

Thermodynamics of layered isotropic magnets at low temperatures

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Layered isotropic magnets are considered, in which the interaction J within a layer is large compared to the interaction J' between layers. Expressions are obtained in the low-temperature region $T \ll J$ for the specific magnetic moment in a weak field and for the specific heat for an arbitrary ratio J'/T . In the region $T \ll J' \ll J$ the magnet behaves like a three-dimensional system and in the region $J' \ll T \ll J$ the moment and specific heat behavior is that characteristic of two-dimensional systems. The analysis is based on a division of the spin correlations into shortwave and longwave correlations. The limits of applicability of the spin-wave approximation and the behavior of the correlation functions at large distances are given.

The magnetic properties of two-dimensional or "almost two-dimensional" spin systems have been under diligent investigation of late (from among the extensive literature, see, for example, the experimental papers^[1-4]). Such systems are realized in various types of metallic compounds characterized by a layered structure. The metallic ions, which possess magnetic moment, form in these compounds plane lattices that are separated by layers of nonmagnetic ions (the number of which can be quite large, $n \lesssim 10$). Approximately 20 such compounds are presently known. Colpa^[1] has measured the low-temperature behavior of the specific heat, Longh et al.^[2] performed temperature and field measurements of the magnetic susceptibility, and Bloembergen and Franse^[3] determined experimentally different characteristics of layered spin systems by measuring the mechanical torque in magnetic phase transitions. Experiments have shown that these compounds undergo magnetic transitions at temperatures on the order of 10°K , below which there is magnetic ordering with an easy-magnetization axis in a plane made up of the ions of the metal. Depending on the type of the compound, the sign of the exchange interaction within a plane or between planes can be arbitrary.

Experiments^[1-3] show that these compounds can be described with good approximation by the model of an isotropic Heisenberg magnet, the exchange interactions of which are essentially different in planes made up of the ions of the metal (J interaction) as well as between these planes (J' interaction). The order of magnitude of the ratio J'/J in these compounds amounts as a rule to 10^{-2} , so that these compounds can be called "almost two-dimensional." The anisotropic part of the interaction energy in these compounds is exceedingly small^[1-4].

The smallness of the interaction J' between planes causes the observed thermodynamic quantities of the indicated substances, as functions of the temperature and of the magnetic field, to exhibit a behavior characteristic of two-dimensional systems. Colpa^[1], in particular, observed a linear low-temperature dependence of the specific heat (unlike the $T^{3/2}$ law in the three-dimensional case) and noted that the linear behavior of the specific heat follows from the spin-wave approximation (SWA) for a two-dimensional ferromagnet. In this connection, the question of the applicability of the SWA to two-dimensional magnets and of the role of the weak interplanar interaction in the establishment of the

magnetic order in quasi-two-dimensional systems, which has heretofore had only purely theoretical significance, becomes of practical interest.

Let us recall briefly the history of this question. As is well known, in the two-dimensional case the SWA leads to a divergence of the fluctuation correction to the spontaneous moment, meaning a destruction of the long-range order by the fluctuations (Bloch, Peierls, Landau). The absence of a spontaneous moment in a two-dimensional Heisenberg magnet was rigorously proved by Mermin and Wagner^[5]. On the other hand, numerical calculations by Stanley and Kaplan^[6] and by others, based on high-temperature expansions, point to the presence of a certain phase transition. This conclusion was confirmed by the results of one of us^[7-9], that in a wide class of two-dimensional systems with a continuous symmetry group, including a system of Heisenberg spins, the correlations show a power-law decrease with distance (with an exponent proportional to T as $T \rightarrow 0$) and the sample has an infinite susceptibility at sufficiently low temperatures¹⁾. This behavior differs qualitatively from the high-temperature behavior (exponential decrease of the correlations and finite susceptibility).

The approach developed in^[7-9] is based on the fact that the SWA is valid for the considered two-dimensional system at sufficiently small but macroscopic distances (i.e., large in comparison with the lattice constant a). Namely, at low temperatures ($T \ll J$), the dimensions of the region of applicability of the SWA are determined by the condition (6) (see below) and can greatly exceed the atomic dimensions. It was shown earlier^[7] how to obtain the asymptotic correlations at large distances by making use of this circumstance. By matching these asymptotic forms with those obtained from the SWA, we can easily calculate all the thermodynamic characteristics of interest at $T \ll J$.

In the present paper we used the same methods of^[7-9] to analyze almost-two-dimensional isotropic Heisenberg magnets that can be realized in compounds with a layered structure of the type indicated above. We also calculated the thermodynamic characteristics of these systems in the low-temperature region, i.e., under the conditions

$$T \ll J, \quad J' \ll J, \quad (\text{A})$$

but for an arbitrary ratio J'/T .²⁾ The behavior of the system turns out to be "almost three-dimensional" in

the case $T \ll J'$, and "almost two-dimensional" in the opposite case $T \gg J'$.³⁾ It is precisely the last case which calls for a special examination, since the usual SWA is applicable at $T \lesssim J'$. There are also a high-temperature region and a phase-transition region ($T \sim J$). In these regions, however, the SWA does not hold and they remain outside the scope of our analysis.

In Sec. 1 we summarize in a simpler and more lucid manner the earlier results^[7-9], including the dependence on the external field, and discuss the question of the matching of the asymptotic results at large distances to the asymptotic results of the SWA. This analysis is generalized in Sec. 2 to include a three-dimensional layered system and the thermodynamic characteristics are calculated for an "almost two-dimensional" isotropic ferromagnet under the conditions (A). Section 3 contains a summary of the results for systems in which the exchange interactions in the plane of the layer and between the layers have opposite signs.

1. SPIN-WAVE APPROXIMATION FOR A TWO-DIMENSIONAL SYSTEM

The Hamiltonian of an isotropic Heisenberg magnet is

$$H = -\frac{1}{2} J \sum_r \sum_{\delta} (S_r \cdot S_{r+\delta}), \quad (1)$$

where the summation is over all the points r of the lattice, which we assume for simplicity to be quadratic (dimensionality $d = 2$) or cubic ($d = 3$); δ are vectors joining neighboring lattice points; S_r is the spin operator of the atom at the site r . The spins of all the atoms are assumed to be of the same magnitude: $S_r^2 = S(S+1)$, where S is an integer or a half-integer (in a system of units where $\hbar = 1$).

In the three-dimensional case at low temperatures, when

$$\alpha = T / 2\pi JS^2 \ll 1 \quad (2)$$

(the factor $(2\pi)^{-1}$ has been introduced for convenience; the symbol for the parameter α is used throughout the article), the spin-wave approximation (it is legitimate to use the spin-wave approximation (SWA), wherein the operators S_r are approximately represented in terms of Bose operators a_r^\pm (regarded as small quantities) using the formulas

$$S_r = S - a_r^+ a_r^-, \quad S_r^\pm = S_r^\pm \pm iS_r^y = (2S)^{1/2} a_r^\pm, \quad (3)$$

In the approximation (3), the Hamiltonian (1) is diagonalized by changing over to the operators a_k^\pm , which are the Fourier components of the operators a_r^\pm . The magnon energy is equal to $JS\Delta(k)$, where

$$\Delta(k) = \sum_{\delta} (1 - e^{ik\delta}) = 2 \sum_{i=1}^d (1 - \cos k_i) = k^2 + O(k^4). \quad (4)$$

The condition for the applicability of the initial approximation (3), and consequently of the SWA, is, as is sufficiently evident from physical considerations (and can be proved quite rigorously), the smallness of the spin deviations

$$\langle \delta S_r^z \rangle = \langle S_r^z - S \rangle \ll S; \quad \langle (S_r^z - S)^2 \rangle \ll S^2,$$

which reduces to the condition (the integral with respect to k is taken within the limits of the reciprocal-lattice cell)

$$\langle a_r^+ a_r^- \rangle = \int \frac{dk}{(2\pi)^d} e^{ik(r-r')} [e^{JS\Delta(k)/T} - 1]^{-1} \ll S. \quad (5)$$

In the three-dimensional case, the integral in (5) converges and as $T/J \rightarrow 0$ it is proportional to $(T/J)^{3/2}$, i.e., in the three-dimensional case the SWA always holds at low temperatures. In the two-dimensional case, this integral diverges at small k . The divergence appears on going to the limit of infinite dimensions ($R/a \rightarrow \infty$, where R is the dimension of the system). Allowance for finite dimensions is obviously equivalent to a cutoff at small $k \sim a/R$, so that the SWA is valid in a region whose dimensions satisfy the inequality

$$\alpha \ln(R/a) \ll 1. \quad (6)$$

We now call attention to the arbitrariness of the choice of the direction of the quantization axis in spin space when the SWA is introduced in accordance with (3). We associate with the quantization axis a coordinate system characterized by a unit vector n in the axis direction and by unit vectors $e_y(n)$ and $e_\varphi(n)$ orthogonal to it. Then the explicit dependence of (3) on the direction n takes the form

$$S_r \approx (S - a_r^+ a_r^-) n + S^{1/2} (a_r^+ e^-(n) + a_r^- e^+(n)), \quad (7)$$

where we have introduced linear combinations of the vectors

$$e^\pm(n) = (e_y(n) \pm ie_\varphi(n)) / \sqrt{2},$$

having the following properties:

$$n e^\pm(n) = 0, \quad (e^\pm(n))^2 = 0, \quad e^+(n) e^-(n) = 1. \quad (8)$$

Obviously, the SWA Hamiltonian, as well as the correlator $\langle a_r^+ a_r^- \rangle$, does not depend on the direction n , whereas according to (7) such a dependence does arise in the calculation of various types of mean values. This is a consequence of the degeneracy of the system with respect to rotations in spin space, and is eliminated by specifying external (or boundary) conditions. In fact, the direction of the quantization axis is determined by the direction of the spontaneous moment. In the most general case compatible with the applicability of the SWA the external conditions call for a random distribution of the directions of the spontaneous moment, while the oscillations of the spins about this direction are small and are described by the SWA formulas⁴⁾.

We consider now a large two-dimensional system, the dimensions R of which do not satisfy the SWA applicability condition (6). Such a system can be arbitrarily broken up into physical regions Ω_i whose dimensions ρ satisfy the conditions

$$\rho \gg a, \quad \text{but} \quad \alpha \ln(\rho/a) \ll 1. \quad (9)$$

Inside each region, the SWA is valid but the direction n_i is random. Then the calculation of the mean values in the large system obviously consists of a determination of the mean values within each region Ω_i by means of the SWA formulas (7), followed by averaging over the random directions n_i . The problem thus reduces to a determination of a compatible distribution of the directions n_i .

Before we proceed to determine the factors that govern this distribution, we note that it is expedient to change over to an equivalent formulation of the problem, since the use of subdivision into regions Ω_i leads to difficulties in a formal sense. From the physical point of view, the representation of correlations in the form of superpositions of small oscillations of the spins in regions Ω_i near the directions of the average moment n_i on random changes of the directions n_i on going to

neighboring regions corresponds to differentiation between short-range and long-range correlations. The dimensions ρ of the region then correspond to the demarcation wavelength between the regions of the long- and short-wave correlations.

This demarcation can be formulated mathematically in the following fashion. We introduce the random vector function

$$\mathbf{n}_r = \{\sin \vartheta(r) \cos \varphi(r); \sin \vartheta(r) \sin \varphi(r); \cos \vartheta(r)\} \quad (10)$$

($\vartheta(r)$ and $\varphi(r)$ are angles specifying the direction of \mathbf{n}_r in a spherical coordinate system), and define this function in such a way that its Fourier expansion contains only components with wavelengths $\gg \rho$:

$$\mathbf{n}_r = \sum_{|\mathbf{k}| < 2\pi a/\rho} \mathbf{n}_k e^{i\mathbf{k}r/a}. \quad (11)$$

By virtue of the definition (11), the function \mathbf{n}_r is specified by a finite number R^2/ρ^2 of the parameters \mathbf{n}_k . If we consider a system of any R^2/ρ^2 points \mathbf{r}_i , then the values $\mathbf{n}_i = \mathbf{n}(\mathbf{r}_i)$ at these points form a system of parameters that are uniquely (and furthermore linearly) connected with \mathbf{n}_k . It will be more convenient for us to choose \mathbf{n}_i in place of \mathbf{n}_k as the random parameters determining the function (11), so that a specification of the random function (11) reduces to a specification of the joint distribution R^2/ρ^2 of the quantities \mathbf{n}_i .

Another circumstance which we shall utilize is that inasmuch as the function (11) contains only long-wave components, it can be regarded as a continuous and differentiable function of spatial coordinates \mathbf{r} that vary slowly over distances on the order of a . In particular, we can write

$$\mathbf{n}_{r+\delta} \approx \mathbf{n}_r + \gamma_{r\delta}^- e^+(\mathbf{n}_r) + \gamma_{r\delta}^+ e^-(\mathbf{n}_r), \quad \mathbf{n}_r e^\pm(\mathbf{n}_{r+\delta}) \approx -\gamma_{r\delta}^\pm, \quad (12)$$

where $\gamma_{r\delta}$ are the components of the expansion of the increment of \mathbf{n}_r on going to the neighboring site, and are regarded as small quantities:

$$\gamma_{r\delta}^\pm = 2^{-1/2} [(\delta \nabla) \vartheta(r) \pm i \sin \vartheta(r) (\delta \nabla) \varphi(r)]. \quad (13)$$

Taking into account the indicated separation into long- and short-wave variables, the spin operators S_r , can be represented in the form

$$S_r = (S - a_r^+ a_r^-) \mathbf{n}_r + S^{1/2} [a_r^+ e^-(\mathbf{n}_r) + a_r^- e^+(\mathbf{n}_r)], \quad (14)$$

where the Fourier expansion of the operators a_r^\pm now contains only components with small wavelengths $\ll \rho$:

$$a_r^\pm = \sum_{|\mathbf{k}| > 2\pi a/\rho} a_k^\pm e^{i\mathbf{k}r/a}. \quad (15)$$

After substituting (14) in (1) and taking (12) and (8) into account, we represent the Hamiltonian as a quadratic form of the quantities $\gamma_{r\delta}^\pm$ (see (13)) and $a_{r+\delta}^\pm - a_r^\pm$. The crossing terms, which contain the products of $\gamma_{r\delta}$ and $a_{r+\delta}^\pm - a_r^\pm$ vanish, since the corresponding Fourier components are concentrated in different regions of \mathbf{k} -space. As a result, the Hamiltonian breaks up into a sum of terms corresponding to long- and short-wave degrees of freedom, averaging over which can be carried out independently. The long-wave energy can, in accordance with (13), be written in the form

$$E_{LW}(\mathbf{n}) = \frac{JS^2}{2} \sum_{r\delta} \gamma_{r\delta}^+ \gamma_{r\delta}^- = \frac{JS^2}{2} \iint \frac{d\mathbf{r}}{a^2} \{(\nabla \vartheta)^2 + \sin^2 \vartheta (\nabla \varphi)^2\}. \quad (16)$$

The statistical distribution of the random function (11) is determined by the energy (16) in the form

$$dZ_\rho(\mathbf{n}) = \exp \left[-\frac{E_{LW}(\mathbf{n})}{T} \right] \prod_i (dn_i), \quad (17)$$

where $(dn_i) = \sin \vartheta_i d\vartheta_i d\varphi_i$, and the subscript ρ emphasizes the implicit dependence of the statistical distribution on the cutoff parameter.

Unlike systems with a commutative symmetry group, such as the plane XY model or a Bose liquid (see^[7-9]), the distribution (17) is not Gaussian (owing to the fact that $\sin^2 \vartheta$ in front of $(\nabla \varphi)^2$ in (16)). It is therefore difficult to calculate directly the correlation functions of the quantities \mathbf{n}_r on the basis of (17). We can, however, establish the law whereby these correlation functions become transformed when all the distances are stretched, including the dimensions of the system.

In fact, the functional (16) is invariant against the similarity transformation $\mathbf{r} \rightarrow \lambda \mathbf{r}$, $\mathbf{n}_r \rightarrow \mathbf{n}_{\lambda r}$ (the factor λ^{-2} due to the squares of the gradients is offset by the factor λ^2 from the volume element). From this and from the form of (17) it is easy to deduce that the correlation functions \mathbf{n}_r should not change if the cutoff parameter ρ is increased by the same factor λ by which all the dimensions are homogeneously stretched. For example, the two-point distribution function $P_\rho(\mathbf{n}_1, \mathbf{r}_1; \mathbf{n}_2, \mathbf{r}_2)$ (the probability density for \mathbf{n}_r to assume the values \mathbf{n}_1 and \mathbf{n}_2 at the points \mathbf{r}_1 and \mathbf{r}_2 , respectively) should depend only on the ratios \mathbf{r}_1/ρ , \mathbf{r}_2/ρ , and R/ρ , and analogously for n -point distribution functions. Therefore the law governing the change of these distribution functions after a homogeneous stretching of all the distances (and at fixed ρ) is uniquely connected with the law governing their change after a change of ρ (and at fixed \mathbf{r}_1 and R). The latter law can be established from the requirement that the statistical distribution of the initial quantities S_r be independent of the auxiliary parameter ρ with the aid of which, according to (14), (11), and (15), the subdivision into short- and long-wave correlations was introduced. Therefore the change of the statistical distribution of the quantities (11) as a result of the transition from ρ to ρ' should be offset by a corresponding change in the distribution of the quantities (15), so as to leave the distribution of the physical quantities (14) unchanged.

Indeed, the asymptotic form of the mean values $\langle a_r^\pm a_r^\mp \rangle$, with allowance for the fact that $a \ll \rho \ll R$, can be represented in the form

$$\langle a_r^+ a_r^- \rangle_\rho = \alpha S \ln \frac{\rho}{a(2\pi a)^{1/2}}, \quad \langle a_r^- a_r^+ \rangle = 1 + \alpha S \ln \frac{\rho}{a(2\pi a)^{1/2}}. \quad (18)$$

For the asymptotics of the correlators at remote points ($|\mathbf{r} - \mathbf{r}'| \gg \rho$) we obtain

$$\langle a_r^\pm a_{r'}^\mp \rangle_\rho \rightarrow 0 \quad \text{and} \quad |\mathbf{r} - \mathbf{r}'| \gg \rho. \quad (19)$$

When the cutoff parameter is changed from ρ to ρ' , the correlators (18) change by an amount $\alpha S \ln(\rho'/\rho)$. This change can be represented as a result of a shift of the Bose operators a_r^\pm by classical (commuting) random quantities δa_r^\pm , the mean values of which are equal to zero and whose second moment is equal to

$$\langle \delta a_r^+ \delta a_r^- \rangle = \alpha S \ln(\rho'/\rho).$$

On the other hand, it is easily seen that the result of the transformation $a_r^\pm \rightarrow a_r^\pm + \delta a_r^\pm$ in (14) is offset by the transformation

$$\mathbf{n}_r \rightarrow \mathbf{n}_r + \delta \gamma_r^+ e^-(\mathbf{n}_r) + \delta \gamma_r^- e^+(\mathbf{n}_r), \quad (20)$$

where $\delta \gamma_r^\pm = -\delta a_r^\pm S^{-1/2}$, i.e., $\delta \gamma_r^\pm$ are random quantities

with zero mean values and with second moments in the form

$$\langle \delta Y_r^+ \delta Y_r^- \rangle = \alpha (\ln \rho - \ln \rho'). \quad (21)$$

Formulas (20) and (21) yield the law governing the variation of the random quantities (11) with changing parameter ρ . This law can be compared with the law governing the variation of a random unit vector \mathbf{n}_l that executes Brownian motion over a unit sphere (see, e.g., [14]). The latter law can be represented in the form

$$\mathbf{n}_{t+\Delta t} = \mathbf{n}_t + \delta Y_{\Delta t}^+ \mathbf{e}^-(\mathbf{n}) + \delta Y_{\Delta t}^- \mathbf{e}^+(\mathbf{n}), \quad (22)$$

where δY_t^\pm are random quantities with zero mean values and with a second moment

$$\langle \delta Y_{\Delta t}^+ \delta Y_{\Delta t}^- \rangle = D \Delta t,$$

where D is the diffusion coefficient. Comparing (22) with (20) and (21), we see that the change of the random quantity \mathbf{n}_r with changing ρ can be regarded as Brownian motion over a sphere, where the role of the time is played by the quantity $t = -\ln \rho$, and the role of the diffusion coefficient is played by the parameter α . As a result, the change in the distribution function of the quantity \mathbf{n}_r is described by a Fokker-Planck equation for rotational Brownian motion [14], where the role of the time is played by $t = -\ln \rho$.

For n -point distribution functions pertaining to remote points \mathbf{r}_i we obtain by virtue of (20) a similar equation, in the right-hand side of which is contained the sum of operators

$$-\hat{L}_i^2 = \frac{1}{\sin \theta_i} \frac{\partial}{\partial \theta_i} \left(\sin \theta_i \frac{\partial}{\partial \theta_i} \right) + \frac{1}{\sin^2 \theta_i} \frac{\partial^2}{\partial \varphi_i^2} \quad (23)$$

for all the points \mathbf{r}_i ($i = 1, 2, \dots, n$). Further, since the n -point distribution functions, as indicated, depend only on the ratios \mathbf{r}_i/ρ and R/ρ , the operator $\rho \partial / \partial \rho$ as applied to them can be replaced by the Euler operator

$$\sum_i \mathbf{r}_i \frac{\partial}{\partial \mathbf{r}_i} + R \frac{\partial}{\partial R}.$$

Thus, the equation for the distribution function is

$$\left(R \frac{\partial}{\partial R} + \sum_{i=1}^n \mathbf{r}_i \frac{\partial}{\partial \mathbf{r}_i} \right) P(\dots, \mathbf{n}_i, \mathbf{r}_i, \dots) = -\frac{\alpha}{2} \left(\sum_{i=1}^n \hat{L}_i^2 \right) P(\dots, \mathbf{n}_i, \mathbf{r}_i, \dots). \quad (24)$$

The quantity ρ no longer enters explicitly in this equation. The dependence on ρ should generally drop out in the calculation of the mean values of (14) as a result of averaging over the short-wave operators (15). As seen from (18), the last averaging is equivalent to replacement of ρ by $\rho_0 = a(2\pi\alpha)^{1/2}$. We can therefore omit the operators \mathbf{a}_r^\pm from (14) in the calculation of long-range correlations, assuming that $\rho = \rho_0$, i.e., we can put

$$\hat{S}_{\mathbf{r}_i} \approx S \mathbf{n}_{\mathbf{r}_i} \quad (|\mathbf{r}_i - \mathbf{r}_j| \gg a). \quad (25)$$

Of direct interest are in fact not the many-point distribution functions but the mean values of the products of the quantities (25), pertaining to the remote points \mathbf{r}_i , which are calculated with the aid of these distribution functions. We can write down the law of variation of these mean values under similarity transformations not in the infinitesimal form (as in (24)), but in finite form. To this end it is necessary to consider the mean products of the quantities $Y_{l_i m_i}(\mathbf{n}_i)$, where $\mathbf{n}_i = \mathbf{n}_{\mathbf{r}_i}$ and $Y_{lm}(\mathbf{n}_i) = Y_{lm}(\vartheta_i, \varphi_i)$ are spherical harmonics. Using the fact that $Y_{lm}(\mathbf{n}_i)$ are the eigenfunctions of the operators (23) with eigenvalues $l_i(l_i + 1)$,

we can easily obtain from (24) the law for the transformation of n -point mean values:

$$\langle Y_{l_1 m_1}(\mathbf{n}_{\mathbf{r}_1}) \dots Y_{l_n m_n}(\mathbf{n}_{\mathbf{r}_n}) \rangle_{\lambda R} = |\lambda|^{-\mu} \langle Y_{l_1 m_1}(\mathbf{n}_{\mathbf{r}_1}) \dots Y_{l_n m_n}(\mathbf{n}_{\mathbf{r}_n}) \rangle_R, \quad (26)$$

$$\mu = \frac{\alpha}{2} \sum_{i=1}^n l_i(l_i + 1).$$

We note that we shall henceforth consider only one- and two-point mean values, and only of products in which not more than one component of $S_{\mathbf{r}_i}$ pertains to each point. Therefore in the examples considered below we have $n = 1$ or 2 , and all $l_i = 1$.

We consider, for example, the two-point correlation functions $\langle S_{\mathbf{r}} S_{\mathbf{r}'} \rangle$ at large distances in a system of infinite dimensions ($R = \infty$). If it is recognized that $\langle S_{\mathbf{r}} S_{\mathbf{r}'} \rangle$ should depend only on $r = |\mathbf{r} - \mathbf{r}'|$, then we can obtain from (26)

$$\langle S_{\mathbf{r}} S_{\mathbf{r}'} \rangle \approx S^2 \langle \mathbf{n}_{\mathbf{r}} \mathbf{n}_{\mathbf{r}'} \rangle = C(\alpha) r^{-2\alpha}.$$

The coefficient $C(\alpha)$ is determined from the condition of matching with the asymptotic correlation function for $r \gg a$, calculated in the SWA. It is equal to $S^2(1 - 2\alpha \ln(r/\rho_0))$, and the results of the matching can be written in the form

$$\langle S_{\mathbf{r}} S_{\mathbf{r}'} \rangle \approx S^2 (2\pi\alpha)^2 \left(\frac{a}{r} \right)^{2\alpha}, \quad r = |\mathbf{r} - \mathbf{r}'| \gg a. \quad (27)$$

As a second interesting example, let us find the total moment for a system of finite dimensions $R \gg a$. For the mean value $\langle S_{\mathbf{r}}^2 \rangle$ we obtain from (26) ($f(r/R)$ is a certain function)

$$\langle S_{\mathbf{r}}^2 \rangle_R = S \langle n_{\mathbf{r}}^2 \rangle_R = R^{-\alpha} f(r/R). \quad (27')$$

Integrating this expression and determining the constant in the obtained expression from the condition of matching with the SWA, we obtain for the total moment of the system

$$M(R) = \int \langle S_{\mathbf{r}}^2 \rangle \frac{dr}{a^2} = (2\pi\alpha)^{\alpha/2} \left(\frac{R}{a} \right)^{2-\alpha}. \quad (28)$$

Thus, even an ideal two-dimensional magnet (were it to exist) would have a macroscopic magnetic moment at finite dimensions. This does not contradict the theorem of Mermin and Wagner (see [5]), since the specific moment

$$m(R) = M(R) / R^2 \sim R^{-\alpha}$$

tends to zero as $R \rightarrow \infty$.

Relations (26) enable us also to consider the behavior of the system in weak fields. We shall discuss this in greater detail since, as will be shown below, weak interaction between layers in a layered system can be accounted for in similar fashion.

In a magnetic field there is added to the Hamiltonian (1) a term that assumes, in terms of the formula (14), the form

$$\delta H = \sum_{\mathbf{r}} h S_{\mathbf{r}} \approx \sum_{\mathbf{r}} \{ (S - a_r^+ a_r^-) \mathbf{n}_r + S^{1/2} (a_r^+ \mathbf{e}^-(\mathbf{n}_r) + a_r^- \mathbf{e}^+(\mathbf{n}_r)) \} h, \quad (29)$$

where h is the magnetic field in the corresponding units. It would be a rather complicated matter to take into account in (29) terms containing the Bose operators \mathbf{a}_r^\pm . We, however, have in mind the case of a magnetic field that is weak in comparison with the exchange energy J . In this case, the main term in (29) is the sum

$$S \sum_{\mathbf{r}} \mathbf{n}_r h = S \int \frac{d\mathbf{r}}{a^2} \mathbf{n}_r h, \quad (30)$$

which depends only on the long-wave variables, and

allowance for terms with the operators $a_{\mathbf{r}}^{\pm}$ lead to inessential corrections, the character of which will be indicated below.

It is easy to see the changes resulting from the addition of the term (30) to the energy (16). To maintain invariance it is now necessary to perform, in addition to the transformation $\mathbf{r} \rightarrow \lambda\mathbf{r}$ and $R \rightarrow \lambda R$, also the transformation $|\mathbf{h}| \rightarrow |\mathbf{h}|/\lambda^2$. Accordingly, a term $-2\hbar\partial/\partial\hbar$ is added to the Euler operator in the left-hand side of (24), and the correlation functions acquire an additional dependence on the argument $\hbar r^2/T$, where r is an arbitrary quantity with the dimension of length. Thus, the expressions for the total moment of a system of finite dimensions R and for the pair correlation function as $R \rightarrow \infty$ take the form

$$M(R) = R^{2-\alpha} f_1\left(\frac{\hbar R^2}{T}\right), \quad \langle S_0 S_r \rangle = r^{-2\alpha} f_2\left(\frac{\hbar r^2}{T}\right), \quad (31)$$

where $f_{1,2}$ are certain functions (that depend also on α).

These functions can be determined by matching with the asymptotic forms of the SWA, but we confine ourselves to a determination, at $\hbar \neq 0$, of the specific spontaneous moment, defined as the limit

$$m = \lim_{R \rightarrow \infty} \frac{M(R)}{R^2} \quad \text{or} \quad m^2 = \lim_{|r-r'| \rightarrow \infty} \langle S_r S_{r'} \rangle, \quad (32)$$

It turns out that the very existence of the limits (32), together with the form of the functional dependences (31), makes it possible to determine the dependence of the moment on the field. Indeed, since the transformation $R \rightarrow \lambda R$ with arbitrary $\lambda > 0$ does not change the limit (32), we obtain for the dependence the functional equation

$$m(\hbar) = \lambda^{-\alpha} m(\lambda^2 \hbar),$$

whence

$$m(\hbar) = \hbar^{\alpha/2} C(J, T). \quad (33)$$

The proportionality coefficient $C(J, T)$ is determined by matching with the SWA in the region $\alpha \ln |h/T| \ll 1$. In this region, (33) takes the form

$$m(\hbar) = (C_0 + \alpha C_1 + \dots) \hbar^{\alpha/2} = C_0 (1 + \frac{1}{2} \alpha \ln \hbar) + C_1 + \dots$$

On the other hand, the SWA yields

$$m(\hbar) = S - \frac{T}{2\pi J S} \int_0^{\infty} \frac{k dk}{\exp(k^2 + \hbar S/T) - 1} \approx S \left(1 + \frac{\alpha}{2} \ln \frac{\hbar S}{T} \right).$$

Comparing these expressions, we obtain

$$m(\hbar) = S (\hbar S / T)^{\alpha/2}. \quad (34)$$

We note that expression (34) pertains to an infinite system and corresponds to asymptotics of the functions f_1 and f_2 in (31) for $(\hbar/J)(R/a)^2 \gg 1$. If the dimensions of the system R/a are of the same order as $(\hbar/J)^{1/2}$, then it is necessary to know the functions f_1 and f_2 at all values of the argument. Expressions (31) are obtained by matching with the asymptotic forms of SWA, and take the form

$$\langle S_0 S_r \rangle_{R, \hbar} \approx m^2(\hbar) \exp\{\alpha K_0(r\sqrt{\hbar/J})\}, \\ \langle S_r S_{r'} \rangle_{R, \hbar} \approx m(\hbar) \exp\{\alpha Q_R(r, r'/\sqrt{\hbar/J})\},$$

where $m(\hbar)$ is given by (34), $K_0(x)$ is a Macdonald function, and $Q_R(r, r'/k)$ is the regular part of the Green's function of the Helmholtz operator $\Delta + k_0^2$ for the considered region with dimension R . At $R/a \ll (\hbar/J)^{1/2}$, these expressions go over into (27) and (27').

A more accurate analysis, which takes into account

terms with operators $a_{\mathbf{r}}^{\pm}$ in (29), can be carried out by the methods developed earlier^[7-9] and lead to a replacement of the exact invariance against the transformations $\hbar \rightarrow \hbar/\lambda^2$ by an approximate one relative to $\hbar \rightarrow \hbar/\lambda^{2-\alpha}$. As a result of this refinement, first, the exponent $\alpha/2$ in (34) is replaced by $\alpha/(2-\alpha) = \alpha/2 + O(\alpha)$, and second, the expression (34) itself now turns out to be the principal term of an asymptotic expansion whose succeeding terms are of the order of $O(\hbar^{1+\alpha/2})$. Since we confine ourselves to the low-temperature region $\alpha \ll 1$, these corrections are immaterial.

2. LAYERED PARAMAGNET IN LOW-TEMPERATURE REGION

We consider now a three-dimensional layered system of isotropic spins with predominant interaction J in the plane of the layer and weak interaction J' between layers. The Hamiltonian of such a system, which includes the interaction with the magnetic field, can be written in the form

$$H = \sum_{\mathbf{r}, \delta} \{ J S_{\mathbf{r}\nu} S_{\mathbf{r}+\delta\nu} + J' S_{\mathbf{r}\delta} S_{\mathbf{r}+\delta} \} + \sum_{\mathbf{r}} \hbar S_{\mathbf{r}\nu}, \quad (35)$$

where the previous symbols \mathbf{r} and δ are retained to indicate the positions of the sites in the plane, while the integer index ν numbers different planes of the lattice. The interactions J and J' are assumed to be ferromagnetic ($J > 0, J' > 0$).

The spontaneous moments of a system with the Hamiltonian (35) is obviously oriented in arbitrary fashion relative to the plane of the layers. In a real system, of course, there is always anisotropy. For example, in the experiments of^[1-4] the spontaneous moment, as a rule, was oriented in the plane of the layer. In our case, however, this circumstance is not of principal significance, and to avoid cumbersome expressions we did not include the anisotropy in the Hamiltonian (35). The anisotropy can be accounted for in a manner perfectly equivalent to the allowance for the magnetic field, and the corresponding formulas can be easily written out.

We consider the thermodynamics of such a system under the conditions

$$T \ll J, \quad J' \ll J \quad (36)$$

and for an arbitrary ratio of the quantities T and J' . Obviously, the behavior of the system will be different, depending on the ratio J'/T . Indeed by virtue of the condition $T \ll J$, a strong correlation takes place, in the entire considered region, between the directions of the spins in one plane. If $T \ll J'$, the interaction J' turns out to be sufficiently strong to establish a correlation also between spins pertaining to different planes, so that the spins in the entire system will oscillate relative to the general direction of the spontaneous moment with small amplitudes. In other words, the three-dimensional situation is realized at $T \ll J'$. To the contrary, in the case $T \gg J'$, the spin directions in the neighboring planes are almost uncorrelated with one another, and the system should behave like a set of independent two-dimensional subsystems. Thus, with increasing ratio T/J' , the character of the thermodynamic behavior of the system should change from three-dimensional to two-dimensional.

It follows from the foregoing that the usual three-dimensional SWA holds when $T \ll J'$. It turns out, however, that the SWA holds in a larger temperature

region, which includes both the region $T \sim J'$ and the region where

$$T \gg J', \text{ but } \alpha \ln(J'S/T) \ll 1. \quad (37)$$

In fact, let us consider the region of applicability of the SWA, using the criterion (5). The expression for the specific moment of the considered system in the region of applicability of the SWA takes the form

$$m = S - \int \frac{dk}{(2\pi)^2} \int \frac{d\kappa}{2\pi} \left[\exp \left\{ \frac{1}{T} (J\Delta(\mathbf{k}) + J'\Delta(\kappa) + h)S \right\} - 1 \right]^{-1}. \quad (38)$$

Here \mathbf{k} is the projection of the dimensionless wave vector on the plane of the layer, κ is the wave-vector component in the perpendicular direction, the quantity $\Delta(\mathbf{k})$ is given by expression (4), and $\Delta(\kappa) = 2(1 - \cos \kappa)$. The integration in (38) is over the reciprocal-lattice cell.

Expression (38) can be simplified in the usual fashion, by using the smallness of the ratio T/J (we make a change of variables $\mathbf{k} = (T/JS)^{1/2} \mathbf{k}'$, then $\Delta(\mathbf{k}') \approx \mathbf{k}'^2$ and the limits of integration with respect to \mathbf{k}' extend to infinity). No such simplification can be made when it comes to integration with respect to κ , since the parameter is assumed to be arbitrary. After integrating with respect to \mathbf{k}' we obtain

$$m = S \left(1 + \frac{\alpha}{2} \int_{-\pi}^{\pi} \frac{d\kappa}{2\pi} \ln \left(1 - \exp \left[-\frac{J'S}{T} \Delta(\kappa) - \frac{hS}{T} \right] \right) \right) \quad (39)$$

(we recall that $\alpha = T/2\pi JS^2$).

The integral in (39) is the fluctuation contribution to the spontaneous moment. This integral diverges as $J' \rightarrow 0$ and $h \rightarrow 0$. The asymptotic form of (39) at $J'/T \ll 1$ and $h/T \ll 1$ is

$$\frac{m}{S} \approx 1 + \frac{\alpha}{2} \ln \left\{ \frac{S}{T} \left[J' + \frac{h}{2} + \left(\left(J' + \frac{h}{2} \right)^2 - J'^2 \right)^{1/2} \right] \right\}. \quad (40)$$

According to (5), the condition for the validity of the SWA is smallness of the fluctuation increment of the moment, which leads at $h = 0$ to the condition (37).

On the other hand, the specific moment at $J' \ll J$ and $h \ll J$ can be determined from renormalization-invariance considerations similar to those used in the preceding section for the two-dimensional system. Namely, at $J' \ll J$ the interaction J' can be taken into account in analogy with the allowance for the interaction with the weak magnetic field at the end of Sec. 1. In view of this analogy, we shall not repeat the corresponding arguments in detail, and present only their brief scheme and the results.

The layered three-dimensional system with Hamiltonian (35) is analyzed in the following manner. We separate the long-wave degrees of freedom from the short-wave ones by introducing a cutoff parameter ρ and expressing the operators $S_{\mathbf{r}\nu}$ in terms of the random function $\mathbf{n}_{\mathbf{r}\nu}$ and the operators $a_{\mathbf{r}\nu}^{\pm}$ by means of formulas analogous to (14), (11), and (15). The statistical distribution of $\mathbf{n}_{\mathbf{r}\nu}$, in analogy with (17), is determined by the energy of the long-wave fluctuations, which is equal to

$$E_{LW}(n) = \sum_{\nu} \int \frac{d\mathbf{r}}{a^2} (JS^2 [(\nabla_{\mathbf{r}}\phi(\mathbf{r}, \nu))^2 + \sin^2 \phi(\mathbf{r}, \nu) (\nabla_{\mathbf{r}}\phi(\mathbf{r}, \nu))^2] + J'S^2 \mathbf{n}_{\mathbf{r}\nu} \cdot \mathbf{n}_{\mathbf{r}\nu+1} + S h n_{\mathbf{r}\nu}), \quad (41)$$

where, in accord with the statements made in the end of the preceding section, we have discarded terms containing the products of J' and h by the operators $a_{\mathbf{r}\nu}^{\pm}$.

The statistical distribution determined by the energy (41) is invariant against the transformations $\mathbf{r} \rightarrow \lambda \mathbf{r}$, $R \rightarrow \lambda R$, $\rho \rightarrow \lambda \rho J'$, $J' \lambda^{-2}$, and $h \rightarrow h \lambda^{-2}$. On the other hand, the dependence of the statistical distribution on the parameter ρ satisfies as before the Fokker-Planck equation with respect to $t = -\ln \rho$. By virtue of this, we obtain for the n -point distribution functions of the quantities $\mathbf{n}_i = \mathbf{n}(\mathbf{r}_i, \nu_i)$ equations of the type (24), in the right-hand side of which there are added to the Euler operator the terms

$$-\left(2J' \frac{\partial}{\partial J'} + 2h \frac{\partial}{\partial h} \right).$$

For the mean values of the products of the quantities $Y_{lm}(\mathbf{n}_i)$ we obtain the transformation law (26), to the left-hand side of which it is necessary to add besides the transformation

$$\mathbf{n}_{\mathbf{r}_i, \nu_i} \rightarrow \mathbf{n}_{\lambda \mathbf{r}_i, \nu_i}, \quad R \rightarrow \lambda R$$

also $J' \rightarrow J' \lambda^{-2}$ and $h \rightarrow h \lambda^{-2}$.

From this we obtain for the per-unit moment

$$m_{r,\nu} = \langle S_{r,\nu}^z \rangle_R \approx S \langle n_{r,\nu}^z \rangle = R^{-\alpha} f(\alpha, r/R, hR^2, J'R^2). \quad (42)$$

In the limit as $R \rightarrow \infty$ we obtain in analogy with (33) the representation

$$m = \lim_{R \rightarrow \infty} \langle S_{r,\nu}^z \rangle_R = (J')^{\alpha/2} C(\alpha, T, J'/h),$$

where $C(\alpha, T, J'/h)$ is determined (accurate to first order in α) by matching with the SWA result (40).

Ultimately we get for J' , $h \ll J$:

$$m(h, J') = S \left(\frac{S}{T} \right)^{\alpha/2} \left[J' + \frac{h}{2} + \left(\left(J' + \frac{h}{2} \right)^2 - J'^2 \right)^{1/2} \right]^{\alpha/2}. \quad (43)$$

At $J' = 0$ this expression goes over into (34), and $h = 0$ it determines the specific spontaneous moment of an almost-two-dimensional magnet:

$$m(J') = S \left(1 - \frac{\alpha}{2} \ln \frac{T}{S} \right) (J')^{\alpha/2} \approx S \left(\frac{SJ'}{T} \right)^{\alpha/2}. \quad (44)$$

We calculate now the specific heat of an almost-two-dimensional magnet. Since the average energy E and accordingly the specific heat $c = dE/dt$ are determined by short-range correlations⁵⁾, the calculation of the average energy by the SWA formulas yields the correct result (this is valid in general for all mean values that depend only on the short-range correlations). Thus, taking into account the smallness of the ratio J'/J , we can represent the average energy, in analogy with (38), in the form

$$E = \frac{T^2}{4\pi JS} \int_{-\pi}^{\pi} \frac{d\kappa}{2\pi} \int_{y_0}^{\infty} \frac{y dy}{e^y - 1}, \quad y_0 = \frac{J'S}{T} \Delta(\kappa). \quad (45)$$

In the region $T \ll J'$, an important role is played in the integral of (45) by small κ , $\Delta(\kappa) \approx \kappa^2$, and (45) takes the asymptotic form

$$E = \frac{\zeta(3/2)}{8\pi^{3/2}} \frac{T^{3/2}}{JS(J'S)^{1/2}} \quad (46)$$

where $\zeta(x)$ is the Zeta function. The specific heat in this region behaves like $T^{3/2}$, as is typical of a three-dimensional magnet.

In the region $T \gg J'$, the lower limit y_0 in (45) can be set equal to zero, and the energy turns out to be

$$E = \pi T^2 / 24JS. \quad (47)$$

The specific heat in the region $J' \ll T \ll J$ varies linearly with the temperature, i.e., it has potentially a two-dimensional character.

3. LAYERED SYSTEMS WITH ANTIFERROMAGNETIC INTERACTIONS

In conclusion, we present a brief summary of the formulas for the thermodynamic characteristics of layered systems in the presence of antiferromagnetic interaction inside the layer or between layers. The formulas that follows do not include the anisotropy energy, since we are dealing with the case of temperatures $T > E_{an}$. In addition, for the sake of concreteness we have written out the formulas only for the case of nearest-neighbor interaction both within the layer and between layers.

The calculations for the cases $J < 0$ or $J' < 0$ are similar to those given above and will be omitted. Account must be taken, however, of the additional circumstance connected with the fact that zero-point spin oscillations in the ground state exist in the antiferromagnetic case. This calls for a certain modification of the reasoning advanced above (we recall that the problem of determining the ground-state wave function in an antiferromagnet has not yet been solved). In fact, however, all our arguments are based only on the form of the energy of the longwave fluctuations, which is similar to (16). (The vector \mathbf{n}_r denotes the direction of the spins in one of the sublattices, and then the spin direction in the other sublattice is $-\mathbf{n}_r$.) The only difference is that the exchange energy J is replaced by the effective energy $J_{eff} = \gamma J$, where $\gamma = 1 - O(zS^{-1})$ (z is the number of nearest neighbors) is a certain factor determined by the (unknown) wave function of the ground state; the same factor expresses the decrease of the magnetization of the sublattices and of the susceptibility at $T = 0$, which is connected with the zero-point oscillations. In accordance with the foregoing, in the case of antiferromagnetic interaction, the exponent α is replaced by

$$\alpha' = \frac{\alpha}{\gamma} = \frac{T}{2\pi\gamma JS^2}.$$

We present now formulas for different cases.

A. Antiferromagnetic interaction with a layer ($J < 0$)

In a weak magnetic field, the distribution of the magnetization in directions perpendicular and parallel to the magnetic field are expressed by the formula

$$\langle S_{r,\pm}^{\perp} \rangle = Sm'(\pm 1)^{\nu} \cos \frac{\mathbf{Qr}}{a}, \quad \langle S_{r,\pm}^{\parallel} \rangle \cong \chi h, \quad (48)$$

where the upper sign pertains to the case $J' > 0$ and the lower to $J' < 0$ (and similarly for formula (49) below); $\mathbf{Q} = \{1, 1\}$ is the wave vector of the ferromagnetic structure in the plane of the layer ($\cos \mathbf{Q} \cdot \mathbf{r}/a = \pm 1$, depending on the sublattice), m' is the sublattice magnetization, and χh is the induced moment (see formula (51) below).

The magnon spectrum near the minima at $\mathbf{k} = 0$ and at $\mathbf{k} = \pm \mathbf{Q}$ is given by

$$E_{\mathbf{k}\alpha}^{(1)} = [2z(JSk)^2 + 4z|J'| |J|S^2(1 \mp \cos \kappa) + h^2]^{\frac{1}{2}}, \quad (49)$$

$$E_{\mathbf{k}\alpha}^{(2)} = [1 - (h/2z|J|S)^2]^{\frac{1}{2}} [2z(JSk)^2 + 4z|J'| |J|S^2(1 - \cos \kappa)]^{\frac{1}{2}}$$

(the numbers of neighbors for a quadratic lattice is $z = 4$).

The spontaneous magnetization of the sublattices in the "two-dimensional" region $T \gg |J'|$ is (cf. (43))

$$m' = \gamma S \left(\frac{S|J'|}{T} \right)^{\alpha'/2}. \quad (50)$$

The susceptibility $\chi = \partial M / \partial h$ in a constant field at an arbitrary ratio $|J'|/T$ in the region (37) where the SWA is applicable is

$$\chi = \frac{\gamma}{2z|J|} \left\{ 1 + \frac{T}{2\pi|J|S} \int_{-\pi}^{\pi} \frac{d\kappa}{2\pi} \ln \left[1 - \exp \left(-\frac{S}{T} [2|J'| |J| (1 \mp \cos \kappa)]^{\frac{1}{2}} \right) \right] \right\}. \quad (51)$$

In particular, at $T \ll |J'|$ (three-dimensional situation)

$$\chi = \frac{\gamma}{2z|J|} \left(1 - \frac{\pi}{6} \frac{T^2}{S^2|J|^{\frac{1}{2}}|J'|^{\frac{1}{2}}} \right). \quad (52)$$

At $T \gg |J'|$ we obtain by matching with the asymptotic form of (50)

$$\chi = \frac{\gamma}{2z|J|} \left(\frac{|J'| |J| S^2}{T^2} \right)^{\alpha}. \quad (53)$$

The average energy is given by

$$E = \frac{2T^3}{2\pi z J^2 S^2} \int_{-\pi}^{\pi} \frac{d\kappa}{2\pi} \int_{\Phi(\kappa)}^{\infty} \frac{x^2 dx}{e^x - 1}, \quad (54)$$

$$\Phi(\kappa) = S[2z|J'| |J| (1 - \cos \kappa)]^{\frac{1}{2}} / T.$$

The asymptotic forms of (54) are

$$E = \begin{cases} \frac{\pi^2}{15} \frac{T^4}{2zJ^2 (2z|J| |J'|)^{\frac{1}{2}} S^3} & \text{if } T \ll (2z|JJ'|)^{\frac{1}{2}} S \\ & \text{(three-dimensional situation)} \\ \frac{4T^3}{2zJ^2 S^2} & \text{if } T \gg (2z|JJ'|)^{\frac{1}{2}} S \\ & \text{(two-dimensional situation)} \end{cases} \quad (55)$$

With increasing $T/|J'|$, the behavior of the specific heat changes from the usual $\sim T^3$ dependence of the three-dimensional antiferromagnet to the $\sim T^2$ dependence of the two-dimensional case.

B. Ferromagnetic interaction within the layer ($J > 0$)

Inasmuch as a layered ferromagnet ($J' > 0$) was considered at the end of the preceding section, we present here formulas for the case $J' < 0$. The distribution of the magnetic moment in a weak external field $h \ll 2z|J'|$ is given by

$$\langle S_{r,\pm}^{\perp} \rangle = Sm'(-1)^{\nu}, \quad \langle S_{r,\pm}^{\parallel} \rangle = S\chi h. \quad (56)$$

At $h > 2z|J'|$ the system becomes ferromagnetic.

The energy spectrum near $\mathbf{k} = 0$ is determined by the expression

$$E_{\mathbf{k}\alpha} = [JSk^2 + 2|J'|S(1 - \cos \kappa)]^{\frac{1}{2}} \left[JSk^2 + 2|J'|S(1 + \cos \kappa) - \frac{h^2}{4|J'|} \cos \kappa \right]^{\frac{1}{2}}. \quad (57)$$

The spontaneous magnetization of the layers m' in the "two-dimensional" region $T \gg |J'|$ is given by the same formula (49), where $\gamma = 1$. The susceptibility is expressed by the integral

$$\chi = \frac{1}{4|J'|} \left\{ 1 + \frac{T}{2\pi J} \int_{-\pi}^{\pi} \frac{d\kappa}{2\pi} \cos \kappa \int_0^{\infty} \frac{k dk}{\exp(\Delta_{\pm}) - 1} \left(\frac{\Delta_{\pm}}{\Delta_{\mp}} \right)^{\frac{1}{2}} \right\}, \quad (58)$$

$$\Delta_{\pm} = k^2 + \frac{2|J'|S}{T} (1 \pm \cos \kappa).$$

At $|J'| \gg T$ (three-dimensional situation) we obtain

$$\chi = \frac{1}{4z|J'|S} \left(1 - \frac{1}{96\pi} \left(\frac{T}{J'S} \right)^2 \right), \quad (59)$$

and at $|J'| \ll T$ (two-dimensional situation)

$$\chi = \frac{1}{4z|J'|S} \left(1 + \frac{T}{4\pi|J'|S} \right). \quad (60)$$

The average energy is given by

$$E = \frac{T^2}{2\pi JS} \int_{-\pi}^{\pi} \frac{d\kappa}{2\pi} \int_0^{\infty} k dk \frac{(\Delta_+ \Delta_-)^{1/2}}{\exp(\Delta_+ \Delta_-)^{1/2} - 1}. \quad (61)$$

In the three-dimensional situation ($T \ll |J'|$)

$$E = \frac{1}{120} \frac{\pi^2 T^4}{JJ'^2 S^3} \quad (62)$$

and in the two-dimensional ($T \gg |J'|$) the energy obviously coincides with expression (47).

¹Similar results for a Bose liquid were obtained earlier by Kane and Kadanoff [¹⁰] and by Lasher [¹¹], and also by Wegner [¹²] for a planar magnet (XY model).

²As $J'/J \rightarrow 0$, the Curie point of a layered magnet tends to a phase-transition point in a two-dimensional magnet, the existence of which follows from the results of the earlier papers [⁷⁻⁹]. The temperature of the latter is of the order of J , so that under conditions (A) we are always below the Curie point of the layered magnet, regardless of the ratio J'/T .

³We note the obvious analogy with the thermal properties, considered by I. M. Lifshitz, of nonmagnetic layered structures [¹³].

⁴More accurately, for spins located far from the boundaries (at distances $\gg a$), the specification of arbitrary boundary conditions reduces effectively to specification of a certain probability distribution $P(n)$ for the directions n of the moment, as can be seen from formulas (A.1)-(A.6) of [⁷].

⁵The average energy is the mean value of the Hamiltonian (35), which depends only on the correlations of the neighboring spins. The direction n drops out in general from this mean value, as becomes obvious if the products of the neighboring spins is rewritten in the form

$$S_i S_{i+\delta} = S(S+1) - 1/2(S_{i+\delta} - S_i)^2.$$

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