Amplification of waves during reflection from a rotating "black hole"

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(Submitted July 27, 1972)
Zh. Eksp. Teor. Fiz. 64, 48-57 (January 1973)

The effect whereby waves are amplified during reflection from a rotating "black hole" is investigated. This process is connected with the extraction of energy and momentum from a "black hole." The conditions for the existence of the effect are found. The amplification factor is computed for the model case of scalar waves. For waves with frequencies which do not satisfy the conditions for the existence of the effect, partial cross sections for their capture by the "black hole" are computed.

1. Interest in the investigation of the physical processes which take place in the vicinity of "black holes," i.e., collapsed stars, has of late increased considerably. The existence of such objects is predicted by the general theory of relativity (just as neutron stars, now identified as pulsars, are). It is quite probable that a "black hole" is the final phase of the evolution of any sufficiently massive star. It is also possible that "black holes" of cosmological origin exist.

Among the diverse phenomena which can occur in the strong gravitational field near a "black hole," processes leading to the extraction of energy from a rotating "black hole" at the expense of its rotational energy and momentum are of special interest. The first such process was discovered by Penrose[1] and investigated by Christodoulou[2]. It is connected with the disintegration of a particle, which has entered the exosphere of a rotating "black hole," into two particles one of which is absorbed by the "black hole," while the other escapes to infinity with part of the rotational energy and momentum of the "black hole." Thus, the Penrose process requires the consideration of composite or unstable particles.

Zel'dovich has recently pointed out[3] that the extraction of energy from a rotating "black hole" can in principle be accomplished with the aid of classical multipole waves, e.g., electromagnetic waves. Such a process does not need composite particles and is a distinctive wave analog of the Penrose process. Zel'dovich considered the scattering of an electromagnetic wave by a conducting cylinder rotating with an angular velocity Ω and showed that a wave with an orbital momentum n and frequency ω is reflected from the cylinder with an amplitude exceeding the amplitude of the incident wave if $\omega < n\Omega$.

In the present paper we investigate directly the amplitude amplification effect for a wave reflected from a rotating "black hole" whose gravitational field is described by the Kerr metric (see formula (1)), and show that the condition for existence of the effect also has the form $\omega < n\Omega$ if we introduce an angular velocity Ω of rotation of the "black hole" (see formula (12)). The magnitude of this effect is found for the model case of a massless scalar field satisfying the Klein-Gordon equation. A scalar field is used here because, unlike the electromagnetic field, its equation, as shown in Sec. 2, allows complete separation of the variables in the Kerr metric[1)]. This fact is not apparent, since a rotating "black hole" possesses only axial symmetry.

We investigate with the aid of the exact solutions of the Klein-Gordon equation in the Kerr metric the behavior of the coefficient of reflection of the wave for low frequencies $\omega \to 0$, as well as in the vicinity of the critical point $\omega = n\Omega$, where the effect vanishes. Under certain conditions (see Sec. 3) the reflection coefficient strongly oscillates in the vicinity of the critical point.

For a scalar wave the amplification effect turns out to be small: the reflection coefficient differs from unity by not more than a few per cent. Estimates show that the effect under consideration is also small for the electromagnetic wave[2)].

The formulas obtained also allow the computation of the partial cross sections for capture by a "black hole" of waves with ω and n which do not satisfy the condition for existence of the amplification effect. In particular, a physically clear result is obtained which indicates that the total cross section for capture by a "black hole" of scalar waves with wavelengths much greater than the gravitational radius of the "black hole" is equal to the surface area of the event horizon of the "black hole" (the dominent contribution then is, as always, made by the s-wave).

In the computations, the gravitational field of the "black hole" was considered as an external field, i.e., the reaction of the scalar field on the metric was not considered. Under actual physical conditions the changes in the mass and momentum of the "black hole" on account of the wave amplification effect are negligible compared to the initial values. Indeed, it is easy to show that the mass of the collapsar can appreciably change as compared to M only in a time

$$\tau_1 \sim c^5 / G^2 M \varepsilon,$$

where M is the initial mass of the "black hole," ϵ is the energy density of the wave field around the "black hole," and G is the gravitational constant. For $M \sim 1$–$100 M_\odot$ this time is much greater than the cosmological time $t_H \sim 10^{18}$ sec even when extreme assumptions are made about the magnitude of ϵ. Thus, the investigated effect does not prevent the existence of rotating "black holes" in the Universe.[3)]

Furthermore, τ_1 is $Mc^2/\epsilon r_g^3 \gg 1$ times greater than the characteristic time $\tau_0 \sim r_g/c \sim GM/c^3$ during which the difference between the nonstationary metric of the collapsing star and the steady-state "black-hole" metric vanishes[7,8].

Finally, notice that in[3], Zel'dovich puts forward the hypothesis that there exists a quantum analog of the above-considered classical amplification effect, viz., the process of spontaneous pair production in the Kerr metric, one of the particles being absorbed by the "black hole" and the other escaping to infinity with part of the rotational energy and momentum of the "black hole." According to the correspondence principle, the probability for the creation of a boson pair with given quantum numbers should be proportional to the difference between the amplification factor found in the present paper for a classical wave and unity. Estimates show that the characteristic time during which the momentum of the "black hole" changes owing to the creation of a massless pair

$$\tau_2 \sim G^2M^3/\hbar c^4, \quad \tau_2 < t_H \text{ for } M < 10^{16}\text{ g};$$

for larger masses the loss in momentum and rotational energy is negligible even during the cosmological time t_H.

2. The Kerr metric, which describes the gravitational field of a rotating "black hole," is used in this paper in the form found by Boyer and Lindquist[9] ($G = \hbar = c = 1$; the electric charge of the "black hole" $e = 0$):

$$ds^2 = \rho^{-2}(\Delta - a^2\sin^2\theta)dt^2 - \rho^2dr^2/\Delta - \rho^2d\theta^2 - \rho^{-2}\sin^2\theta[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]d\varphi^2 + 4Mar\rho^{-2}\sin^2\theta dt d\varphi, \quad \rho^2 = r^2 + a^2\cos^2\theta; \quad \Delta = r^2 - 2Mr + a^2, \tag{1}$$

where M is the mass of the "black hole," $L = Ma$ $(0 \le a \le M)$ is its momentum oriented in the direction $\theta = 0$. We also introduce the notation

$$r_{1,2} = M \pm (M^2 - a^2)^{1/2}.$$

The equation for the surface S_{hor} of the event horizon is $r = r_1$, i.e., $\Delta = 0$.

The scalar field ψ is described by the Klein-Gordon equation (a semicolon denotes a covariant derivative; since for the metric under consideration $R_{ik} = 0$, the question as to whether we should add the term $R\psi/6$ to the scalar-field equation does not arise here):

$$\psi_{;i}^{;i} = 0. \tag{2}$$

All the variables in Eq. (2) expressed in the Kerr metric are separable (the complete separability of the variables in the Hamilton-Jacobi equation for the motion of a test particle when the equation is expressed in the Kerr metric was first pointed out by Carter[10] [4)]); the same is true for the Klein-Gordon equation with a mass term. Thus, we can seek ψ in the form $(\omega > 0)$

$$\psi = R(r)P(\theta)e^{in\varphi - i\omega t}. \tag{3}$$

$P(\theta)$ satisfies the eigenvalue equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \left(\frac{n^2}{\sin^2\theta} + \omega^2a^2\sin^2\theta\right)P + \lambda P = 0 \tag{4}$$

with the boundary conditions $|P(0)| < \infty$, $|P(\pi)| < \infty$.

Equation (4) is the equation for a spheroidal wave function of the first kind[11]. The eigenvalues $\lambda_l^n = \lambda_l^{-n}$ $(l \ge |n|)$ are not expressible in analytic form in terms of l and n. The smallest eigenvalue for a given $n - \lambda_n^n$ satisfies the inequality, which will subsequently be used:

$$n(n+1) < \lambda_n{}^n \le n(n+1) + (\omega a)^2\frac{n+1}{n+{}^3/_2}. \tag{5}$$

The left-hand side of the inequality is obvious, while the right-hand side is obtained when the associated Legendre polynomial $P_l^n(\cos\theta)$ is substituted into the variational integral for λ_n^n as a trial function. For $\omega a \ll 1$

$$\lambda_l{}^n = l(l+1) + (\omega a)^2\cdot\frac{2}{3}\left[1 + \frac{3n^2 - l(l+1)}{(2l-1)(2l+3)}\right] + \ldots \tag{6}$$

The radial function $R(r)$ satisfies the equation

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) + [\omega^2(r^2+a^2)^2 - 4aMrn\omega + a^2n^2 - \lambda\Delta]R = 0. \tag{7}$$

In the presence of the mass term $\mu^2\psi$ in Eq. (2), we should replace ω^2 by $\omega^2 - \mu^2$ in Eq. (4) and λ by $\lambda + \mu^2(r^2 + a^2)$ in Eq. (7). By changing the variable r and the function $R(r)$ with the aid of the expressions,

$$\frac{dr}{dy} = \frac{\Delta}{r^2+a^2}, \quad -\infty < y < \infty, \quad R = \frac{b}{(r^2+a^2)^{1/2}}, \tag{8}$$

we can reduce Eq. (7) to the form

$$\frac{d^2f}{dy^2} + k^2(y)f = 0, \tag{9}$$

$$k^2(y) = \left(\omega - \frac{an}{r^2+a^2}\right)^2 - \frac{\Delta}{(r^2+a^2)^2}\left\{\lambda - 2an\omega + \sqrt{r^2+a^2}\frac{d}{dr}\left[\frac{\Delta r}{(r^2+a^2)^{3/2}}\right]\right\}, \quad r = r(y). \tag{10}$$

As $y \to \infty$ $(r \to \infty)$ the function $k^2(y) \to \omega^2$, while as $y \to -\infty$ $(r \to r_1)$

$$k^2(y) \to \left(\omega - \frac{an}{r_1^2+a^2}\right)^2.$$

Christodoulou[2] has obtained a formula expressing the mass of a "black hole" in terms of its momentum L and "irreducible" mass M_0, which is so named because it cannot decrease in any process involving the "black hole." This formula has the form

$$M^2 = M_0^2 + L^2/4M_0^2. \tag{11}$$

Let us define the angular velocity of the "black hole," in accordance with the general rules, as

$$\Omega = \frac{\partial M}{\partial L} = \frac{a}{4M_0^2} = \frac{a}{r_1^2+a^2}. \tag{12}$$

Let us now determine the physical boundary condition for Eq. (9) for $r \to r_1$ $(y \to -\infty)$. It follows from the fact that the surface $r = r_1$ is a surface-trap which can capture only physical objects. Therefore, the boundary condition should be chosen such that the group velocity of the wave for $y \to -\infty$ is directed towards the surface-trap, i.e.,

$$f \sim e^{-i(\omega - n\Omega)y} \quad \text{as} \quad y \to -\infty. \tag{13}$$

Notice that when $\omega < n\Omega$ the phase and group velocities of the wave for $y \to -\infty$ have different directions: $v_g = -1$ and $v_{ph} = \omega/(n\Omega - \omega)$. From the mathematical point of view the boundary condition (13) is the analytical continuation with respect to a through the critical point $\omega = n\Omega$ of Matzner's boundary condition[12] for the Schwarzschild metric $(a = 0)$.

Let the condition (13) be fulfilled, and let, as $y \to \infty$ $(r \to \infty)$, f has the form

$$f = Ae^{-i\omega y} + Be^{i\omega y}. \tag{14}$$

In Eq. (14) the first term describes the incident wave; the second, the reflected wave. Then it follows from the Wronskian of Eq. (9) that

$$|A^2| - |B|^2 \propto \operatorname{sign}(\omega - n\Omega). \tag{15}$$

Thus, the condition for existence of the effect whereby a wave is amplified during reflection from a rotating "black hole" has the form

$$|B|^2 > |A|^2, \text{ i.e., } \omega < n\Omega. \tag{16}$$

In the electromagnetic-wave equation obtained recently by Ipser[5] in the Kerr metric, the variables r and θ are apparently inseparable. However, the condition for existence of the amplification effect for these waves also have the form (16). This follows from the fact that the electromagnetic-wave equation differs from Eqs. (2) and (9) only by a term of the form

$$\frac{M\Delta}{(r^2+a^2)^2}\frac{r+ia\cos\theta}{(r-ia\cos\theta)^2}f,$$

which vanishes both as $r \to r_1$ ($\Delta \to 0$) and as $r \to \infty$.

Finally, the condition (16) preserves its form for a massive was field (in this case we only have to bear in mind that $\omega > \mu$, and not that $\omega \to 0$), as well as for the case when the "black hole" has an electric charge e (in this case in formula (12) $r_1 = M + (M^2 - a^2 - e^2)^{1/2}$).

3. Let us find the magnitude of the amplification factor, i.e., the coefficient R of reflection of a wave from the potential barrier created by the space-time curvature in the vicinity of the "black hole." Let us first investigate the behavior of R at low frequencies: $\omega M \ll 1$. Under this condition we can, in the first approximation, assume that $\lambda_l^n = l(l+1)$. Then R has the form (for derivation, see the Appendix)

$$R_{ln}-1=-D_{ln}=4Q\frac{(l!)^4}{[(2l)!]^2[(2l+1)!!]^2}\times\prod_{k=1}^{l}\left(1+\frac{4Q^2}{k^2}\right)[\omega(r_1-r_2)]^{2l+1}, \quad (17)$$

$$Q=\frac{r_1^2+a^2}{r_1-r_2}(n\Omega-\omega),$$

where $D = 1 - R$ is the coefficient of transmission through the potential barrier. As it should be, $R_{ln} - 1 > 0$ when $\omega < n\Omega$.

The formula (17) encompasses all the particular cases, including the cases $a = 0$ and $a = M$; the region of its applicability is defined by $\omega M \ll 1$. The asymptotic form of this formula for $l \gg 1$ has the form

$$R_{ln}-1=\operatorname{sign} Q\cdot\left[\frac{\omega(r_1-r_2)e(1+4\beta^2)^{1/2}}{8(l+1/2)}\right]^{2l+1}\times\exp\left(4|Q|\operatorname{arc\,tg}\frac{1}{2\beta}\right)[1-e^{-4\pi|Q|}], \quad (18)$$

where $\beta = |Q|/(l + 1/2)$.

For $n \neq 0$ we can replace Q in the formulas (17) and (18) by

$$Q_1=\frac{r_1^2+a^2}{r_1-r_2}n\Omega=\frac{an}{r_1-r_2}.$$

with the exception of the two extreme cases:

1. $a \ll M$ (including $a = 0$), $|n| \ll M/a$, $\omega \geqslant n\Omega$;
2. $a \approx M$, $\frac{M-a}{M} \lesssim (\omega M)^2 \ll 1$, $\frac{l}{|n|}\omega M \gg 1$.

For fixed l and n, R_{ln} monotonically grows with increasing a. Proceeding to the limit, we find that when $a = M$ and $n \neq 0$

$$R_{ln}-1=\operatorname{sign} n\cdot\frac{4|n|^{2l+1}(\omega M)^{2l+1}}{[(2l-1)!!(2l+1)!!]^2}. \quad (19)$$

At very large l ($l \gtrsim |n|/\omega M \gg |n|$) we should, as indicated above, replace n in this formula by $n - 2\omega M$. Thus, for $n \neq 0$ and $\omega \to 0$, $|R_{ln} - 1| \sim \omega^{2l+1}$.

The formulas (17)—(19) are also applicable for $n \leq 0$, when $R < 1$ and the amplification effect is absent. In this case the incident wave is partially reflected and partially captured by the "black hole." The corresponding partial capture cross section for $\omega M \ll 1$[5)] is determined from the formula (see[13])

$$\sigma_{ln}=\pi D_{ln}/\omega^2.$$

In particular, for $n = 0$ the transmission coefficient D_{l0} has the form

$$D_{l0}=\frac{4(l!)^4}{[(2l)!]^2[(2l+1)!!]^2}\omega^2(r_1^2+a^2)[\omega(r_1-r_2)]^{2l}, \quad a\neq M, \quad (20)$$

$$D_{l0}=\frac{2^{2l+3}(\omega M)^{4l+2}}{[(2l-1)!!(2l+1)!!]^2}, \quad a=M. \quad (21)$$

For $\omega \to 0$, $D_{l0} \sim \omega^{2l+2}$ when $a \neq M$ and $D_{l0} \sim \omega^{4l+2}$ when $a = M$. The formula (20) is valid for $\omega M \ll ((M-a)/M)^{1/2}$; formula (21) is also valid for a close to M if

$$l^{3/2}((M-a)/M)^{1/2} \ll \omega M \ll 1.$$

The transition region is described by the general formula (17).

It follows from (20) and (21) that

$$D_{00}=4\omega^2(r_1^2+a^2) \quad \text{for} \quad a\leqslant M,$$

$$\sigma_{\text{cap}}\approx\sigma_0=\pi D_{00}/\omega^2=4\pi(r_1^2+a^2)=S_{\text{hor}}.$$

Thus, for $\omega \to 0$ the cross section for the capture of scalar waves is equal to the surface area of the event horizon. Notice that for $\omega \to \infty$ the capture cross section for any waves, as well as for ultrarelativistic particles, is determined from geometrical optics and also tends to a constant value, equal, for example, to $(27/4)\pi r_1^2$ for the Schwarzschild field ($a = 0$)[7]. Finally, for $a = r_2 = 0$ formula (20) gives an expression for all (n-independent) D_{ln} in the Schwarzschild field.

Let us now investigate the behavior of the effect near the critical point $\omega = n\Omega$. For $a \ll M$ and $|n| \ll M/a$ (then $|n|\Omega M \ll 1$), we can directly use formula (17), which in the present case has the form ($a \approx r_2 \approx 0$, $r_1 \approx 2M$)

$$R_{ln}-1=4\frac{(l!)^4}{[(2l)!]^2[(2l+1)!!]^2}(\omega r_1)^{2l+1}r_1(n\Omega-\omega). \quad (22)$$

It can be seen that $R_{ln} - 1$ passes linearly through zero at the critical point. In the range $0 < \omega < n\Omega$ the function R_{ln} has only one maximum at

$$\omega=\frac{2l+1}{2l+2}n\Omega \quad (l\geqslant n\geqslant 1).$$

For a comparable with but less than M, formula (17) is inapplicable near the point $\omega = n\Omega$, but it can be shown that here also $R_{ln} - 1$ passes linearly through zero at the critical point, i.e., that $R_{ln} - 1 \sim \alpha$ in the region $|\alpha| Q_1 \ll 1$, where

$$\alpha=1-\frac{\omega}{n\Omega}, \quad Q_1=\frac{an}{r_1-r_2}>0.$$

There are two essentially different cases for $a = M$. Let Let $\lambda = \lambda_l^n$ ($\omega = n/2M$); then, if

$$\lambda > 2n^2 - 1/4,$$

we have for $\alpha \to 0$

$$R_{ln}-1\infty\operatorname{sign}\alpha\cdot\exp[2(\lambda-2n^2+1/4)^{1/2}\ln|\alpha|]. \quad (23)$$

In this case R_{ln} passes monotonically through the critical point. When $n = 1$ each $\lambda_l^1 > 2$, as follows from the inequality (5). Therefore, for $n = 1$ there are no oscillations near the critical point.

If, on the other hand,

$$\delta^2 \equiv 2n^2-\lambda-1/4>0, \quad (24)$$

then in the region $|\alpha| n^2 \ll 1$, if $\delta \sim n$ (or $|\alpha| n^4 \ll 1$, if $\delta \lesssim 1$),

$$(R_{ln}-1)^{-1}=\operatorname{sign}\alpha\cdot\left\{\frac{\operatorname{ch}^2\pi(n-\delta)}{\operatorname{sh}^2 2\pi\delta}e^{\pi(n+\delta)(\operatorname{sign}\alpha-1)}\right.$$

$$+\frac{\operatorname{ch}^2\pi(n+\delta)}{\operatorname{sh}^2 2\pi\delta}e^{\pi(n-\delta)(\operatorname{sign}\alpha-1)}-\frac{2\operatorname{ch}\pi(n-\delta)\operatorname{ch}\pi(n+\delta)}{\operatorname{sh}^2 2\pi\delta}e^{\pi n(\operatorname{sign}\alpha-1)}$$
$$\times\cos[\varphi_0-2\delta\ln(2n^2|\alpha|)]\Big\}, \qquad (25)$$
$$\varphi_0(\delta)=4\arg\Gamma(1+2i\delta)+2\arg\Gamma(1/2+in-i\delta)+2\arg\Gamma(1/2-in-i\delta).$$
(for the derivation of this formula, see the Appendix).

In the vicinity of the point $\alpha = 0$ the quantity $R_{ln} - 1$ has an infinite number of oscillations in the region $|\alpha| n^2 \ll 1$ (provided δ is not small compared to 1, which is an exceptional case). For $\alpha > 0$

$$\max(R_{ln}-1)=\frac{\operatorname{ch}^2\pi\delta}{\operatorname{sh}^2\pi n},\quad \min(R_{ln}-1)=\frac{\operatorname{sh}^2\pi\delta}{\operatorname{ch}^2\pi n}.$$

For $\alpha < 0$ the barrier can be totally transparent:

$$\max D_{ln}=1,\quad \min D_{ln}=\frac{\operatorname{sh}^2 2\pi\delta}{(\operatorname{ch}2\pi\delta+e^{-2\pi n})^2}.$$

The condition (24) is satisfied by all λ_l^n with $n \geq 2$ and not very large l ($l < n\sqrt{2}$)—in particular, by λ_2^2.

For a not equal, but close to M, the quantity $R_{ln} - 1$ varies linearly in the region $Q_1|\alpha| \ll 1$ and oscillates in the interval $Q_1^{-1} < |\alpha| \lesssim n^{-2}$ (if, of course, such an interval exists)[6]. Therefore, an additional condition for the existence of oscillations (besides (24)) when $a \neq M$ will be

$$n \ll (M/(M-a))^{1/4}.$$

The physical cause of these oscillations is connected with the presence of quasi-stationary bound states near the event horizon in the Kerr metric when a is close to M. An analogous effect arises near the creation threshold for oppositely charged particles[14].

Finally, by comparing the formulas (17), (23), and (25), we can show that the quantity $R_{ln} - 1$ does not exceed 5% for n = 1, and 0.5% for n ≥ 2.

The author is grateful to Ya. B. Zel'dovich for continuous attention to the work and valuable hints, and also to his colleagues at the Institute of Theoretical Physics and the Astrophysics Division of the Institute of Applied Mathematics for a discussion of the paper.

APPENDIX

1. DERIVATION OF FORMULA (17)

We use the method used in the investigation of the scattering of slow particles (see[13]). Let us make in Eq. (7) the change of variable:

$$x=\frac{r-r_1}{r_1-r_2}\quad (a\neq M).$$

For $x \ll l/\omega(r_1 - r_2)$ and $\omega M \ll 1$ we can neglect in this equation all the terms containing ω, with the exception of the term with ω which is contained in Q. Then Eq. (7) reduces to the form ($\lambda_l^n \approx l(l+1)$)

$$x(x+1)\frac{d}{dx}\left[x(x+1)\frac{dR}{dx}\right]+[Q^2-l(l+1)x(x+1)]R=0. \qquad (A.1)$$

The solution of this equation satisfying the boundary condition (13) for $r \to r_1$ ($x \to 0$) has the form

$$R=(-1)^l\left(\frac{x}{x+1}\right)^{iQ}F(-l;\ l+1;\ 1-2iQ;\ x+1), \qquad (A.2)$$

where $F(\alpha, \beta, \gamma; z)$ is the hypergeometric function. Then, for $x \gg \max(l; Q)$

$$R=C_1x^l+C_2x^{-l-1};$$
$$C_1=\frac{(2l)!}{l!}\frac{\Gamma(1-2iQ)}{\Gamma(l+1-2iQ)},$$
$$C_2=-iQ\frac{(1-iQ)\ldots(l-iQ)}{(l+1)!}(-1)^l\cdot {}_3F_2(-l,l+1,l+1-iQ;\ l+2,1-2iQ;\ 1); \qquad (A.3)$$

${}_3F_2(\alpha, \beta, \gamma; \delta, \epsilon; z)$ is the generalized hypergeometric function (for its definition, see[11]); in the present case it is a polynomial of degree l in z. C_1 and C_2 satisfy the following condition:

$$C_1C_2^*-C_1^*C_2=iQ(l+1/2).$$

On the other hand, for $x \gg \max(Q, l)$ we can drop in Eq. (7) all the terms except those which describe free motion with momentum l. We obtain

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dR}{dx}\right)+\left[\omega^2(r_1-r_2)^2-\frac{l(l+1)}{x^2}\right]R=0. \qquad (A.4)$$

The solutions of this equation are the spherical Bessel functions. Matching the solutions of Eqs. (A.1) and (A.4) in the region

$$\max(Q,l)\ll x\ll l/\omega(r_1-r_2),$$

we obtain formula (17). We can verify in the process that in the region under consideration all the terms omitted in Eq. (7) do not influence the answer (they can only distort the phase of the wave function, a quantity which does not enter into the expression for the transmission coefficient D).

2. DERIVATION OF FORMULAS (23) AND (25)

Let a = M and $|\alpha| n^2 \ll 1$. Let us set $x = r/M - 1$. For $x \ll 1/n$, we can by dropping the small terms, reduce Eq. (7) to the form

$$\frac{d^2R}{dz^2}+\left(\alpha^2n^2-\frac{2\alpha n^2}{z}+\frac{2n^2-\lambda}{z^2}\right)R=0, \qquad (A.5)$$

where $z = x^{-1}$. Its solution is of the form

$$R=C_1z^{1/2-i\delta}e^{-i\alpha nz}F(1/2-in-i\delta;\ 1-2i\delta;\ 2i\alpha nz)+C_2z^{1/2+i\delta}e^{-i\alpha nz}F(1/2-in+i\delta;\ 1+2i\delta;\ 2i\alpha nz), \qquad (A.6)$$

where $F(\alpha, \gamma, z)$ is the confluent hypergeometric function.

The coefficients C_1 and C_2 should be chosen such that the solution satisfies the boundary condition (13) for $r \to r_1$ ($z \to \infty$). We obtain:

$$|C_2/C_1|\propto|\alpha|^{2|\delta|}\ \text{for}\ \delta^2<0,$$
$$|C_2/C_1|=e^{-\pi\delta\operatorname{sign}\alpha}\left(\frac{\operatorname{ch}\pi(n+\delta)}{\operatorname{ch}\pi(n-\delta)}\right)^{1/2}\ \text{for}\ \delta^2>0. \qquad (A.7)$$

In the region $x \gg |\alpha| n$ the dominant terms in Eq. (7) have the form

$$\frac{1}{x^2}\frac{d}{dx}\left[x^2\frac{dR}{dx}\right]+\left[\frac{n^2}{4}+\frac{n^2}{x}+\frac{2n^2-\lambda}{x^2}\right]R=0. \qquad (A.8)$$

It has the following solution:

$$R=C_3x^{-1/2+i\delta}e^{-inx/2}F(1/2+in+i\delta;\ 1+2i\delta;\ inx)+C_4x^{-1/2-i\delta}e^{-inx/2}F(1/2+in-i\delta;\ 1-2i\delta;\ inx). \qquad (A.9)$$

Matching the solutions (A.6) and (A.9) in the region $|\alpha| n \ll x \ll n^{-1}$, we obtain the formulas (23) and (25).

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1) As became known to the author, the last result has recently been obtained independently by Brill et al. [4]

2) Thus, scalar and electromagnetic waves are not generated in the Kerr metric (cf. [5]).

3) As has recently been shown by Press [6], a rotating "black hole" can also lose momentum if by chance it is placed in a time-independent scalar field produced by sources outside the "black hole." The characteristic time during which the momentum changes is then the same as indicated above. In this case, however, the rotational energy of the

"black hole" is not extracted from it, but converted into an "irreducible" (see [2]) "black hole" mass.

[4]See also the paper [4] by Brill et al.

[5]In this case the angular part of the function ψ is, in the first approximation, simply a spherical harmonic.

[6]In this case the number of oscillations is finite.

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Translated by A. K. Agyei
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