Stationary Nonlinear Ion-Acoustic Oscillations in a Dense Weakly Ionized Current-Carrying Plasma

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The nonlinear stage of development of ion-acoustic instability of a weakly ionized, current-carrying plasma is investigated under the conditions of applicability of the hydrodynamic approximation. The amplitude of the stationary wave, its period and propagation velocity, and also the development of instability can all be studied near the instability threshold, when only the fundamental mode increases. It is shown that the growth of the ion-acoustic oscillations is limited by the ion viscosity. The case corresponding to conditions of large excess above the threshold is also studied.

1. INTRODUCTION

It is well known that, excitation of ion-acoustic oscillations is possible in a spatially homogeneous, nonisothermal plasma, with $T_e \gg T_i$, which is located in an external electric field. The linear theory of the ion-acoustic instability of plasma of both the collision-free, and collision-dominated varieties has been considered in detail in an external electric field in a number of researches[1] (see also[2]). In contrast with the linear theory, the nonlinear theory of this instability, which describes the stationary ion-acoustic oscillations and the process of their establishment, is still far from complete. The results obtained in this region refer only to collision-free plasma,[3] or to a plasma with a small number of collisions.[4] The reason for the instability of the ion-acoustic oscillations under these conditions is the Cerenkov effect on electrons; the mechanism of the stabilization of the instability is due to the quasi-linear action of the excited oscillations on the electron distribution function.

The present research is devoted to the investigation of the nonlinear stage of development of the ion-acoustic instability in a dense, weakly ionized plasma with a large number of collisions, under the conditions when the hydrodynamic approximation is applicable. The cause of the instability here is the drift of the electrons relative to the quiescent ions, which changes the character of the electron heat conductivity and diffusion; the growth of the ion-acoustic oscillations is limited by the weak ion viscosity. The mechanism of stabilization in a bounded system is connected with the appearance in the nonlinear stage of damped higher harmonics of the fundamental growing mode of the ion-acoustic wave. We have succeeded here in tracking the distortion of the wave profile, and of determining the stationary amplitude, period and propagation velocity of a wave of finite amplitude.

2. INITIAL ASSUMPTIONS AND FUNDAMENTAL EQUATIONS

We now consider a weakly ionized plasma in an external constant electric field E_0. Under the action of this field, the electrons of the plasma will drift relative to the ions and the neutral particles with velocity

$$\mathbf{u} = e\mathbf{E}_0 / m\nu_e, \tag{2.1}$$

where ν_e is the collision frequency of ions with neutral particles. Moreover, thanks to the ohmic heating, the temperature of the electrons in such a plasma will deviate from the temperature of the ions, so that $T_e \gg T_i$. We shall neglect the effect of the electric field on the ions.

Under the conditions when the drift velocity u is larger than the velocity of ion sound $v_s = (T_e/M)^{1/2}$, excitation of ion-sound oscillations in a weakly ionized plasma is possible with a wavelength greater than the free path length of the electrons $(kv_{Te} < \nu_e)$ and with a frequency greater than the frequency of ion-neutral collisions $(\omega > \nu_i)$. For the description of the picture of the development of such an instability of the plasma, the following hydrodynamic equations are applicable:

$$\begin{gathered} e\nabla\Phi - T_e\nabla\ln n - m\nu_e\mathbf{v}_e = 0, \\ M d\mathbf{v}_i/dt = -e\nabla\Phi - M\nu_i\mathbf{v}_i + M\eta_0\Delta\mathbf{v}_i, \\ \partial\ln n/\partial t + \mathbf{v}_i\nabla\ln n + \nabla\mathbf{v}_i = 0, \\ \partial\ln n/\partial t + \mathbf{u}\nabla\ln n + \nabla\mathbf{v}_e = 0, \end{gathered} \tag{2.2}$$

where Φ is the potential of the field of the ion-acoustic wave, v_e and v_i are perturbations of the velocities of the electrons and ions, and $\eta_0 = T_i/M\nu_i$ is the specific ion viscosity. In the description of the system (2.2), the temperature of the electrons was assumed to be constant, which is valid under the conditions of large electron heat conductivity, when $\omega\nu_e \ll k^2v_{Te}^2$.

Introducing the quantities

$$\psi = e\Phi/T_e, \quad \rho = \ln n, \quad D_e = T_e/m\nu_e \tag{2.3}$$

and limiting ourselves to a consideration of the one-dimensional case, we write down the system (2.2) in the form

$$\begin{gathered} v_e = D_e\frac{\partial}{\partial x}(\psi - \rho), \\ \frac{dv_i}{dt} = -v_s^2\frac{\partial\psi}{\partial x} - \nu_i v_i + \eta_0\frac{\partial^2 v_i}{\partial x^2}, \\ \frac{\partial\rho}{\partial t} + v_i\frac{\partial\rho}{\partial x} + \frac{\partial v_i}{\partial x} = 0, \\ \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \frac{\partial v_e}{\partial x} = 0. \end{gathered} \tag{2.4}$$

Eliminating v_e and ψ from this set, we obtain a set of two equations (dropping the subscript i of v_i):

$$\partial\rho / \partial t + v\partial\rho / \partial x + \partial v / \partial x = 0, \tag{2.5}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + \nu_i v - \eta_0 \frac{\partial^2 v}{\partial x^2}\right) = \left[\frac{m}{M}\nu_e\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) - v_s^2 \frac{\partial^2}{\partial x^2}\right]\rho.$$

The set (2.5) can now be reduced approximately to a single equation by replacing $\partial\rho/\partial t$ in the small component, which is proportional to m/M, by $-\partial v/\partial x$ (in accord with the linear approximation of the first equation), and by replacing the quantity $\partial\rho/\partial x$ in the small nonlinear component of the first equation by $-v_s^{-2}\partial v/\partial t$ (which is a consequence of the second equation if small dissipative components are neglected). As a result, we get

$$\frac{\partial^2 v}{\partial t^2} - v_s^2\frac{\partial^2 v}{\partial x^2} = \left[-\frac{m}{M}\nu_e\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) - \nu_i\frac{\partial}{\partial t} + \eta_0\frac{\partial^3}{\partial t\partial x^2}\right]v - \frac{\partial^2 v^2}{\partial t \partial x}. \tag{2.6}$$

This equation is the basis of the entire further analysis of nonlinear ion-acoustic oscillations.

In the linear approximation, for excitations of the form $e^{-\omega t + ikx}$, we get the spectrum of ion-acoustic waves from (2.6) ($\omega \to \omega + i\gamma$)

$$\omega^2 = k^2 v_s^2, \quad \gamma = \frac{1}{2}\frac{m}{M}\left(\frac{u}{v_s} - 1\right)\nu_e - \frac{1}{2}\nu_i - \frac{1}{2}k^2\eta_0. \tag{2.7}$$

It is then seen that at drift velocities

$$\frac{u}{v_s} > 1 + \frac{M}{m}\frac{\nu_i}{\nu_e} + \frac{M}{m}\frac{k^2\eta_0}{\nu_e} \tag{2.8}$$

the ion-acoustic oscillations become unstable for a given k. It is evident that in a bounded system, the greatest wavelength oscillations, with $k_{min} \sim 1/L$, have the minimal threshold for oscillation. Here L is the linear dimension of the system.[1)]

3. DEVELOPMENT OF THE ION-ACOUSTIC INSTABILITY NEAR THRESHOLD

Let us consider a plasma near the instability threshold $u \approx u_{min}$, such that obey the fundamental mode with $k = k_{min}$ was increasing, and all higher modes are damped. In such a state, we can expect the establishment of nonlinear ion-acoustic oscillations of finite amplitude. The existence of a steady state here is guaranteed by the transfer of energy from the fundamental, increasing mode of oscillations to the higher, damped mode. In the study of the development of the ion-acoustic instability near threshold, it is convenient to introduce the above-threshold parameter

$$\varepsilon = \frac{\mu\nu_e(u/v_s - 1) - \nu_i - k^2\eta_0}{k^2\eta_0}, \tag{3.1}$$

where $\mu = m/M$. Near threshold of instability, when $\epsilon \ll 1$, the solution of the exact nonlinear equation of ion-acoustic oscillations (2.6) can be sought in the form

$$v = v_1(t)e^{ik(x - v_s t)} + v_2(t)e^{2ik(x - v_s t)} + \text{c.c.}, \tag{3.2}$$

where v_1 and v_2 are slowly changing amplitudes of the first two modes of oscillations (for simplicity, we have neglected possible nonlinear distortion of the frequency)

Substituting (3.2) in (2.6), and equating the coefficients in the same exponents, we get

$$\frac{da_1}{d\tau} = a_1 - \frac{i}{\varepsilon}a_1^* a_2, \quad \varepsilon\frac{da_2}{d\tau} = -3a_2 - ia_1^2, \tag{3.3}$$
$$\tau = \gamma t, \quad a_{1,2} = kv_{1,2}/\gamma_t, \quad \gamma_t = \gamma + k^2\eta_0/2.$$

From (3.3) we get a nonlinear equation which describes the time development of the fundamental mode of ion-acoustic oscillations, $A_1 = a_1 a_1^*$:

$$\frac{dA_1}{d\tau} = 2A_1 - \frac{2}{3}\frac{A_1^2}{\varepsilon}. \tag{3.4}$$

The solution of this equation has the form

$$A_1(\tau) = \left\{\frac{1}{3\varepsilon} + \left(\frac{1}{A_1(0)} - \frac{1}{3\varepsilon}\right)e^{-2\tau}\right\}^{-1}, \tag{3.5}$$

where $A_1(0)$ is the initial value of the amplitude of the ion-acoustic waves, which is determined by the thermal noise in the plasma.

From (3.5) we find the stationary amplitude of the fundamental mode of the ion-acoustic waves,

$$A_1(\infty) = 3\varepsilon. \tag{3.6}$$

It is easy to see that $A_2(\infty) = a_2 a_2^* \sim \epsilon^2$ and so on. As was to have been expected, the proposed method of determination of ion-acoustic waves in a weakly ionized plasma near the instability threshold that is an expansion of the solution of the nonlinear equation (2.6) in powers of the parameter ϵ: the square of the amplitude of the n-th harmonic is proportional to ϵ^n.

4. ION-ACOUSTIC WAVES OF FINITE AMPLITUDE IN THE CASE OF LARGE SUPERCRITICALITY

In the general case of arbitrary superthreshold, the solution of the nonlinear equation (2.6) presents serious difficulties. This equation is relatively easily analyzed in a case of large superthreshold, when $\epsilon \gg 1$. This is in contrast to the case considered above. We shall assume here that in the limit as $t \to \infty$, an ion-acoustic wave of finite amplitude is established in the plasma and therefore all the quantities can be assumed to be dependent on the argument $\xi = kx - \omega t$. This allows us to reduce Eq. (2.6) to the usual equation of second order, which describes the stationary ion-acoustic wave of finite amplitude,

$$\frac{\eta_0 k^2}{\omega}\frac{d^2 w}{d\xi^2} + \left(1 - \frac{k^2 v_s^2}{\omega^2} - 2w\right)\frac{dw}{d\xi} + \left[\frac{m}{M}\frac{\nu_e}{\omega} \times \left(\frac{ku}{\omega} - 1\right) - \frac{\nu_i}{\omega}\right]w = 0, \tag{4.1}$$

where $w = kv/\omega$.

Equation (4.1) is identical in form with that investigated in[5] for a stationary drift wave. Therefore, following[5,6], we write (4.1) in the form

$$\frac{dw}{d\xi} = g \equiv P(g, w), \tag{4.2}$$
$$\frac{\eta_0 k^2}{2\omega}\frac{dg}{d\xi} = -\Delta g + gw - \Gamma w \equiv Q(g, w);$$
$$2\Delta = 1 - \frac{k^2 v_s^2}{\omega^2}, \quad \Gamma = \frac{\mu}{2}\frac{\nu_e}{\omega}\left(\frac{ku}{\omega} - 1\right) - \frac{\nu_i}{2\omega} > 0.$$

[1)]We note that in a rigorous consideration of the boundary problem, the longitudinal wave number k_z is quantized: $k_z \approx \pi n/L$, where $n = 1, 2, \ldots$.

Using the criterion of Dulac,[2)][7] we find the condition for the existence of periodic solutions of the system (4.2):

$$\omega^2 = k^2 v_s^2. \tag{4.3}$$

Thus, the dispersion relation for ion-acoustic waves remains in force for arbitrary excess above threshold and amplitude of the wave.

Equation (4.1) is materially simplified with account of the relation (4.3), and takes the form

$$\frac{1}{\varepsilon+1}\frac{d^2\rho}{d\xi^2} - \frac{\rho}{\Gamma}\frac{d\rho}{d\xi} + \rho = 0. \tag{4.4}$$

In writing down of this equation, we have taken it into account that, in accord with (2.5), with accuracy to small dissipative terms, $w = \rho$. Finally, by substitution of the variables

$$\xi = \frac{\tau}{[2(\varepsilon+1)]^{1/2}}, \quad \rho = \left(\frac{2}{\varepsilon+1}\right)^{1/2}\Gamma\frac{dh}{d\tau} \tag{4.5}$$

Eq. (4.4) reduces to the form studied in[6]:

$$2\frac{d^2h}{d\tau^2} - \left(\frac{dh}{d\tau}\right)^2 + h = 0. \tag{4.6}$$

This equation has a first integral

$$g^2 \equiv \left(\frac{dh}{d\tau}\right)^2 = Ce^h + h + 1, \tag{4.7}$$

where C is the constant of integration. According to the analysis carried out previously,[6] the periodic solutions (4.6) correspond to closed integral curves in the (g, h) plane, to which correspond the values $0 > C > -1$. The values $C \approx -1$ correspond to almost sinusoidal solutions, which in turn correspond to the case considered above of small superthreshold $\epsilon \ll 1$. Actually, inasmuch as[6] $C = -\exp(-g_m^2)$, where g_m is the amplitude value of g, then $g_m \ll 1$. Recognizing, moreover, that $|h| \ll 1$ here we find from (4.7)

$$\tau \approx \left(\frac{2}{1-g_m^2}\right)^{1/2} \arcsin\frac{(1-g_m^2)h - g_m^2}{2^{1/2}g_m}. \tag{4.8}$$

Finally, according to (4.5) we have

$$\rho(\xi) = \frac{2^{1/2}\Gamma g_m}{[(\varepsilon+1)(1-g_m^2)]^{1/2}}\cos[(1+\varepsilon)(1-g_m^2)]^{1/2}\xi. \tag{4.9}$$

Since the period of this function in ξ should be equal to $T_\xi = 2\pi$, then satisfaction of the condition

$$(1-g_m^2)(1+\varepsilon) = 1, \tag{4.10}$$

is necessary, whence it follows for the case considered, $\epsilon \approx g_m^2 \ll 1$, and

$$\rho(\xi) = (2\varepsilon)^{1/2}\Gamma\cos\xi, \tag{4.11}$$

which corresponds to the result (3.6).

In the opposite limit, when $C \to -0$, the integral curve essentially passes close to the separatrix corresponding to C = 0, and only for a large positive value of $h = h_m \sim h_0 = g_m^2 \gg 1$, which corresponds to the larger root of the equation

$$Ce^h + h + 1 = 0, \tag{4.12}$$

does it drop down sharply, undergoing transition from the upper branch of the separatrix to the lower. In this case, the oscillations take on a sawtoothed form with period[6]

$$T_\tau = 4(g_m + (2\ln g_m)^{1/2}/g_m + \ldots). \tag{4.13}$$

From the condition $T_\xi = 2\pi$, it follows that $g_m = 2^{-1/2}\pi(\epsilon+1)^{1/2}$, whence, with account of (4.5), in the limit $\epsilon \gg 1$, we find the amplitude of the wave

$$\rho_m = \pi\Gamma - O(\varepsilon^{-1/2}). \tag{4.14}$$

The physical meaning of the sawtoothed shape of the solution thus found is easily understood if we take it into account that, for larger superthreshold ($\epsilon \gg 1$), the excitation of a large number of different harmonics occurs and the oscillations have an essentially nonharmonic character.

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2)Choosing $F = \Delta/(g - \Gamma)$ as an arbitrary function, we see that the expression

$$\frac{\partial}{\partial w}(FP) + \frac{\partial}{\partial g}(FQ) = \frac{\Delta^2\Gamma}{(g-\Gamma)^2} \geqslant 0.$$

Therefore, according to Dulac's criterion, the system (4.2) has periodic solutions only for $\Delta = 0$, i.e., for the condition (4.3)

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