. Fiz. **52,** 1303 (1967) [Sov. Phys. JETP **25,** 867 (1967)].

4S. L. Adler, J. N. Bahcall, C. G. Callan, and M. N. Rosenbluth, Phys. Rev. Lett. **25,** 1061 (1970).

5Z. Bialynicka-Birula and I. Bialynicki-Birula, Preprint NYO-3829-54, Univ. of Pittsburg, 1970; Phys. Rev. D **2,** 2341 (1970).

6D. V. Gol'tsov and V. V. Skobelev, Zh. Eksp. Teor. Fiz., Pis'ma Red. **13,** 173 (1971) [JETP Lett. **13,** 122 (1971)].

7V. I. Ritus, FIAN Preprint No. 20, 1971; Ann. of Phys. (in print)

8J. Schwinger, Phys. Rev. **82,** 664 (1951).

9V. I. Ritus, Zh. Eksp. Teor. Fiz. **57,** 2176 (1969) [Sov. Phys. JETP **30,** 1181 (1970)].

10V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Relyativistskaya kvantovaya teoriya (Relativistic Quantum Theory), part I, Nauka, 1968.

11A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46,** 776 (1964) [Sov. Phys. JETP **19,** 529 (1964)].

Translated by J. G. Adashko
232