# CONTRIBUTION TO THE THEORY OF PARAMETRIC RESONANCE IN AN INHOMOGENEOUS PLASMA

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A kinetic theory of the dispersive properties of longitudinal oscillations in an inhomogeneous-plasma layer located in an external high-frequency (HF) electromagnetic field is developed. The range of external-field frequencies close to the natural frequencies of the electron surface or volume Langmuir oscillations is studied. The cases of transparent and opaque plasma layers are considered. Threshold values of the HF field intensity are found and the near-threshold increments are determined. It is shown that the buildup of the ion-sound waves takes place throughout the plasma, despite the fact that the surface electron and localized Langmuir oscillations exist in small regions of the plasma layer. It is found that volume ion-sound oscillations can be excited in an opaque plasma when the thickness of the plasma layer considerably exceeds the dimensions of the skin layer. The effect of inhomogeneity on parametric instability in a strong HF field is studied.

1. The theory of parametric resonance is at present sufficiently developed only for the case when the plasma can be regarded as homogeneous and unbounded[1-5]. Under experimental conditions, however, the interaction of high-frequency (HF) electromagnetic fields with a plasma may essentially depend on the nature of the inhomogeneity of the charged particle density distribution. In the present paper we investigate the parametric excitation of volume and surface quasistatic waves in a layer of an inhomogneous plasma located in an HF field whose intensity vector is parallel to the plasma boundary. We determine the threshold values of the intensities of the external field beginning from which the plasma becomes unstable during a parametric resonance at surface electron oscillation frequencies. The increments of the above-threshold instability are also determined.

It is found that the volume ion-sound oscillations interact with the surface electron oscillations, in particular, when the oscillation interaction region, limited bv the extent of localization of the surface wave, is smaller than the thickness of the layer. Despite the fact that generation of ion sound takes place only in the region of localization of the surface wave, the drift of the ion-sound oscillation from the region of interaction of the waves leads to the growth of the amplitude of the ionic sound throughout the plasma. An analogous situation is realized in the case when the thickness of the plasma layer considerably exceeds the penetration depth of the external HF field.

We also consider the case of localized Langmuir oscillations, when the density profile has a minimum at the middle of the layer. It is shown that the ratio of the threshold HF field intensity (for the buildup of the Langmuir and ion-sound oscillations) in an inhomogeneous plasma to the corresponding value for the case of uniform density distribution is equal to the square root of the ratio of the characteristic dimensions of the localization of the ion-sound and Langmuir oscillations. At the same time, for an aperiodic instability, during which there is a buildup of the Langmuir oscillations and the spatial charge separation waves of zero frequency, the magnitude of the threshold external-field intensity turns out to be coincident with the value corresponding to the case of a homogeneous plasma. Such a coincidence is connected with the fact that the spatial charge separation waves are not natural waves and grow only in the region of localization of the Langmuir waves. The threshold external-field intensity for the excitation of the aperiodic instability may then turn out to be considerably smaller than the corresponding value for the periodic instability, even in the case of strong nonisothermy when the electron temperature is considerably higher than the ion temperature.

The effect of plasma inhomogeneity on parametric instability in a strong HF field is also elucidated. It is shown that the presence of spatial dispersion of the surface waves in an inhomogeneous plasma leads to the broadening of the range of frequencies of the external field in which instability with respect to the buildup of the surface waves becomes possible. At the same time the unique damping of the surface waves, which is connected with the plasma inhomogeneity, leads to a situation in which the parametric instability may turn out to be dissipative even in the case of large values of the intensity of the external field.

2. Let us consider a quasineutral-plasma layer of thickness d, bounded by a medium of permittivity $\epsilon_0$[1)] We shall assume that the electron density $n_e(z)$ increases rapidly to the value $n_e(a)$ in a transition region of dimension a ($a \ll d$) close to the plasma boundaries; in the rest of the region the plasma will be assumed to be mildly inhomogeneous with a characteristic dimension of the inhomogeneity:

$$L = (\partial \ln n_e(z) / \partial z|_{z=a})^{-1} \gtrsim d.$$

The vector of the external electric field $\mathbf{E}_0(z) \sin \omega_0 t$

[1)]It is assumed that the space-time dispersion of the permittivity $\epsilon_0$ is insignificant.

is assumed to be directed along the boundary of the layer.

Being interested in the question of plasma instability with respect to the buildup of high-frequency surface and low-frequency ion-sound oscillations when the plasma is located in a HF electric field, we shall study the evolution in time of small initial perturbations of the electromagnetic field. In doing this we shall assume that the component of the wave vector of the oscillations along the plasma boundary $k_{\parallel}$ satisfies the condition $a^{-1} \gg k_{\parallel} \gg L^{-1}$. This corresponds in the absence of the HF field to slowly decaying surface waves[6]. Under the assumption that the plasma Debye radius $r_D$ is small in comparison with the dimension a of the transition region for high-frequency surface oscillations, the thermal motion of the particles turn out to be insignificant. This permits us to describe the surface waves on the basis of the equations of a cold plasma.

To investigate the low-frequency ion-sound oscillations we used the kinetic equation with the collision integral in the form suggested by Landau. Since the spectrum of the ion-sound oscillations whose wavelength is large compared to the dimension a of the transition layer is determined by the volume effects, we can in studying low-frequency perturbations neglect the presence of the transition region close to the boundaries. We shall, furthermore, assume that no absorption of the plasma particles takes place at the boundaries. This satisfies any boundary condition that assumes the vanishing of the normal component of the particle fluxes at the boundaries $z = 0, d$. An example of such a boundary condition is the simplest condition used below that the particles be reflected specularly from the plasma surface.

The intensity $\mathbf{E}_0$ of the external field is assumed to be relatively small, so that the amplitude $r_E = eE_0/m_e\omega_0^2$ of the electron oscillations in the external HF field is smaller than the wavelength $\lambda\!\!\!^{-} = 1/k_{\parallel}$ of the oscillations under study, and the frequency of the external field $\omega_0$ is assumed to be close to the natural frequencies of the symmetric $\omega_s^{(+)}$ or antisymmetric $\omega_s^{(-)}$ quasi-static ($k_{\parallel}c \gg \omega_s^{(\pm)}$ surface waves[7] 2). Here

$$\omega_s^{(\mp)} = \omega_{Le}(a)\left(\frac{1 \pm \exp\{-k_{\parallel}d/2\}}{1+\varepsilon_0 \pm (1-\varepsilon_0)\exp\{-k_{\parallel}d/2\}}\right)^{1/2}, \tag{2.1}$$

$$\omega_{Le}^2(a) = 4\pi e^2 n_e(a)/m_e. \tag{2.2}$$

In the case of low-frequency instabilities ($|\omega + i\gamma| \ll \omega_0$) being studied, only the amplitudes $\mathbf{E}^{(0)}$ and $\mathbf{E}^{(\pm)}$ in the expansion of the electric field of the perturbations in terms of the harmonics of the frequency of the external field do not turn out to be small.

Depending on the relation between the thickness d of the layer and the depth $l = c(\omega_p^2 - \omega_0^2)^{-1/2}$ to which the external field penetrates into the plasma, two cases are possible; first, the case of thin layers, when the penetration depth of the external field considerably exceeds the dimensions d of the layer and, secondly, the case of opaque—for the HF field—layers ($l \ll d$). Taking into account the condition $k_{\parallel}r_E \ll 1$, as well as the closeness of $\omega_0$ to the natural frequencies of the surface oscillations (2.1), we can derive for the amplitudes of the electric field at the plasma boundaries the following expression:

$$\begin{gathered}\mathbf{k}_{\parallel}\mathbf{E}^{(n)}(z=0,\mathbf{k}_{\parallel},\omega+n\omega_0) \pm \mathbf{k}_{\parallel}\mathbf{E}^{(n)}(z=d,\mathbf{k}_{\parallel},\omega+n\omega_0) \\ = -\Xi^{-1}(\omega,\mathbf{k}_{\parallel})\sum_{\nu}\int d\mathbf{v}\sum_{\alpha}\frac{4\pi e_\alpha F_\alpha(t=0,\mathbf{v},\mathbf{k}_\nu)k_{\parallel}^2}{k_\nu^2\varepsilon^{(n)}(\mathbf{k}_\nu)(\omega+n\omega_0-\mathbf{k}_\nu\mathbf{v})D_n^{(\pm)}}, \\ k_\nu^2 = k_{\parallel}^2 + k_{z\nu}^2, \qquad k_{z\nu} = \pi\nu/d,\end{gathered} \tag{2.3}$$

where to the plus sign corresponds summation over even $\nu$ (symmetric modes), and to the minus sign summation over odd $\nu$ (antisymmetric modes). Here

$$D_n^{(\pm)} = 1 + \frac{2\varepsilon_0 W_n}{\sigma_n d}\sum_{\nu}\left\{\frac{k_{\parallel}^2}{k_\nu^2\varepsilon^{(n)}(\mathbf{k}_\nu)} - \frac{n^2\omega_0^2 k_{z\nu}^2}{c^2 k_\nu^4}\right\}; \tag{2.4}$$

$$\sigma_n = (k_{\parallel}^2 - \omega_0^2 n^2\varepsilon_0/c^2)^{1/2}, \qquad W_0 = 1,$$

$$W_{n\neq 0} = 1 + \frac{k_{\parallel}}{\varepsilon_0}\int_0 dz\,\varepsilon(\omega+n\omega_0,z) - \varepsilon_0 k_{\parallel}\int_c \frac{dz}{\varepsilon(\omega+n\omega_0,z)}$$

$$-\frac{1+\varepsilon_0}{2\varepsilon_0 k_{\parallel}}\left[\frac{\partial}{\partial z}\ln\varepsilon(\omega+n\omega_0,z)\right]_{z=a}; \qquad \omega + i\gamma \to \omega;$$

where $\delta\epsilon_\alpha^{(n)}(\mathbf{k}_\nu) \equiv \delta\epsilon_\alpha(\omega + n\omega_0, \mathbf{k}_\nu)$ is the partial contribution of particles of the type $\alpha$ to the longitudinal permittivity $\epsilon^{(n)}(\mathbf{k}_\nu) \equiv \epsilon(\omega + n\omega_0, \mathbf{k}_\nu)$; the function $F_\alpha(t = 0, v)$ is, up to quantities proportional to the intensity $\mathbf{E}_0$ of the external field, the initial perturbation of the distribution function of the $\alpha$-type particles.

As can be seen from (2.3), to solve the initial problem and elucidate, in particular, the question of plasma stability with respect to the buildup of the waves under consideration, we must find the solution of the dispersion equation $\Xi(\omega, \mathbf{k}_{\parallel}) = 0$, which describes the parametric connection of the low-frequency oscillations with the frequency $\omega$, and of the high-frequency oscillations with the frequencies $\omega_0 \pm \omega$. In the case of thin plasma layers, when the external HF field can be assumed to be uniform, we have for the quantity $\Xi(\omega, \mathbf{k}_{\parallel})$ two expressions, describing respectively the connection between the even or odd surface waves and the ion-sound waves:

$$\Xi(\omega,\mathbf{k}_{\parallel}) = \Xi_{\pm}(\omega,\mathbf{k}_{\parallel}) \equiv 1 + \frac{(\mathbf{k}_{\parallel}\mathbf{r}_E)^2}{4}\frac{[\delta\varepsilon_i^{(0)}(\omega)]^2}{[\varepsilon(\omega_0)]^2}\frac{D_0^{(\pm)}-1}{D_0^{(\pm)}}\left\{\frac{1}{D_1^{(\pm)}} + \frac{1}{D_{-1}^{(\pm)}}\right\}. \tag{2.5}$$

Here, $\delta\epsilon_i^{(0)}(\omega)$ and $\epsilon(\omega_0)$ are the permittivities of a cold plasma.

When the plasma is opaque to the HF field ($l \ll d$), but the quasi-static condition ($\lambda\!\!\!^{-} \ll l$) is fulfilled, the surface waves are excited close to only one of the plasma boundaries. The circumstance that the region of localization of the surface waves, where the interaction of the oscillations occurs, is considerably smaller than the depth of penetration of the external HF field into the plasma, turns out to be important in this case, for it permits us to consider the HF field as a uniform field near one of the plasma boundaries. The symmetric and antisymmetric modes then turn out to be coupled. Such a coupling is due to an exponentially small difference in the frequencies of the symmetric and antisymmetric surface waves (2.1) when $k_{\parallel}d \gg 1$, the expression for $\Xi(\omega, \mathbf{k}_\nu)$ having the form

$$\Xi(\omega,\mathbf{k}_{\parallel}) = \Xi_+(\omega,\mathbf{k}_{\parallel}) + \Xi_-(\omega,\mathbf{k}_{\parallel}) - 1, \tag{2.6}$$

where we must set $D_{\pm1}^{(+)} = D_{\pm1}^{(-)}$.

Notice that in accordance with the above-employed

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2)We shall write below $\omega_{Le,\,Li}(a)$ as $\omega_{Le,\,Li}$.

condition of indistinguishability of the frequencies of the symmetric and antisymmetric surface waves, the expression (2.6) for $\Xi(\omega, \mathbf{k}_{\|})$, correctly describes the variation in time of the electric fields only for sufficiently small times $t < T \equiv \omega_{Le}^{-1} \exp(k_{\|} d)$. We shall henceforth assume that the characteristic time of development of parametric instabilities for this case is smaller than T. Such an interval of time T corresponds to the appearance of beats of frequency equal to the difference between the frequencies of the symmetric and antisymmetric surface modes (2.1) and, consequently, to the appearance of a surface-wave field also near the boundary to which the HF field does not penetrate.

3. Let us turn to a discussion of the possibility of exciting surface and volume waves in specific cases.

A. Let us consider first of all the case of the parametric resonance of the external field at the frequency $\omega_S^{(+)}$, (2.1), of the symmetric surface electron oscillations. In the limit $\omega_0 \ll \omega_{Le} \times (1+\epsilon_0)^{-1/2}$ the wavelengths of the excited oscillations considerably exceed the dimension of the layer $(k_{\|} d \ll 1)$, while $\omega_0 \approx \omega_{Le} \times (k_{\|} d/2\epsilon_0)^{1/2}$. At the same time from the condition that the surface waves be quasi-static follows the inequality $d \ll c/\omega_{Le}$. This means that we must in the given frequency range use the dispersion equation (2.5), which corresponds to the case of uniform external field. The excitable low-frequency ion-sound oscillations possess a discrete frequency spectrum $\omega = \omega_S(k_{2n})$:

$$\omega_s^2(\mathbf{k}_m) = \frac{\omega_{Li}^2 k_m^2 r_{De}^2}{1 + k_m^2 r_{De}^2}, \quad k_m^2 = k_{\|}^2 + \varkappa_m^2. \tag{3.1}$$

Here the wave number $\varkappa_m$ is determined by the equations

$$1 + \varepsilon_0 k_{\|}^2 r_{De}^2 \frac{k_{\|}}{\varkappa_m} \operatorname{tg} \frac{\varkappa_m d}{2} = 0,$$

$$1 - \varepsilon_0 k_{\|}^2 r_{De}^2 \frac{k_{\|}}{\varkappa_m} \operatorname{ctg} \frac{\varkappa_m d}{2} = 0$$

for the symmetric and antisymmetric vibration modes, respectively. In the absence of the HF field the damping constant of these oscillations is determined by the usual expression (see, for example,[8]) (3.2)

$$\gamma_s = \left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_{Li}}{\omega_{Le}} \omega_s \left[1 + \frac{r_{De}^2 v}{r_{Di}^2 v_{Ti}} \exp\left(-\frac{\omega_s^2}{2k_{2n}^2 v_{Ti}^2}\right)\right] + \frac{4}{5} \frac{r_{Di}^2}{r_{De}^2} \nu_{ii}.$$

Assuming that $\gamma_S \ll \omega$, and that $\omega$ differs slightly from the frequency of the ion-sound oscillations, we write the quantities entering into (2.5) in the following form:

$$D_0^{(+)} = \frac{\gamma_s + \gamma - i\delta\omega}{\gamma_s + \gamma + iA - i\delta\omega}, \quad \frac{1}{D_{-1}^{(+)}} + \frac{1}{D_1^{(+)}} = -\frac{\omega_0 \Delta}{\Delta^2 - (\omega + i\gamma + i\tilde{\gamma})^2},$$

$$\tilde{\gamma} = \frac{1}{2}\left\{\omega_{\text{п}}^{(+)} \varepsilon_0 k_{\|} \pi \int_0^a dz \delta[\varepsilon(\omega_{\text{п}}^{(+)}, z)] + \nu_{ei}\right\}, \tag{3.3}$$

$$A = 2\omega_{Li} \frac{r_{De}}{d} k_{\|} r_{De}^2 k_{2n}, \quad \delta\omega = \omega - \omega_s, \quad \Delta = \omega_0 - \omega_{Le}\left(\frac{k_{\|} d}{2\varepsilon_0}\right)^{1/2}.$$

In accordance with these expressions, it follows the dispersion equation (2.5) that the smallest value of the threshold field intensity (and the maximum increment) is attained for $n = 0$. In the case when $\tilde{\gamma} \ll \omega_S$, we obtain for the maximum increment the expression

$$\delta\omega = 0, \quad \gamma = -\frac{\tilde{\gamma} + \gamma_s}{2} + \left[\frac{(\gamma_s - \tilde{\gamma})^2}{4} + \frac{(\mathbf{k}_{\|} \mathbf{r}_E)^2}{16\varepsilon_0} \frac{d}{r_{De}} \omega_0 \omega_{Li}\right]^{1/2}, \tag{3.4}$$

corresponding to the condition of disintegration into surface and ion-sound waves:

$$\omega_0 = \omega_{Le}\left(\frac{k_{\|} d}{2\varepsilon_0}\right)^{1/2} + \omega_s(\mathbf{k}_{\|}). \tag{3.5}$$

Only oscillations with frequencies $\omega$ and $\omega_0 - \omega$ are then effectively excited.

The condition $\tilde{\gamma} \ll \omega_S$ leads to the following limitation on the frequency of the external field:

$$\frac{\nu_{ei}}{2} + \frac{\pi \varepsilon_0^2 \omega_0^3}{\omega_{Le}^2 d} \int_0^d dz\, \delta[\varepsilon(\omega_0, z)] < 2\omega_{Li} \frac{\omega_0^2}{\omega_{Le}^2} \frac{\varepsilon_0 r_{De}}{d}. \tag{3.6}$$

We obtain from (3.4) the following expression for the threshold intensity of the HF field:

$$\frac{E_0^2 \,\text{thr}}{4\pi n_e T_e} = \frac{4}{\varepsilon_0} \frac{d}{r_{De}} \frac{\tilde{\gamma}\gamma_s}{\omega_0 \omega_{Li}}. \tag{3.7}$$

In the case when the condition $\tilde{\gamma} \ll \omega_S$, (3.6), is not fulfilled, but the frequency $\omega$ slightly differs from $\omega_S(k_{2n})$ $(\delta\omega \ll \omega)$, the instability becomes undamped. We then obtain from Eq. (2.5) the following expression for the increment:

$$\gamma = -\gamma_s + \frac{(\mathbf{k}_{\|} \mathbf{r}_E)^2}{16\varepsilon_0} k_{\|} d \frac{\omega_0 \Delta \tilde{\gamma} \omega_{Li}^2}{(\Delta^2 + \tilde{\gamma}^2)^2} \tag{3.8}$$

and for the corresponding change in frequency

$$\delta\omega = \frac{\gamma_s + \gamma}{2\omega_s \tilde{\gamma}} (\Delta^2 + \tilde{\gamma}^2). \tag{3.9}$$

It can be seen from formula (3.8) that the maximum of the increment

$$\gamma = -\gamma_s + \frac{3\sqrt{3}}{32} \varepsilon_0^2 \frac{r_{De}^2}{d^2} \frac{\omega_0^7 \omega_{Li}^2}{\omega_{Le}^6 \tilde{\gamma}^2} \tag{3.10}$$

and the minimum of the threshold intensity of the HF field

$$\frac{E_{0,\,\text{thr}}^2}{4\pi n_e T_e} = \frac{8\sqrt{3}}{9\varepsilon_0^2} \frac{\tilde{\gamma}^2}{\omega_{Le}^2} \frac{\gamma_s}{\omega_{Li}} \frac{d^2}{r_{De}^2} \frac{\omega_{Le}^4}{\omega_{Li} \omega_0^3} \tag{3.11}$$

are attained at $\Delta = \tilde{\gamma}/\sqrt{3}$.

Notice that the condition that the oscillations should be quasi-static, $k_{\|} c \gg \omega_0$, imposes the following limitation on the frequency of the external field:

$$\omega_0 \gg \omega_{Le} \frac{v_{Te}}{c} \frac{d}{r_{De}}. \tag{3.12}$$

The approximation $\omega_S \gg \delta\omega$ used leads, as can be seen from (3.7), to the following inequality:

$$\tilde{\gamma}(\gamma_s + \gamma) \ll \omega_s^2. \tag{3.13}$$

B. When $d \ll l$ surface oscillations of short wavelengths $(k_{\|} d \gg 1)$ are nonresonantly excited as the frequency $\omega_0$ of the external field increases and approaches $\omega_{Le}(1+\epsilon_0)^{-1/2}$. Let us, however, in this case consider not too short wavelengths:

$$k_{\|}^3 d^3 \ll \frac{\pi^2 (2n+1) d^2}{r_{De}^2}, \tag{3.14}$$

which corresponds to the values $\varkappa_{2n} = 2\pi n/d$. Then out of the quantities entering into (2.5) $\Delta$, $\tilde{\gamma}$, and $\epsilon(\omega_0)$ will differ from those given in (3.2) and (3.3). These quantities will, in the present case, have the form

$$\tilde{\gamma} = \frac{1}{2}\nu_{ei} + \omega_s^{(+)} \frac{\pi \varepsilon_0^2}{2(1+\varepsilon_0)} k_{\|} \int_0^a dz\, \delta[\varepsilon(\omega_s^{(+)}, z)],$$

$$\Delta = \omega_0 - \omega_s^{(+)}\left\{1 - \frac{(\omega_s^{(+)})^2}{8k_{\|}^2 c^2} + \frac{k_{\|}}{2(1+\varepsilon_0)} \int_0^a \frac{dz\,[\varepsilon_0^2 - \varepsilon^2(\omega_s^{(+)}, z)]}{\varepsilon(\omega_s^{(+)}, z)} + \frac{1}{2k_{\|}}\left[\frac{\partial}{\partial z} \ln \varepsilon(\omega_s^{(+)}, z)\right]_{z=a}\right\}, \quad \varepsilon(\omega_0) \approx -\varepsilon_0.$$

Since the frequency of the ion-sound oscillations is considerably smaller than the damping constant (3.15), the parametric excitation of the oscillations is, in the case under consideration, accomplished under undamped conditions. Close to the threshold of the instability, the spectrum of the excitable low-frequency oscillations slightly differs from the ion-sound spectrum and is given by formula (3.9). The threshold value of the intensity of the external field and the maximum value of the increment are respectively given by the formulas

$$\frac{E_{0,\,\mathrm{thr}}^2}{4\pi n_e T_e} = \frac{16\sqrt{3}}{9\varepsilon_0} \frac{\tilde{\gamma}^2 \gamma_s \omega_0}{\omega_s^2 \omega_{Le}^2} k_\parallel d, \tag{3.17}$$

$$\gamma_{max} = -\gamma_s + \frac{3\sqrt{3}}{16} \frac{r_E^2}{r_{De}^2} \frac{\omega_0 \omega_s^2}{\tilde{\gamma}^2} \frac{1}{k_\parallel d}. \tag{3.18}$$

To such dependence of $E_{0\mathrm{thr}}$ and $\gamma_{\max}$ on $k_\parallel d$ may be compared a simple physical picture. The field of a surface wave is localized near the plasma boundary over a distance of the order of $1/k_\parallel$. It is precisely in this region that the interaction leading to the excitation of ion-sound oscillations occurs between the surface mode and the external HF field. Outside the region of interaction of the waves the ion-sound oscillations attenuate with the decrement $\gamma_S$ (3.2). The growth constant $\Gamma$ of the ion-sound oscillations for the case when the dimension of the region of interaction of the waves is of the order of the dimensions of the layer (see (3.10)) has the form

$$\Gamma \approx \frac{\omega_0 \omega_s^2}{\tilde{\gamma}^2} \frac{r_E^2}{r_{De}^2} - \gamma_s. \tag{3.19}$$

Thus, by building up near the plasma surface in a region of the order of $1/k_\parallel$ and attenuating in the remaining plasma layer of thickness d ($d \gg 1/k_\parallel$), the ion-sound oscillations, as is easy to understand, attenuate on the average (grow) with the effective decrement (increment)

$$\gamma = -\gamma_s + \Gamma / k_\parallel d, \tag{3.20}$$

which corresponds to the formulas (3.17) and (3.18).

Finally, notice that in the range of frequencies of the external field being considered, the excitation of the antisymmetric ion-sound oscillations does not differ from the buildup of the symmetric modes. In the case of an opaque plasma ($d \gg l$), when the frequency $\omega_{Le}(1+\epsilon_0)^{-1/2}$, Eq. (2.6) is to be used in the study of parametric instability. This equation describes the excitation of both the symmetric and antisymmetric ion-sound modes which, it turns out, grow in intensity everywhere in the plasma, in spite of the fact that the plasma is opaque to the external HF field. Taking into account the fact that the frequencies $\omega$ close to $\omega_S$ (3.1) are the solution to the dispersion equation in the limit (3.14) of interest to us and that the damping constant of the surface waves $\tilde{\gamma}$ (3.15) considerably exceed the frequency $\omega_S$, we obtain from the dispersion equation (2.6) the expressions (3.17) and (3.18) for the growth constant and the value of the threshold intensity of the external field. Notice that the assumption $\gamma \gg \omega_{Le} \exp(-k_\parallel d)$, used in the derivation of the dispersion equation (2.6), leads, for sufficiently short-wave-length oscillations when $k_\parallel d \gg \ln(\omega_{Le}/\gamma_S)$, to an exponentially small difference between the threshold intensity of the HF field and the value given by (3.17).

C. At still higher frequencies of the external field when

$$\omega_{Le}(1+\varepsilon_0)^{-1/2} < \omega_0 \lessapprox \omega_{Le},$$

a parametric resonance is realized on the antisymmetric surface wave (2.1). For such a range of external-field frequencies,

$$k_\parallel d \ll 1, \quad \omega_S^{(-)} \approx \omega_{Le}(1 - \varepsilon_0 k_\parallel d/4),$$

and the condition that the surface wave should be of quasi-static nature leads to the inequality $d \ll l$, which implies the transparency of the plasma to the external HF field. Substituting into the dispersion equation (2.5) the value of $\Delta$, $\tilde{\gamma}$, and $\epsilon(\omega_0)$ given by the formulas

$$\Delta = \omega_0 - \omega_S^{(-)}, \quad \tilde{\gamma} = \omega_{Le}\varepsilon_0^2 \frac{k_\parallel^2 d\pi}{4} \int_0^a dz\, \delta[\varepsilon(\omega_0, z)] + \frac{1}{2}\nu_{ei},$$
$$\varepsilon(\omega_0) = -\frac{\varepsilon_0 k_\parallel d}{2}, \quad \frac{1}{D_{-1}^{(-)}} + \frac{1}{D_{+1}^{(-)}} = -\frac{\varepsilon_0 k_\parallel d}{2} \frac{\omega_0 \Delta}{\Delta^2 - (\omega + i\gamma + i\tilde{\gamma})^2}, \tag{3.21}$$

we obtain the following expression for the increment of the instability:

$$\gamma = -\gamma_s + \frac{(k_\parallel r_E)^2 k_\parallel}{2k_{2n+1}^3 d^2} \frac{\omega_0 \tilde{\gamma} \omega_{Li}^2 \Delta}{(\Delta^2 + \tilde{\gamma}^2)^2}. \tag{3.22}$$

The correction to the frequency is, in this case, determined by the formula (3.9) into which the expressions (3.21) should be substituted.

It can be seen from (3.22) that the maximum value of the increment

$$\gamma = -\gamma_s + \frac{r_E^2}{r_{De}^2} \frac{3\sqrt{3}}{32} \frac{k_\parallel^3}{k_{2n+1}^3} \frac{\omega_0 \omega_{Li}^2}{\gamma^2} \tag{3.23}$$

and the magnitude of the threshold intensity of the external HF field

$$\frac{E_{0,\,\mathrm{thr}}^2}{4\pi n_e T_e} = \frac{32\sqrt{3}}{9} \frac{k_{2n+1}^3}{k_\parallel^3} \frac{\gamma_s \tilde{\gamma}^2}{\omega_0 \omega_{Li}^2} \frac{d^2}{r_{De}^2} \tag{3.24}$$

are attained at the wave numbers determined from the condition $\Delta = \tilde{\gamma}/\sqrt{3}$. It can be seen that the value of the intensity of the threshold field increases with the number of the mode of the excitable low-frequency oscillations.

4. In the present section we shall show that the above-investigated effect connected with the localization of the interacting waves appears not only in the case of parametric resonance at surface-wave frequency, but also when the frequency of the external field is close to that of the localized plasma waves. Such a localization arises, for example, in the case when the dependence of the density on the coordinates inside the plasma layer has the form

$$n_\alpha(z) = n_\alpha(0)\left[1 + \frac{(z - d/2)^2}{b^2}\right]. \tag{4.1}$$

We shall, for simplicity, assume that the characteristic dimension b of the inhomogeneity inside the plasma layer exceeds the thickness d of the layer, and that the dimension a of the transition region near the plasma boundaries will, as before, be much smaller than d. Then, as follows from the results of[9], in the absence of the HF field there exist in such a plasma layer trapped natural Langmuir oscillations the potential of whose electric field has the form

$$\Phi(\mathbf{r},\ t)=\Psi_n(z)\exp(ik_\parallel\mathbf{r}-i\omega_n t-\tilde{\gamma}t),\quad k_\parallel r_{De}\ll 1,$$
$$\Psi_n(z)=A_n\exp(-\xi^2/2)H_n(\xi),\ \xi=(z-d/2)(3b^2r_{De}^2)^{-1/4}. \tag{4.2}$$

Here, $A_n$ is the oscillation amplitude, $H_n(\xi)$ is a Hermite polynomial of the n-th order, $k_\parallel$ is the component of the wave vector in the plane perpendicular to the z axis, the corresponding wavelength $\lambda\!\!\!^{-} = 1/k_\parallel$ being assumed to be considerably smaller than the characteristic dimension of the oscillations of the functions $\Psi_n(z)$, i.e., $k_\parallel(3b^2r_{De}^2)^{1/4} \gg (2n+1)^{1/2}$, while the quantum numbers n are such that the characteristic region of localization of the Langmuir oscillations does not exceed the thickness of the plasma layer:

$$(2n+1)^{1/2}(3b^2r_{De}^2)^{1/4}\ll d.$$

The expressions for the frequency and the damping constant of the Langmuir oscillations then have, in the absence of the HF field, the form

$$\omega_n^2=\omega_p^2(1+3k_\parallel^2r_{De}^2+3^{1/2}(2n+1)r_{De}/b),$$
$$\omega_p^2=\omega_{Le}^2(d/2)+\omega_{Li}^2(d/2), \tag{4.3}$$
$$\tilde{\gamma}=\frac{1}{2}\nu_{ei}+\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega_p}{k_\parallel^3r_{De}^3}\exp\left[-\frac{3}{2}-\frac{1}{2k_\parallel^2r_{De}^2}\right].$$

Owing to the condition d < b, the spectrum of the ion-sound oscillations is described by the formulas (3.1) and (3.2), while the potential of the field of the oscillations of amplitude $B_m$ have, in the absence of the HF field, the form

$$\Phi(\mathbf{r},\ t)=\chi_m(z)\exp\{ik_\parallel\mathbf{r}-i\omega t-\gamma_s t\},$$
$$\chi_m(z)=B_m\cos\varkappa_m z. \tag{4.4}$$

In the presence of a HF electric field whose intensity vector is oriented along the boundary of the layer and whose frequency $\omega_0$ is close to the plasma frequency $\omega_p$, there arises a parametric coupling between the Langmuir and the ion-sound oscillations, the increment of the decay-parametric instability being given by the following expression

$$(\gamma+\gamma_s)(\gamma+\tilde{\gamma})=\frac{1}{16}\frac{(k_\parallel r_E)^2}{k_\parallel^2r_{De}^2}\omega_0\omega_s\Lambda_{nm}, \tag{4.5}$$

$$\Lambda_{nm}=\left[\int_0^d dz\Psi_n(z)\chi_m(z)\right]^2\left[\int_0^d dz\Psi_n^2(z)\right]^{-1}\left[\int_0^d dz\chi_m^2(z)\right]^{-1}. \tag{4.6}$$

The expression (4.5) differs from the corresponding expression for the case of an unbounded homogeneous plasma of coefficient $\Lambda_{nm}$, whose maximum with respect to the number m

$$\max\Lambda_{nm}=2(3b^2r_{De}^2)^{1/4}/d \tag{4.7}$$

is attained at n = 0; for n ≫ 1, we have for $\Lambda_{nm}$

$$\max\Lambda_{nm}\approx 1.27\cdot 2(3b^2r_{De}^2)^{1/4}/n^{1/6}d.$$

It is not difficult to see from the expressions (4.5) and (4.7) that the maximum threshold value of $E_0$thr, which is equal to

$$\frac{E_{0,\,\mathrm{thr}}^2}{4\pi n_eT_e}=\frac{16\tilde{\gamma}\gamma_s}{\omega_0\omega_s}\frac{d}{2(3b^2r_{De}^2)^{1/4}}, \tag{4.8}$$

exceeds the corresponding value for the unbounded homogeneous plasma by a factor equal to the ratio of the dimensions of the regions of localization of the ion-sound and Langmuir oscillations. In the more general case, the region of interaction of the waves may turn out to be smaller than the characteristic dimensions of the localization of the oscillations, as obtains when the number of the Langmuir modes n ≫ 1. In this case the coefficient $\Lambda_{nm}$ is proportional to the ratio of the square of the characteristic dimension of the region of interaction of the ion-sound and Langmuir modes to the product of the localization lengths of these oscillations. In the case of an aperiodic instability, during which Langmuir oscillations and the charge-separation waves of frequency $\omega = 0$ are excited, the expressions for the increment and minimum value of the threshold intensity of the external field

$$\frac{E_{0\,\mathrm{thr}}^2}{4\pi(n_eT_e+n_iT_i)}=\frac{4\nu_{ei}}{\omega_p} \tag{4.9}$$

coincide with the corresponding values obtained for the case of an unbounded homogeneous plasma[4]. Such a coincidence is connected with the fact that the oscillations with the frequency $\omega = 0$ are not natural oscillations and have a potential configuration that coincides with the distribution of the field of a Langmuir wave. It becomes clear from comparison of the expressions (4.8) and (4.9) that in an inhomogeneous plasma the minimum threshold intensity of the HF field for the excitation of a decay-parametric instability considerably exceeds the corresponding value $E_0$thr for the buildup of an aperiodic instability when

$$\gamma_s/\omega_s>(3b^2r_{De}^2)^{1/4}/d.$$

5. In conclusion let us consider the effect of plasma inhomogeneity on the vibrational spectrum and parametric instability in strong fields when the thermal motion of the particles can be completely neglected. Let us first discuss the spectra of the slowly decaying short-wavelength surface waves ($k_\parallel d \gg 1$) for the case when the external-field frequencies are considerably higher than the plasma frequency $\omega_p$. In this case the existence of both high-frequency and low-frequency surface waves of frequency

$$\Omega^2=\omega_s^2\left\{1-\frac{\omega_n^2}{4k_\parallel^2c^2}+\frac{k_\parallel}{1+\varepsilon_0}\int_0^a\frac{dz}{\varepsilon(\omega_n,z)}[\varepsilon_0^2-\varepsilon^2(\omega_s,z)]\right. \tag{5.1}$$
$$\left.+\frac{1}{2k_\parallel}\left[\frac{\partial}{\partial z}\ln\varepsilon(\omega_s,z)\right]_{z=a}\right\}$$

and damping constant

$$\tilde{\gamma}=\omega_s\frac{\pi\varepsilon_0^2}{2(1+\varepsilon_0)}k_\parallel\int_0 dz\,\delta[\varepsilon(\omega_s,z)]. \tag{5.2}$$

turns out to be possible.

For the low-frequency oscillations whose frequency is considerably smaller than the plasma frequency, we should for $\omega_s$ and $\epsilon(\omega, z)$ use the expressions

$$\omega_s^2=\frac{\omega_{Li}^2}{1+\varepsilon_0}(1-J_0^2),\qquad \varepsilon(\omega,z)=1-\frac{\omega_{Li}^2(z)}{\omega^2}(1-J_0^2), \tag{5.3}$$

whereas for the high-frequency oscillations these quantities have the form

$$\omega_s^2=\frac{\omega_{Le}^2}{1+\varepsilon_0},\qquad \varepsilon(\omega,z)=1-\frac{\omega_{Le}^2(z)}{\omega^2}. \tag{5.4}$$

Here, $J_n$ is an n-th order Bessel function whose argument is $k_\parallel r_E$. The HF field does not up to terms proportional to the ratio of the electron and ion masses change the spectrum of the high-frequency surface waves. Therefore, for such waves the second and third terms in the curly brackets in formula (5.1), as well as the damping constant (5.2), can be found from the dispersion equation obtained by Stepanov[6], while the last term corresponds to the Romanov[10] correction connected with the plasma inhomogeneity in the absence of the HF field.

As an overtone $s\omega_0$ of the frequency of the HF field approaches $\omega_S$ (5.4), there develops a parametric resonance leading to the growth (with increment $\gamma$) of the low-frequency (with frequency $\omega$) and high-frequency (with frequency $\omega \pm s\omega_0$) surface oscillations. The dispersion equation for the increment $\gamma$ and frequency $\omega$ has in this case the form

$$(\omega+i\gamma)^2 - J_s^2 \frac{\omega_{Li}^2}{1+\varepsilon_0} \frac{s\omega_0\Delta}{\Delta^2-(\omega+i\gamma+i\tilde{\gamma})^2} = 0. \quad (5.5)$$

Here, $\Delta = s\omega_0 - \Omega$, where $\Omega$ is given by the formulas (5.1) and (5.4).

The dispersion equation (5.5) differs from the equation obtained in[11] by the allowance of the small corrections to the frequency $\omega_S$ which determine the spatial dispersion of the high-frequency surface waves, as well as by the allowance for the damping of the surface oscillations. The consequence of the dependence of the detuning $\Delta$ on the wave number is that the values of the maximum increments, which coincide with the values found in[11], are attained in a relatively broad range of external-field frequencies. Thus, for example, in contrast to the sharp-boundary case, considered in[11], an aperiodic instability, whose increment exceeds the corresponding value for a periodic instability, turns out to be possible not only for $s\omega_0 < \omega_S$, but also in the range of external-field frequencies where $s\omega_0 > \omega_S$.

For values of the detuning $s\omega_0 - \omega_S$ such that $\omega$ and $\gamma \ll \tilde{\gamma}$, the instability becomes dissipative in character. The solution to the dispersion equation (5.5) in the case of a periodic instability then has form

$$\omega^2 = \frac{\omega_{Li}^2}{1+\varepsilon_0} J_s^2 \frac{s\omega_0\Delta}{\Delta^2+\tilde{\gamma}^2},$$

$$\gamma = \frac{\omega_{Li}^2}{1+\varepsilon_0} J_s^2 \frac{s\omega_0\Delta\tilde{\gamma}}{[\Delta^2+\tilde{\gamma}^2]^2}, \quad \Delta > 0.$$

Correspondingly, we obtain for the increment of the aperiodic instability the following expression:

$$\gamma = \left(-\frac{\omega_{Li}^2}{1+\varepsilon_0} J_s^2 \frac{s\omega_0\Delta}{\Delta^2+\tilde{\gamma}^2}\right)^{1/2}, \quad \Delta < 0.$$

It should be noted that in the case when the characteristic dimension of the inhomogeneity of the transition region is large compared to the Debye radius, the thermal motion has an insignificant effect on the spectra of the oscillations studied. The opposite limiting case of a homogeneous plasma with a sharp boundary has been studied in our paper[12].

Notice, finally, that the expressions for the increments of periodic and aperiodic instabilities during resonance at the frequency of the volume Langmuir oscillations differ from the previously obtained expressions[13,14] only by the allowance made for the discreteness of the natural frequencies of the Langmuir oscillations.

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