Expansion of the equation of state in the $(4-\epsilon)$-dimensional Heisenberg model

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By a method analogous to that of Wilson, the first three terms of the series expansion of the equation of state in powers of $\epsilon = 4 - d$, where d is the dimensionality of space, are found for the Heisenberg model with an n-component spin. It is found that the convergence of the series in ϵ for the universal function f(x) is somewhat worse than for the critical indices. In the weak-field region and below the transition temperature, the ϵ-expansion becomes meaningless because of the existence of spin waves in an ideal ferromagnet. The possibility of studying the behavior of a Bose gas on the basis of the equation of state obtained is discussed.

1. INTRODUCTION

There is no doubt at the present time of the correctness of the phenomenological theory of second-order phase transitions, based on the hypothesis of scaling of the correlations in the critical region. The two most important results of this theory are that 1) in the critical region, the behavior of all physical quantities A is described by simple power laws: $A \propto |T - T_c|^{\lambda}$, and 2) the indices λ for corresponding quantities are the same or almost the same in widely different transitions.

The description of a transition becomes complete if we can find the two constants in terms of which all the critical indices are expressed. The answer to this question, however, should be given by a microscopic theory of the phenomenon. The study of phase transitions by means of the general methods of quantum field theory (unitarity conditions, Ward identities, etc.) has made it possible to justify and generalize the scaling hypotheses. But the calculation of the critical indices, like the construction of a quantitative theory, has encountered considerable mathematical difficulties, associated with the absence of a small parameter in the problem and with the associated impossibility of using perturbation theory.

The study of four-dimensional models has shown that, in the critical region, the deviations from the Landau theory tend to zero. In $(4-\epsilon)$-dimensional space, the deviations are of order ϵ, and in this connection Wilson and Fisher[1] have put forward the idea of using the dimensionality of space as a small parameter. Wilson[2] has found the first two terms of the series expansion in ϵ for the critical indices. It was found that, even with $\epsilon = 1$, the first terms of the series give values close to the experimental values for the indices. The idea arises, therefore, of performing an ϵ-expansion for other universal quantities too, and in particular for the equation of state. The first three terms of the ϵ-expansion of the equation of state in the Ising model were found by Migdal and the author[3]. It was found that the equation obtained coincides with the phenomenological equation of Migdal's paper[4]. In the present work, the equation of state is found to second order in ϵ for the Heisenberg model with n-component spin (n = 1 corresponds to the Ising model, n = 2 to the Bose gas, and n = 3 to the Heisenberg model).

2. WILSON'S METHOD

We shall discuss briefly the method of calculation (Wilson[2]). The Hamiltonian of the problem considered has the form

$$H = \int \left\{ \frac{1}{2} r_0 s^2(\mathbf{x}) + \frac{1}{2} [\nabla s(\mathbf{x}) - \nabla \nabla^2 s(\mathbf{x})]^2 + u_0 s^4(\mathbf{x}) \right\} d^d x, \qquad (1)$$

where d is the dimensionality of space and u_0 is the interaction constant of the pair forces. The term $\nabla\nabla^2 s(\mathbf{x})$ ensures the convergence of the integrals at large momenta; this is equivalent to the invariant introduction of a cutoff length into the theory. The constant r_0 is related to the reduced temperature $\tau = (T - T_c)/T_c$: $\tau = r_0 - r_{0c}$, where r_{0c} determines the transition temperature.

The correlation functions to be calculated in the theory are obtained in the form of series in u_0 and ϵ. The integrals encountered in the calculation are generalized to the case of noninteger dimensionality of space in the following way:

$$(2\pi)^{-d} \int d^d k\, f(k^2) = K_d \int_0^\infty k^{d-1} f(k^2)\, dk,$$
$$(2\pi)^{-d} \int d^d k\, f(k^2, \mathbf{k}\mathbf{k}_1) = \frac{1}{2\pi} K_{d-1} \int_0^\infty dk \int_0^\pi k^{d-1} (\sin\theta)^{d-2} f(k^2, kk_1\cos\theta)\, d\theta, \qquad (2)$$

where

$$K_d = 2^{-(d-1)} \pi^{-d/2} [\Gamma(d/2)]^{-1}.$$

In the critical region, the Green function

$$g(k) \propto k^{-2+\eta}.$$

The scaling laws predict a power behavior of the amplitude u_R in the critical region:

$$\bar{u}_R \propto \tau^{\gamma(\epsilon - 2\eta)/(2-\eta)},$$

where the indices γ and η depend on ϵ but do not depend on the bare coupling constant u_0[5,6]. A diagram technique gives the possibility of representing u_R and the Green functions in the form of infinite series in u_0; it is necessary to sum the whole series in order to obtain the critical behavior of these functions, which does not depend on u_0. However, by means of the equations of the renormalization group[7], it can be shown[2] that there exists a unique choice of the coupling constant $u_0(\epsilon)$ for which the expected critical behavior is obtained in the first order of perturbation theory in u_0.

By matching the asymptotic behavior of the amplitude u_R with the perturbation-theory series in u_0, and also taking into account that

$$g(k = 0, \tau) \propto \tau^{-\gamma}, \quad g(k, \tau = 0) \propto k^{-2+\eta},$$

Wilson[2] obtained expressions for the critical indices γ and η and the coupling constant $u_0(\epsilon)$:

$$\gamma = 1 + \frac{(n+2)}{2(n+8)}\varepsilon + \frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\varepsilon^2 + O(\varepsilon^3), \quad (3)$$

$$\eta = \frac{(n+2)}{2(n+8)^2}\varepsilon^2 + \frac{(n+2)}{2(n+8)^2}\left[\frac{6(3n+14)}{(n+8)^2} - \frac{1}{4}\right]\varepsilon^3 + O(\varepsilon^4), \quad (4)$$

$$u_0(\varepsilon) = \frac{2\pi^2}{(n+8)}\varepsilon + \frac{2\pi^2}{(n+8)}\varepsilon^2\left[\frac{9n+42}{(n+8)^2} - \frac{11}{12} + K'\right] + O(\varepsilon^3), \quad (5)$$

where K′ is defined by the relation

$$K_{4-\varepsilon} = K_4(1 - \varepsilon K') + O(\varepsilon^2).$$

3. EQUATION OF STATE IN THE MICROSCOPIC THEORY

The relation under consideration between the magnetic field, magnetic moment and temperature can be obtained by means of the methods of quantum field theory. In the presence of a magnetic field, the free energy of the system can be represented in the form (cf., e.g., [5])

$$F(\tau, H) = F_0(\tau) - \sum_{n=1}^{\infty} \frac{Q_{2n} H^{2n}}{(2n)!}, \quad (6)$$

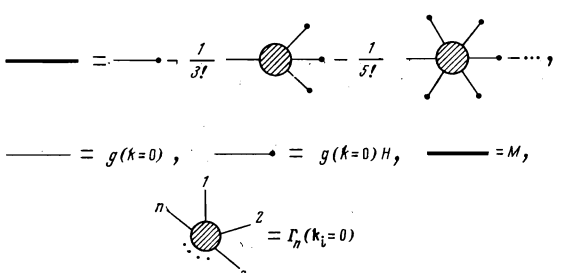
where H is the magnetic field and the expansion coefficients Q_{2n} are defined in the microscopic theory in terms of the vertex parts Γ_{2n} and the Green function:

$$Q_{2n} = \Gamma_{2n}(\mathbf{k}_i = 0) g^{2n}(k = 0) \quad (7)$$

($\Gamma_2 = -g^{-1}$). Then the magnetic moment of the system is

$$M = \frac{\partial F}{\partial H} = -\sum_{n=1}^{\infty} \frac{\Gamma_{2n} g^{2n} H^{2n-1}}{(2n-1)!}. \quad (8)$$

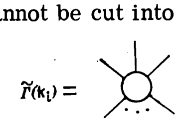
This expression can be written by means of graphs:



(9)

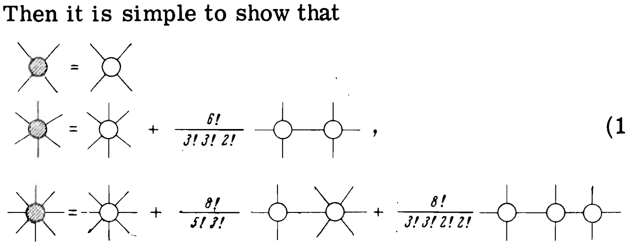
The equation of state determines the magnetic field as a function of the magnetic moment and temperature. Our problem consists, therefore, in solving Eq. (8) for H. This is done conveniently by means of a diagram technique[8], using the graphical representation (9).

We shall introduce the so-called "irreducible vertices" $\tilde{\Gamma}$-vertices which cannot be cut into two through one line:



(10)

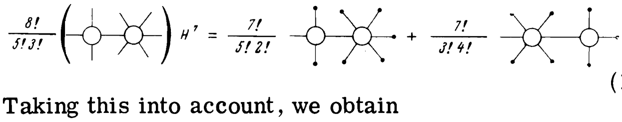
Then it is simple to show that



(11)

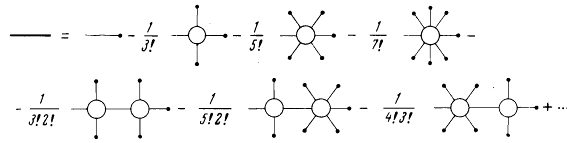
etc.

We shall express the vertices Γ_n in (9) in terms of the irreducible vertices, using the relations (11) and analogous relations. Then graphical multiplication by H leads to diagrams of different types. For example,



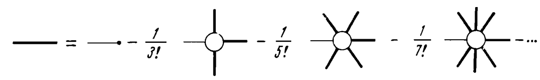
(12)

Taking this into account, we obtain



(13)

By direct inspection it is not difficult to see that the series (13) can be resumed as follows:



(14)

or, in analytical form,

$$M = gH - \sum_{n=2}^{\infty} \frac{g\Gamma_{2n} M^{2n-1}}{(2n-1)!}. \quad (15)$$

Hence, we obtain the required relation

$$H = \sum_{n=1} \frac{\Gamma_{2n} M^{2n-1}}{(2n-1)!} \quad (16)$$

($\tilde{\Gamma}_2 = g^{-1}$). Everywhere below, we again return to the old notation for the vertices—Γ_{2n}, but it must be remembered that they differ from the ordinary vertices.

The phenomenological theory predicts and microscopic theory justifies the power behavior of the vertex parts in the critical region:

$$\Gamma_{2n} = \gamma_{2n}\tau^{[\gamma n(-\eta+2-d)+d\gamma]/(2-\eta)}; \quad (17)$$

γ_{2n} are certain constants. Introducing the index β:

$$2\beta/\gamma = (2 - \eta - d)/(\eta - 2),$$

we obtain after simple transformations

$$H = M^{(\beta+\gamma)/\beta}\mathscr{F}(\tau M^{-1/\beta}),$$

$$\mathscr{F}(x) = x^{\gamma+2\beta}\sum_{n=1}^{\infty}\frac{\gamma_{2n}(x)^{-2n\beta}}{(2n-1)!}. \quad (18)$$

Equation (18) does not fix the units of measurement of the quantities occurring in it, and so the function $\mathscr{F}(x)$ is not universal. In zero magnetic field, the equation $\mathscr{F}(\tau M^{-1/\beta}) = 0$ has the solution $M_s = B|\tau|^\beta$, $\tau < 0$, where B is a non-universal constant. If we measure the magnetic moment in units of B, the function $\mathscr{F}$ will depend on the dimensionless argument $\tau(M/B)^{-1/\beta}$, and $\mathscr{F}(-1) = 0$. In this case,

$$\mathscr{F}(0) = D^{-(\beta+\gamma)/\beta},$$

where D defines the strong-field behavior of the moment:

$$M = DH^{\beta/(\beta+\gamma)}.$$

By an appropriate choice of the units of measurement of the magnetic field (normalization of the function $\mathscr{F}$), we can bring the equation of state to a form in which the universal function f(x) appears:

$$H = \left(\frac{M}{D}\right)^{(\beta+\gamma)/\beta} f\left(\tau\left(\frac{M}{B}\right)^{-1/\beta}\right) \quad (19)$$

($f(-1) = 0$, $f(0) = 1$). In fact, the renormalization-group transformation

$$M' = z^{1/2}M \quad (20)$$

does not change the form of the function f, since, simultaneously with (20),

$$B' = z^{1/2}B, \quad D' = z^{1/2}D.$$

In the model under consideration, the Green function $g_{\alpha\beta}(k, \tau, M^2)$ is a symmetric second-rank tensor in the n-dimensional "spin" space. In the presence of a magnetic field, and also for $\tau < 0$, it makes sense to introduce the longitudinal, $g_{\parallel}$, and transverse, $g_{\perp}$, Green functions. Then,

$$H / M = g_{\perp}^{-1}(k=0, \tau, M^2), \qquad (21)$$

$$g_{\perp}^{-1}(\tau, M^2) = g_{\perp}^{-1}(\tau, 0) - \sum_{n=1}^{\infty} \frac{\Gamma_{2n+2}(\tau) M^{2n}}{(2n)!}. \qquad (22)$$

In fact, the equation of state can be written in the form

$$H_i = M_i \varphi(M^2, \tau).$$

Introducing the inverse Green function $g^{-1}_{\alpha\beta} = \partial H_\alpha / \partial M_\beta$, we see that

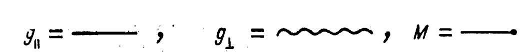
$$g_{\alpha\beta}^{-1} = \delta_{\alpha\beta}\varphi(M^2, \tau) + 2M_\alpha M_\beta \varphi'_{M^2}(M^2, \tau),$$

from which (21) follows.

4. ε-EXPANSION OF THE EQUATION OF STATE

The problem solved in this paper is the determination of the first three terms of the expansion of the function f(x) in a series in ε; the microscopic approach, according to (21), consists in calculating the inverse transverse Green function.

For the following, it is convenient to introduce the notation


(23)

The bare Green functions in zero magnetic field and for $\tau > 0$ will be

$$g_{\parallel}^{(0)}(k, \tau) = g_{\perp}^{(0)}(k, \tau) = \frac{1}{\tau + k^2(1+k^2)^2}. \qquad (23a)$$

For the correct choice of starting approximation, we shall consider the first terms of the series (21) in the coupling constant u_0:


(24)

The expression corresponding to these is (cf. (A.3))

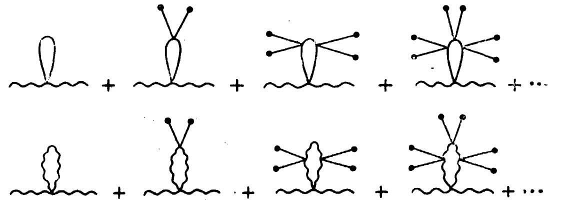
$$g_{\perp}^{-1} = \tau + 4M^2 u_0 + 2u_0 K_4 (n+2)\,[\,(\tau \ln \tau + {}^{11}/_{6}\tau)(1 - K'\varepsilon) - {}^{1}/_{4}\varepsilon\tau \ln^2 \tau + {}^{1}/_{2}\varepsilon\tau] + \ldots \qquad (24a)$$

The spontaneous moment is determined by the equation

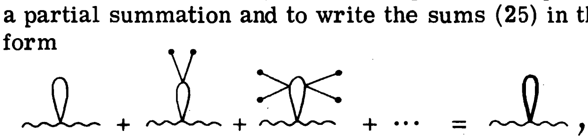
$$g_{\perp}^{-1}(\tau, M_s^2) = 0,$$

$$M_s^2 = -\tau / 4u_0 - {}^{1}/_{2}K_4(n+2)\,[\,(\tau \ln \tau + {}^{11}/_{6}\tau)(1 - K'\varepsilon) - {}^{1}/_{4}\varepsilon\tau \ln^2 \tau + {}^{1}/_{2}\varepsilon\tau] - \ldots,$$

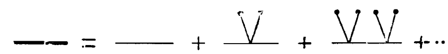
whence it follows that $M^2 \sim M_S^2 \sim 1/u_0 \sim 1/\epsilon$. Therefore, the terms of the series that are proportional to $u_0 M^2$ (or ϵM^2) must be regarded as terms of zeroth order in ε. For the same reason, the diagrams


(25)

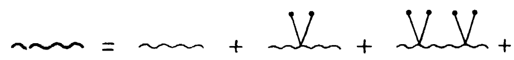
The diagram technique makes it possible to perform a partial summation and to write the sums (25) in the form


(26)

respectively, where



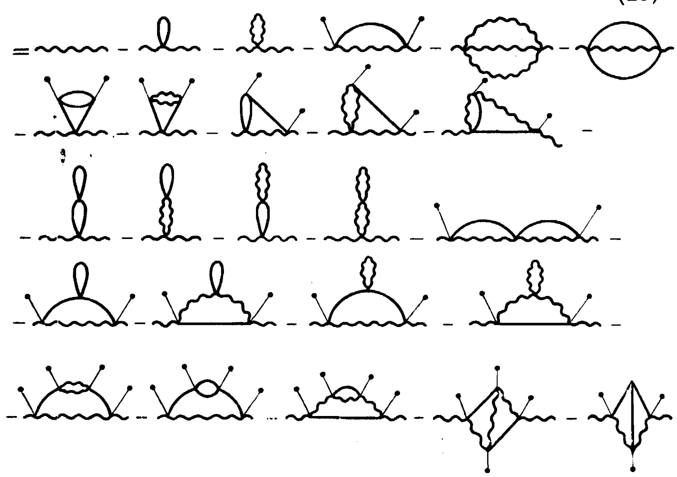
$$= \frac{1}{\tau + 3M^2 + k^2(1+k^2)^2} \qquad (27)$$



$$= \frac{1}{\tau + M^2 + k^2(1+k^2)^2} \qquad (28)$$

(we have introduced the new variable M^2 in place of $4M^2 u_0$). Such a summation must be performed in all the internal lines, after which it is necessary to take into account the following diagrams:

$$g_{\perp}^{-1}(k=0, \tau, M^2) \qquad (29)$$



The internal solid and wavy lines in these diagrams correspond to the Green functions (27) and (28) respectively.

The calculation of the graphs is discussed in more detail in the Appendix; here we give the result for the non-universal function $\mathscr{F}(x)$ (one must put $M^2 = 1$ in (29)):

$$\mathscr{F}(x) = 1 + x + \varepsilon\mathscr{F}_1(x) + \varepsilon^2\mathscr{F}_2(x) = (1+x) + \frac{\varepsilon}{2(n+8)}\Big[3(x+3)\ln(x+3) + (n-1)(x+1)\ln(x+1) + \frac{11}{6}(n+2)x + \frac{11}{6}(n+8)\Big] + \frac{\varepsilon^2}{2(n+8)^2}\Big\{\Big(\frac{10-n}{4}x + \frac{3n+78}{4}\Big) \times \ln^2(x+3) + \Big(\frac{n^2-3n+2}{4}x + \frac{n^2+n-2}{4}\Big)\ln^2(x+1) + [2(n-1)x + 3(n-1)]\ln(x+1)\ln(x+3) + \Big[\frac{9n^2+186n+552}{4(n+8)}x + \frac{27n^2+756n+3240}{4(n+8)}\Big]\ln(x+3) + \Big[\frac{17n^3+249n^2+414n-680}{12(n+8)}x + \frac{17n^3+315n^2+876n-1208}{12(n+8]}\Big]\ln(x+1) + (n-1)(x+1)F_1\Big(\frac{x+1}{x+3}\Big) - 2(n-1)F_2\Big(\frac{x+1}{x+3}\Big) - 3(n-1)F_3\Big(\frac{x+1}{x+3}\Big)\Big\} \qquad (30)$$

(for the definition of $F_i(y)$, see the Appendix). For n = 1, this formula corresponds to the non-universal function $\mathscr{F}(x)$ of the Ising model.

The transformation to the universal function f(x) is effected by the formula

$$
\begin{aligned}
f(x) = 1 + x &- \varepsilon[\mathscr{F}_1(0)(1+x) - \mathscr{F}_1(-1) - \mathscr{F}_1(x)] \\
&+ \varepsilon^2 \mathscr{F}_1(0)[\mathscr{F}_1(0)(1+x) - \mathscr{F}_1(-1) - \mathscr{F}_1(x)] \\
&- \varepsilon^2[-x\mathscr{F}_2(-1) - \mathscr{F}_2(x) + (1+x)\mathscr{F}_2(0)] \\
&+ \varepsilon^2 \mathscr{F}_1(-1) x[\mathscr{F}_1'(x) - \mathscr{F}_1'(-1)],
\end{aligned} \tag{31}
$$

where the functions $\mathscr{F}_1(x)$ and $\mathscr{F}_2(x)$ are defined by (30). Putting $n = 3$ in (30) and using (31), we find $f(x)$ for the Heisenberg model ($n = 3$) in second order in ϵ:

$$
\begin{aligned}
f(x) &= f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) = 1 + x + \frac{\varepsilon}{22}[3(3+x)\ln(3+x) \\
&+ 2(1+x)\ln(1+x) - x(9\ln 3 - 6\ln 2) - 9\ln 3] \\
&+ \frac{\varepsilon^2}{242}\Big\{\Big(\frac{7}{4}x + \frac{79}{4}\Big)\ln^2(3+x) + \frac{1}{2}(x+1)\ln^2(x+1) \\
&+ (4x+10)\ln(x+1)\ln(x+3) + \Big(\frac{293}{22}x + 9x\ln 2 - \frac{27}{2}x\ln 3 \\
&+ \frac{879}{22} - \frac{81}{2}\ln 3\Big)\ln(3+x) + \Big(\frac{171}{11}x + 6x\ln 2 - 9x\ln 3 \\
&+ \frac{171}{11} - 9\ln 3\Big)\ln(1+x) + 2(1+x)F_1\Big(\frac{1+x}{3+x}\Big) - 4F_2'\Big(\frac{1+x}{3+x}\Big) \\
&- 6F_3\Big(\frac{1+x}{3+x}\Big) + \Big[\frac{83}{4}\ln^2 3 + 9\ln^2 2 - 27\ln 3\ln 2 - \frac{879}{22}\ln 3 \\
&+ \frac{293}{11}\ln 2 - 4F_2'(0) - 6F_3(0) - 2F_1\Big(\frac{1}{3}\Big) + 4F_2'\Big(\frac{1}{3}\Big) \\
&+ 6F_3\Big(\frac{1}{3}\Big)\Big]x + \frac{83}{4}\ln^2 3 - \frac{879}{22}\ln 3 - 2F_1\Big(\frac{1}{3}\Big) \\
&+ 4F_2'\Big(\frac{1}{3}\Big) + 6F_3\Big(\frac{1}{3}\Big)\Big\}.
\end{aligned} \tag{31'}
$$

For $x \to \infty$, from (31′) we can find the asymptotic form of $f(x)$ corresponding to weak fields above the transition point:

$$
\begin{aligned}
f(x) \to x &- {}^1/_{22}\varepsilon(9\ln 3 - 6\ln 2)x + {}^1/_{242}\varepsilon^2[{}^{83}/_4\ln^2 3 \\
&+ 9\ln^2 2 - 27\ln 3\ln 2 - {}^{879}/_{22}\ln 3 + {}^{293}/_{11}\ln 2 - 4F_2'(0) \\
&- 6F_3(0) - 2F_1({}^1/_3) + 4F_2'({}^1/_3) + 6F_3({}^1/_3) + 2F_1(1)]x \\
&+ {}^1/_{242}\varepsilon^2[{}^{25}/_4 x\ln^2 x + ({}^{635}/_{22} + 15\ln 2 - {}^{45}/_2\ln 3)x\ln x].
\end{aligned} \tag{32}
$$

It is not difficult to convince oneself that this expression is the ϵ-expansion of the term Cx^γ, where C is a universal constant characterizing the function $f(x)$ and equal to

$$
\begin{aligned}
C = 1 &+ {}^1/_{22}(6\ln 2 - 9\ln 3)\varepsilon + {}^1/_{242}\varepsilon^2[{}^{83}/_4\ln^2 3 + 9\ln^2 2 \\
&- 27\ln 3\ln 2 - {}^{879}/_{22}\ln 3 + {}^{293}/_{11}\ln 2 - 4F_2'(0) \\
&- 6F_3(0) - 2F_1({}^1/_3) + 4F_2'({}^1/_3) + 6F_3({}^1/_3) + 2F_1(1)].
\end{aligned} \tag{33}
$$

in second order in ϵ.

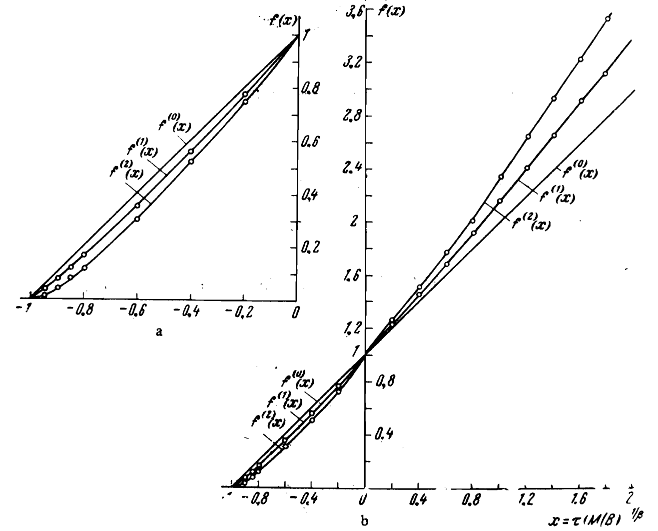
5. DISCUSSION OF THE RESULTS

The expression (31′) that we have obtained for the function $f(x)$ is so cumbersome that we investigate it in general form. In the Figure, graphs of this function are given together with graphs of the zeroth-approximation ($f^{(0)}(x) = f_0(x)$) and first-approximation ($f^{(1)}(x) = f_0(x) + \epsilon f_1(x)$) functions for $\epsilon = 1$. As calculations show, the second-order correction in $f(x)$ is found to be somewhat greater than the first-order correction: $f_1(x)/f_0(x) \sim 0.10$ and $f_2(x)/f_0(x) \sim 0.11$ for $0 \lesssim x \lesssim 5$. For sufficiently large x, and also for x close to -1, these ratios become larger (0.8–0.9). At the same time, the ϵ-expansion for the indices, e.g., for γ in the Heisenberg model, has the form

$$
\gamma = 1 + \gamma_1\varepsilon + \gamma_2\varepsilon^2 + O(\varepsilon^3) = 1 + \frac{5}{22}\varepsilon + \frac{635}{5324}\varepsilon^2 + O(\varepsilon^3).
$$

Here, $\gamma_1 \sim 0.22$ and $\gamma_2 \sim 0.11$. Thus, the ϵ-expansion for the function $f(x)$ converges worse than for the indices, and terms of higher order in ϵ probably have no meaning

Another important question is the question of the region of applicability of the ϵ-expansion of the function $f(x)$. In the Ising model, this region coincides with the



Plot of the universal function f(x) in different approximations in $\epsilon = 1$: a) for $-1 \leqslant x \leqslant 0$; b) for $-1 \leqslant x \leqslant 2$.

whole range of variation of the variable x. For a Heisenberg ferromagnet, the presence of spin waves in the region $\tau < 0$, $H \to 0$ leads to singular behavior of the susceptibility[8]:

$$
\chi \infty H^{-1/2}, \quad H \to 0, \quad \tau < 0.
$$

In fact,

$$
\chi = g_{\parallel}(k=0) \infty
$$

$$
\infty (n-1)\int \frac{d^{4-\varepsilon}k}{(k^2 + H/M)^2} \sim (n-1)H^{-\varepsilon/2}, \tag{34}
$$

i.e.,

$$
g_{\parallel}(k=0) \infty (n-1)H^{-\varepsilon/2} \to \infty \text{ as } H \to 0.
$$

From the standpoint of the ϵ-expansion, this will appear in the form of a series:

$$
g_{\parallel}(k=0) \sim (n-1)[1 - {}^1/_2\varepsilon\ln H + O(\varepsilon^2)] \to \infty \text{ as } H \to 0.
$$

Hence it can be seen that in the region of exponentially small fields and for $\tau < 0$ the ϵ-expansion becomes meaningless (estimates show that this corresponds to $x \approx -1 + 10^{-5}$).

In order to reach a final conclusion on the possibility of using the expansion of the function $f(x)$ in the dimensionality of space ϵ as the small parameter when $\epsilon = 1$, it is necessary to compare the results obtained with experiment. Unfortunately, the author knows of no sufficiently reliable experimental data on the equation of state of an ideal ferromagnet close to the critical point and can only express the hope that the problem of comparison with experiment will be solved in the near future.

Up to now, we have spoken of the Ising and Heisenberg models ($n = 1$ and $n = 3$). The case $n = 2$ corresponds to a Bose gas. In this case, the equation of state, generally speaking, has no meaning, since for a Bose system there is no analog of the magnetic field. However, if we consider, e.g., the behavior of helium in thin films, in this case the inhomogeneities at the boundary play the role of the magnetic field and the equation of state gives the possibility of finding the distribution of the parameter ψ over the thickness of the film. For a Bose gas, it is also meaningful to consider the ratio of the coefficients of the specific heat, A_+/A_-, and the specific-heat discontinuity ΔC near the critical point. The ratios A_+/A_- and $\Delta C/A_+$ can be calculated using the equation of state found.

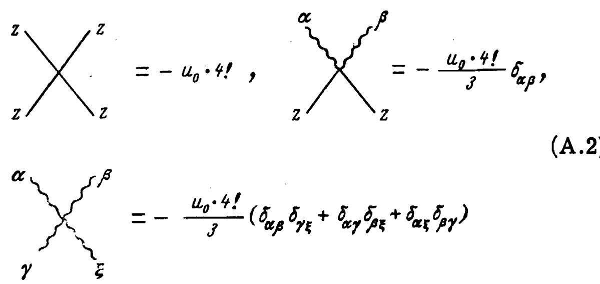
The author expresses her gratitude to A. A. Migdal for useful advice and constant interest in the work.

APPENDIX

1. The interaction Hamiltonian of the Heisenberg model is of the form

$$H_{int} = u_0\varphi^4 = u_0(\varphi_\perp^4 + \varphi_z^4 + 2\varphi_\perp^2\varphi_z^2). \quad (A.1)$$

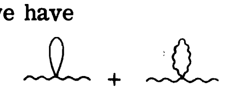
The vertex parts $\Gamma_m^{\alpha_1\cdots\alpha_m}(k_i)$ will be m-th rank tensors in the n-dimensional "spin" space, symmetric with respect to interchanges of any pair of variables $k_i\alpha_i \leftrightarrow k_j\alpha_j$. To calculate the green function, we shall need the four point functions Γ_4; the expressions corresponding to the bare vertices are



$$= -u_0\cdot 4!, \qquad = -\frac{u_0\cdot 4!}{3}\delta_{\alpha\beta}, \qquad = -\frac{u_0\cdot 4!}{3}(\delta_{\alpha\beta}\delta_{\gamma\xi} + \delta_{\alpha\gamma}\delta_{\beta\xi} + \delta_{\alpha\xi}\delta_{\beta\gamma}) \quad (A.2)$$

The third diagram in (24) differs from the fourth diagram only by a factor arising in the latter diagram of a result of the summation over the tensor indices and equal to $n+1$. In the calculation of the integral corresponding to these diagrams, we have taken into account only terms of the type $\tau\ln\tau$, $\epsilon\tau\ln^2\tau$, τ, $\epsilon\tau$. Terms of higher order in τ have not been taken into account (we are working in the critical region $\tau \ll 1$); nor have we taken into account the constant in the integral, which leads to a renormalization of the transition temperature in first order in ϵ.

As a result, we have



$$= \frac{-u_0 4!(n+2)}{3}\left[\int_0^\infty \frac{k^3dk}{\tau + k^2(1+k^2)^2} - K'\varepsilon\int_0^\infty \frac{k^3dk}{\tau+k^2(1+k^2)^2} - \varepsilon\int_0^\infty \frac{k^3\ln k\,dk}{\tau+k^2(1+k^2)^2}\right] = -2u_0K_4(n+2)\left[\left(\tau\ln\tau + \frac{11}{6}\tau\right)(1-K'\varepsilon) - \frac{\varepsilon}{4}\tau\ln^2\tau + \frac{\varepsilon}{2}\tau\right]. \quad (A.3)$$

2. We examine the calculation of certain of the graphs. We introduce the notation

$$\tau + M^2 = b, \quad \tau + 3M^2 = a.$$

Then the expression corresponding, e.g., to the diagram

$$(A.4)$$

is a function of a and b, i.e., (A.4) is equal to

$$\frac{u_0^2(4!)^2}{9\cdot 2}\frac{1}{(2\pi)^8}\int d^4k_1\,d^4k_2[g(a,\mathbf{k}_1)g(a,\mathbf{k}_2)g(b,\mathbf{k}_1+\mathbf{k}_2) - g(0,\mathbf{k}_1)g(0,\mathbf{k}_2)g(0,\mathbf{k}_1+\mathbf{k}_2)] \quad (A.4a)$$

(Here we have made a subtraction at $\tau = 0$, $M^2 = 0$, corresponding to a redefinition of the transition point in second order in ϵ).

The expression in the square brackets can be brought to the form $A_1 + A_2 + A_3$, where

$$A_1 = g(a,\mathbf{k}_1)g(b,\mathbf{k}_1+\mathbf{k}_2)[g(a,\mathbf{k}_2) - g(0,\mathbf{k}_2)], \quad (A.5)$$

$$A_2 = g(0,\mathbf{k}_1)g(b,\mathbf{k}_1+\mathbf{k}_2)[g(a,\mathbf{k}_2) - g(0,\mathbf{k}_2)], \quad (A.6)$$

$$A_3 = g(0,\mathbf{k}_1)g(0,\mathbf{k}_2)[g(b,\mathbf{k}_1+\mathbf{k}_2) - g(0,\mathbf{k}_1+\mathbf{k}_2)]. \quad (A.7)$$

Thus, the problem reduces to calculating three integrals, in which the last integration is conveniently performed over the momentum $\mathbf{k}_2$ in (A.5) and (A.6) and over the momentum $(\mathbf{k}_1+\mathbf{k}_2)$ in (A.7).

We shall examine, e.g., the integral with A_1, which we denote by J_1:

$$J_1 = -a\int\frac{d^4k_2}{[a+k_2^2(1+k_2^2)^2]k_2^2(1+k_2^2)^2} \times \int\frac{d^4k_1}{[a+k_1^2(1+k_1^2)^2][b+(\mathbf{k}_1+\mathbf{k}_2)^2(1+(\mathbf{k}_1+\mathbf{k}_2)^2)^2]} \quad (A.8)$$

It is not difficult to see that the integral obtained is logarithmic. We have already said that, in all the expressions, we shall be interested only in terms of the type $a\ln a$, $b\ln b$, $af(a/b)$, etc. Terms linear in a and b can be disregarded, since they can always be included in the definition of τ and M^2, and for this reason they do not appear in the universal function f(x). Therefore, the terms of interest to us in integrals of the type (A.8) can be selected conveniently as follows. Since we are working in the region $a \ll 1$, $b \ll 1$, we can always choose $\min(a,b) \ll \lambda \ll 1$, and divide the integration over the whole range of momenta into two:

$$\int_0^\infty dq = \int_0^\lambda dq + \int_\lambda^\infty dq. \quad (A.9)$$

In this case, in the integral from zero to λ we can disregard the cutoff factor $((1+k^2)^2 \approx 1 + 0(\lambda^2))$, and in the integral from λ to infinity the constants a and b become unimportant to within terms of order a/λ, b/λ.

Returning to (A.8), we note that for our calculations two regions of momenta are important:

I) $0 \le k_2 \le \lambda$, $\lambda \le k_1 \le \infty$.
II) $0 \le k_2 \le \lambda$, $0 \le k_1 \le \lambda$.

In the first region,

$$J_1^{(I)} \sim -{}^{11}/_6 a\ln a.$$

The calculation of the integral in the second region is complicated by the integration over k_1. Here, it is convenient to perform a Feynman parametrization (cf., e.g., [9]) and take into account that the important $k_2 \ll \lambda$. As a result, we have

$$J_1^{(II)} \sim -a({}^1/_2\ln^2 a - \ln a) - aF(b/a),$$

$$F(x) = \int_0^\infty \frac{dt}{t+1}[-(1-f_1)\ln|1-f_1| - f_1\ln|f_1| + (f_2-1)\ln|f_2-1| - f_2\ln|f_2|],$$

$$f_{1,2} = \frac{t+x-1}{2t} \mp \left[\frac{(t+x-1)^2}{4t^2} + \frac{1}{t}\right]^{1/2}.$$

The integrals with A_2 and A_3 are calculated in the same way.

The functions $F_i(x)$ appearing in the equation of state are defined as follows:

$$F_1(x) = \int_0^\infty \frac{dt}{t+x}[-(1-g_1)\ln|1-g_1| - g_1\ln|g_1| + (g_2-1)\ln|g_2-1| \quad (A.10)$$

$$
- g_2 \ln|g_2| + \frac{1}{t}\ln\frac{1}{t} - \left(1+\frac{1}{t}\right)\ln\left(1+\frac{1}{t}\right)\Big],
$$

$$
F_2(x) = -\int_0 \frac{t\,dt}{(t+x)^2}[-(1-g_1)\ln|1-g_1| - g_1\ln|g_1|
$$

$$
+(g_2-1)\ln|g_2-1| - g_2\ln|g_2|], \quad (A.11)
$$

$$
g_{1,2} = \frac{t+1-x}{2t} \mp \left[\frac{(t+1-x)^2}{4t^2} + \frac{x}{t}\right]^{1/2}; \quad (A.12)
$$

$$
F_3(x) = -2\int_0^\infty \frac{t\,dt}{(t+1)^2}[-u\ln|u| + (u-1)\ln|u-1|], \quad (A.13)
$$

$$
u = \frac{1}{2} - \left(\frac{1}{4} + \frac{x}{t}\right)^{1/2}. \quad (A.14)
$$

For small x, the function $F_2(x)$ contains the term $\frac{1}{2}\ln^2 x$. Therefore, it is convenient to introduce the function

$$
F_2'(x) = F_2(x) - \tfrac{1}{2}\ln^2 x,
$$

so that $F_2'(0) = \pi^2/3$.

It is not difficult to see that $F_3(0) = 0$. The values of $F_i((1+x)/(3+x))$ for those points x used to construct the graph are given in the Table. The integrals F_i were computed on a machine with accuracy 0.01.

Values of the functions $F_i = F_i(1 + x/3 + x)$

x	F_1	F_2	F_3	x	F_1	F_2	F_3	x	F_1	F_2	F_3
–0,95	–29.15	10.38	0.23	0.2	–10.50	3.82	1.53	1,6	–8.88	3.34	2.03
–0,90	–23.22	8.16	0.38	0.4	–10.10	3.70	1.67	1.8	–8,77	3.30	2.06
–0.85	–20.25	7,09	0.50	0.6	–9.79	3.61	1.75	2.0	–8,67	3.28	2.10
–0,80	–18.36	6.42	0.61	0,8	–9,54	3.53	1.82	4.0	–8.08	3.10	2.32
–0,60	–14.54	5.12	0.93	1	–9.33	3.47	1.88	6.0	–7.81	3.02	2.44
–0.40	–12,78	4.54	1.16	1.2	–9,16	3.42	1,94	8.0	–7.65	2.98	2.51
–0,20	–11,72	4.21	1,33	1.4	–9.01	3.37	1.98	100	–7.13	2.83	2.78
0	–11.01	3,98	1,46								

Note added in proof (December 17, 1972). The author has recently learnt of the publication of a preprint by K. Wilson and co-workers, in which analogous results are obtained.* The author thanks A. A. Belavin for help in establishing this correspondence.

*See E. Brézin, D. J. Wallace and K. G. Wilson, Phys. Rev. B7, 232 (1973) [Transl. note].

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Translated by P. J. Shepherd
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