# Nonlinear ion-sound waves in a plasma with three-dimensional random inhomogeneities

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The problem of nonlinear ion-sound waves in a nonisothermal plasma containing weak three-dimensional random inhomogeneities in the electron concentration is considered. The basic integro-differential equation describing the nonstationary wave processes in a weakly nonlinear and randomly inhomogeneous plasma is derived. It is shown that in specific cases it can be reduced to the modified Burgers and Korteweg—de Vries—Burgers equations. Stationary waves are investigated. Analysis shows that random inhomogeneities in the equilibrium concentration lead to oscillations behind the shock-wave front. Furthermore, the appearance of a weak ion-sound shock wavė is possible in certain cases.

Nonlinear waves in homogeneous, dispersive and absorbing media have now been studied sufficiently thoroughly (see, for example,[1-4]). The question has been partially discussed in application to smoothly inhomogeneous, regular media[5-7]. At the same time investigation of the nonlinear wave processes in media with random variations in the parameters is of natural interest. One of the simplest cases—the interaction of quasi-monochromatic waves in a plasma with random one-dimensional inhomogeneities in the electron concentration—has been analyzed by the authors[8]. The effect of allowance for stochastic factors on shock waves in nondispersive media has been studied[9-11]. In particular, in[9] a "stochastic" force was introduced into the nonlinear equations and the behavior of a magnetohydrodynamic wave in the field of the stochastic force was studied. In[10,11] the effect of a random change in the sound velocity on the parameters of a shock wave is studied.

In the present paper, using as an example the problem of nonlinear waves in a nonisothermal plasma (ion sound), we consistently take into account both the dispersion and the random three-dimensional inhomogeneities of the medium. We derive a basic integro-differential equation describing the non-stationary wave processes in a plasma with a slight nonlinearity and with weak random inhomogeneities in the equilibrium electron density. As follows from the analysis, in specific limiting cases this equation can be reduced to the modified Burghers and Korteweg—de Vries—Burgers equations[1-3]. The stationary waves are also investigated: it is shown that random inhomogeneities in the equilibrium concentration lead to oscillations behind the shock-wave front. Furthermore, under certain conditions the appearance of a weak ion-sound shock wave is possible.

Notice that the results obtained in this paper are essentially more general in nature: similar effects should occur in the propagation of nonlinear Alfvén waves in a magnetoactive plasma, of waves in the surface of a liquid, in interspaced transmission lines, in ferromagnets, etc.

In the quasi-hydrodynamic approximation the basic equations for a nonisothermal plasma ($T \gg T_i$) where T and $T_i$ are the electron and ion temperatures)[1)] have the form

$$\Delta\varphi = -4\pi e(N_0 - N_0 e\varphi/\varkappa T + {}^1/_2 N_0 (e\varphi/\varkappa T)^2 - N);$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = \frac{e}{M}\operatorname{grad}\varphi, \quad \frac{\partial N}{\partial t} + \operatorname{div}(N\mathbf{v}) = 0. \tag{1}$$

Here $\varphi$ is the electric potential; $\mathbf{v}$ and N are the velocity and concentration of the plasma ions; e/M is the specific charge of an ion and $\kappa$ is the Boltzmann constant. The electrons in the field of the ion-sound wave have the Boltzmann distribution

$$N_e = N_0 \exp(-e\varphi/\varkappa T).$$

In the first equation of the system (1), $\exp(-e\varphi/\kappa T)$ has been expanded in a series in powers of the small parameter $e\varphi/\kappa T$ up to the quadratic term.

Let us represent the functions $\varphi$, N, and $\mathbf{v}$ in the form[2)]

$$\varphi = \langle\tilde\varphi(x,t)\rangle + \varphi'(\mathbf{r},t),\ N = N_{00} + \delta N(\mathbf{r}) + \langle\tilde N(x,t)\rangle + N'(\mathbf{r},t),$$
$$N_0 = N_{00} + \delta N(\mathbf{r}),\ \mathbf{v} = \langle\tilde v(x,t)\rangle + \mathbf{v}'(\mathbf{r},t),$$

where the sign $\langle\ldots\rangle$ denotes averaging over the ensemble of the three-dimensional inhomogeneities $\delta N(\mathbf{r})$ of the electron and ion concentrations, and $N_{00}$ is the mean value of the charged-particle concentration (the plasma is assumed to be quasi-neutral); a tilde denotes an average wave perturbation, while a prime denotes its deviation due to fluctuation. We shall henceforth assume that the plasma is weakly nonlinear and that the concentration fluctuations are small, i.e. ($v_S^2 = \kappa T/M$),

$$\frac{|\delta N(\mathbf{r})|}{N_{00}} \sim \mu \ll 1, \quad \frac{e|\langle\tilde\varphi_{max}\rangle|}{\varkappa T} \sim \frac{|\langle\tilde N\rangle|}{N_{00}} \sim \frac{|\langle v\rangle|}{v_{3\text{в}}} \sim \mu^2. \tag{2}$$

Averaging the system of equations (1) over the ensemble of the inhomogeneities $\delta N(\mathbf{r})$, we obtain

$$\frac{\partial^2\langle\tilde\varphi\rangle}{\partial x^2} = 4\pi e\left[N_{00}\frac{e\langle\tilde\varphi\rangle}{\varkappa T} + a(x,t) - \frac{N_{00}}{2}\left(\frac{e\langle\tilde\varphi\rangle}{\varkappa T}\right)^2 + \langle\tilde N\rangle\right],$$
$$\frac{\partial\langle\tilde v\rangle}{\partial t} + \langle\tilde v\rangle\frac{\partial}{\partial x}\langle\tilde v\rangle = \frac{e}{M}\frac{\partial}{\partial x}\langle\tilde\varphi\rangle,$$
$$\frac{\partial\langle\tilde N\rangle}{\partial t} + N_{00}\frac{\partial}{\partial x}\langle\tilde v\rangle + b(x,t) + \frac{\partial}{\partial x}(\langle\tilde N\rangle\langle\tilde v\rangle) = 0, \tag{3}$$

where the functions a(x, t) and b(x, t) are equal to

$$a(x,t) = \langle\delta N(\mathbf{r})e\varphi'(\mathbf{r},t)/\varkappa T\rangle, \quad b(x,t) \tag{4}$$

Subtracting from the equations (1) the corresponding equations (3), we obtain a system for the fluctuation components of the wave perturbations:

$$\Delta\varphi' = 4\pi e\left(N_{00}\frac{e\varphi'(\mathbf{r},t)}{\varkappa T} + N' + \delta N(\mathbf{r})\frac{e\langle\tilde\varphi\rangle}{\varkappa T}\right),$$
$$\frac{\partial\mathbf{v}'}{\partial t} = \frac{e}{M}\operatorname{grad}\varphi', \quad \frac{\partial N'}{\partial t} + N_{00}\operatorname{div}\mathbf{v}' + \frac{\partial}{\partial x}(\delta N\langle\tilde{\mathbf{v}}\rangle) = 0. \tag{5}$$

  

In deriving (5) we discarded terms of the type

$$\delta N(\mathbf{r}) e\varphi'/\varkappa T - \langle \delta N e\varphi'/\varkappa T\rangle, \quad \delta N v' - \langle \delta N v'\rangle$$

which are, as is easy to show, of the order of $O(\mu^4)$. Allowance for these terms in (3) would have led to the appearance of small corrections $\sim O(\mu^5)$, which need not be considered in our case, since the nonlinearity is of the order of $\mu^4$.

Below we shall be interested in the behavior of only the averaged quantities described by the system (3) containing the still unknown functions a(x, t) and b(x, t). They can be found from the solution to the system (5). Since the equations of the system (5) are linear, its solution can be found by the Fourier method. We omit the intermediate computations and give the expressions for a(x, t) and b(x, t):

$$a(x,t) = -\frac{e^2\langle(\delta N)^2\rangle}{2\pi\varkappa T}\left\{\int \frac{\gamma_N(\rho)}{\varepsilon_1(\omega)\rho}\exp(i\omega\tau - ik_0\rho)\left[\frac{e}{\varkappa T}\langle\tilde{\varphi}(x-\xi, t-\tau)\rangle + \frac{\xi}{\omega\rho^2}(k_0\rho - i)\langle\tilde{v}(x-\xi, t-\tau)\rangle\right] d\rho\, d\omega\, d\tau, \qquad (6)$$

$$b(x,t) = -\frac{e^2\langle(\delta N)^2\rangle}{2\pi M}\frac{\partial}{\partial x}\int \frac{\gamma_N(\rho)}{\omega\varepsilon_2(\omega)\rho^3}\exp(i\omega\tau - ik_0\rho) \times\left\{\frac{e}{\varkappa T}\xi(k_0\rho - i)\langle\tilde{\varphi}(x-\xi, t-\tau)\rangle + \omega^{-1}[k_0^2\xi^2 + (1+ik_0\rho) \times (1 - 3\xi^2/\rho^2)]\langle\tilde{v}(x-\xi, t-\tau)\rangle\right\} d\rho\, d\omega\, d\tau;$$

$$k_0 = \Omega_0/v_s\, \varepsilon_i^{1/2}(\omega), \qquad \varepsilon_i(\omega) = \Omega_0^2/\omega^2 - 1, \qquad (7)$$

where $\gamma_N(\boldsymbol{\rho})$ is the correlation coefficient for the equilibrium concentration fluctuation and $\boldsymbol{\rho}$ is a vector with the components $(\xi, \eta, \zeta)$.

Let us now turn to the system (3). Assuming the dispersion (the term $\partial^2\langle\tilde{\varphi}\rangle/\partial x^2$) to be small, we obtain for $\langle\tilde{\varphi}(x, t)\rangle$ after a series of computations (see also[12]) an integro-differential equation describing the nonstationary wave processes in the plasma:

$$\frac{\partial\langle\tilde{\varphi}\rangle}{\partial t} + v_s\left(1 - \frac{e\langle\tilde{\varphi}\rangle}{\varkappa T}\right)\frac{\partial\langle\tilde{\varphi}\rangle}{\partial x} + \frac{v_s^2}{2\Omega_0^2}\frac{\partial^3\langle\tilde{\varphi}\rangle}{\partial x^3} - \frac{Mv_s^2}{2eN_{00}}\left(\frac{\partial a}{\partial x} + b\right) = 0. \qquad (8)$$

The investigation of Eq. (8) in its general form presents considerable mathematical difficulty, in virtue of the complicated integral operators a(x, t) and b(x, t); therefore we shall limit ourselves below to an approximate analysis. Since the correlation coefficient $\gamma_N(\boldsymbol{\rho})$ has a characteristic scale of the order of the correlation radius $l$, then assuming that the functions $\langle\tilde{v}(x-\xi, t-\tau)\rangle$ and $\langle\tilde{\varphi}(x-\xi, t-\tau)\rangle$ in (6) and (7) vary more smoothly in space than $\gamma_N(\boldsymbol{\rho})$, we can expand them into series around the point $\xi = 0$. The dependence of these functions on the variable $\tau$ can be treated in analogous fashion. Although the function of the variables $\rho$ and $\tau$ entering into (6) and (7) cannot be computed completely, its characteristic time scale can be estimated, for example, by the stationary-phase method. It turns out to be equal to $\tau_0 \approx \max\{\rho/v_s, \Omega_0^{-1}\}$.

Thus, the integral operators a(x, t) reduce to differential operators. If, in the expansion of the functions $\langle\tilde{v}(x-\xi, t-\tau)\rangle$ and $\langle\tilde{\varphi}(x-\xi, t-\tau)\rangle$ in terms of $\xi$ and $\tau$, we limit ourselves to only the second derivatives with respect to $\xi$ and $\tau$, and assume that the field of the fluctuations is isotropic, then the integrals (6) and (7) can be easily evaluated, and Eq. (8) assumes the form[3)]

$$\frac{\partial}{\partial t}\langle\tilde{\varphi}\rangle + v_s\left(1 + \frac{|e|\langle\tilde{\varphi}\rangle}{\varkappa T}\right)\frac{\partial}{\partial x}\langle\tilde{\varphi}\rangle + \frac{v_s^3}{2\Omega_0^2}\frac{\partial^3\langle\tilde{\varphi}\rangle}{\partial x^3} + \alpha_1\frac{\partial^3\langle\tilde{\varphi}\rangle}{\partial x\,\partial t^2} - \alpha_2\frac{\partial^3\langle\tilde{\varphi}\rangle}{\partial^2 x\,\partial t} + \eta\frac{\partial^2\langle\tilde{\varphi}\rangle}{\partial x\,\partial t} = 0;$$

$$\alpha_1 = \frac{v_s}{3\Omega_0^2}\left\langle\left(\frac{\delta N}{N_{00}}\right)^2\right\rangle\frac{\bar{L}^2}{r_d^2}, \qquad \alpha_2 = v_{3B}\alpha_1, \qquad \eta = \frac{\bar{L}}{3}\left\langle\left(\frac{\delta N}{N_{00}}\right)^2\right\rangle,$$

$$r_d = \frac{v_s}{\Omega_0}, \qquad \bar{L}^2 = \int_0^\infty \rho\gamma_N(\rho)\,d\rho, \qquad \bar{L} = \int_0^\infty \gamma_N(\rho)\,d\rho. \qquad (9)$$

The quantities $\bar{L}^2$ and $\bar{L}$ characterize the integral scales of the inhomogeneity, while the coefficients $\alpha_{1,2}$ characterize the contribution of the random inhomogeneities to the dispersion of the ion-sound waves, which, in terms of the effective permittivity $\epsilon_{eff}(\omega, k)$ (see[13]), corresponds to the real part of $\epsilon_{eff}$ and $\eta$ corresponds to the viscosity of the medium, i.e., to the imaginary part of $\epsilon_{eff}$. Notice that in the present paper we have virtually computed $\epsilon_{eff}$ for ion-sound waves, which is of interest in itself.

The solutions of Eqs. (9) can be investigated for the case of weak viscosity, using well-known methods[3,5,6]. We shall analyze the simplest class of solutions, namely, stationary waves, when the function $\langle\tilde{\varphi}(x, t)\rangle$ depends on the variable $z = x - ut$ (u is the wave velocity). In this case Eq. (9) reduces to a second-order ordinary differential equation and can be investigated in the phase plane $(\langle\tilde{\varphi}\rangle, \langle\tilde{\varphi}\rangle'_z)$. It is easy to show that for a weak wave $(e\langle\tilde{\varphi}\rangle/\kappa T \ll 1)$, when

$$\bar{L}^2/r_d^2\langle(\delta N/N_{00})^2\rangle < 2(M_0 - 1)$$

($M_0 = u/v_s$ is the Mach number and $r_d$ is the Debye radius), the phase plane has the form shown in Fig. 1, i.e., the solution has the form of a shock wave with oscillations behind the wave front. Note that the spatial period of the oscillations should be much greater than the inhomogeneity scale and that this is fulfilled when $r_d/(M_0 - 1)^{1/2} \gg l$.

For a sufficiently strong viscosity

$$\langle(\delta N/N_{00})^2\rangle(\bar{L}/r_d)^2 > 2(M_0 - 1)$$

a weak shock wave type of solution with a frontal width

$$l_{\mathrm{f}} \sim \langle\delta N^2/N_{00}^2\rangle r_d^2/\bar{L} \gg l.$$

is possible.

Thus, there appear in a plasma with random inhomogeneities new effects that are characterized in particular cases by the introduction of viscosity and additional dispersion of the ion-sound waves into the Korteweg–de Vries equation; in the general case dispersion and viscosity are interrelated and their separation is not possible (see (6)–(8)).

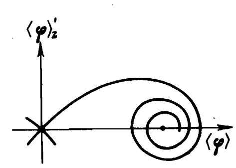

FIG. 1

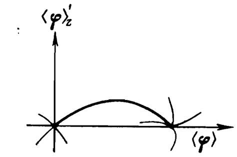

FIG. 2

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1)We shall for simplicity assume below that the ion temperature is equal to zero.

2)The fluctuations $\delta N(\mathbf{r})$ are assumed to be stationary. This is possible if the characteristic time of the nonlinear process $t_0 \sim L/u$ (L is the spatial scale of the wave, u its velocity) is small compared to the exchange time for the realization of the random field $\delta N(\mathbf{r})$.

[3]Since Eqs. (8) and (9) describe near-stationary processes (functions of $(x-ut)$), the derivatives $\partial/\partial t$ can be replaced by the derivatives $-u\partial/\partial x$ (see [3].


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