Transport phenomena in a completely ionized ultrarelativistic plasma

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A set of transport equations for a hot relativistic plasma whose particles may, in particular, possess a thermal energy exceeding the rest energy, is obtained in the presence of electric and magnetic fields. In the limit when the aforementioned condition is satisfied for electrons but the ions remain cold, the transport coefficients are found for a plasma consisting of electrons and one type of positive ions. It is shown that in this case the temperature dependence of the transport coefficients changes strongly, as does the hydrodynamic equation system itself.

In the study of certain phenomena that occur in celestial objects[1], and also in the investigation of strong-current relativistic electron beams[2,3], it is necessary to deal with a plasma whose particles can have relativistic thermal energy, i.e., the condition $T_a \gtrsim m_a c^2$ is satisfied, where m_a and T_a are respectively the rest mass and the temperature of the electrons or ions, measured in energy units (a stands for an electron or an ion). Obviously, such a plasma exhibits new properties, which are due precisely to the relativistic character of the temperature. For example, the spectra of the natural oscillations are altered[4,5,6] and it becomes necessary to investigate anew the questions of stability[7,8], the thermodynamics of the plasma[9], processes connected with transfer phenomena, etc.

This raises the question of obtaining a closed system of hydrodynamic equations for a hot plasma, where the temperatures of the charged particles are arbitrary, and in particular relativistic.

1. The state of an electron-ion plasma can be described with the aid of the particle distribution functions $f_a(t, \mathbf{r}, \mathbf{p}_a)$, which in the presence of electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ satisfy the system of kinetic equations[10]

$$\frac{\partial f_a}{\partial t}+\frac{c^2}{\varepsilon_a}\mathbf{p}_a\frac{\partial f_a}{\partial \mathbf{r}}+e_a\left\{\mathbf{E}+\frac{c}{\varepsilon_a}[\mathbf{p}_a\mathbf{H}]\right\}\frac{\partial f_a}{\partial \mathbf{p}_a}=\sum_b C_{ab}(f_a, f_b), \qquad (1.1)$$

where $\epsilon_a = c(p_a^2 + m_a^2c^2)^{1/2}$, and C_{ab} is a collision term, the explicit form of which will be given below. The function f_a is a relativistic invariant[10], and then the quantity $C_{ab}d\mathbf{p}_a$ is also invariant with respect to the Lorentz transformations.

If we confine ourselves to consideration of only elastic collisions, i.e., if we disregard ionization, recombination, and similar processes, then we can point to several general properties of the collision term, even without knowing its explicit form. It is clear that the laws for the conservation of the number of particles, momentum, and energy make it possible to write down the following relations:

$$\int C_{ab}\,d\mathbf{p}_a=0, \quad \int \mathbf{p}_a C_{aa}\,d\mathbf{p}_a=0, \quad \int (\varepsilon_a-m_ac^2)C_{aa}\,d\mathbf{p}_a=0,$$
$$\int \mathbf{p}_a C_{ab}\,d\mathbf{p}_a+\int \mathbf{p}_b C_{ba}\,d\mathbf{p}_b=0, \qquad (1.2)$$
$$\int (\varepsilon_a-m_ac^2)C_{ab}\,d\mathbf{p}_a+\int (\varepsilon_b-m_bc^2)C_{ba}\,d\mathbf{p}_b=0.$$

As is well known[11], starting from the kinetic equations, we can obtain a system of transport equations for the macroscopic parameters of the plasma (the particle density n_a and the temperature T_a in the rest system of the given plasma component, and also the average velocity $\mathbf{u}_a$). In the relativistic case, however, the question of determining the temperature becomes more complicated. In fact, if we obtain, with the aid of Maxwell's relativistic equilibrium distribution function, the mean value of the kinetic energy of the particles in their proper reference frame, then we obtain a certain complicated function of the temperature[12]. In analogy with the nonrelativistic limit, it is possible to retain this definition in force also in the case when there is no thermal equilibrium and the distribution function is not Maxwellian.

Thus, in the rest system of the given plasma component, we introduce the principal definitions

$$\int f_a\,d\mathbf{p}_a=n_a, \qquad c^2\int\frac{\mathbf{p}_a}{\varepsilon_a}f_a\,d\mathbf{p}_a=0,$$
$$\frac{1}{n_a}\int(\varepsilon_a-m_ac^2)f_a\,d\mathbf{p}_a=m_ac^2(G_a-1)-T_a, \qquad G(z_a)=\frac{K_3(z_a)}{K_2(z_a)}, \qquad (1.3)$$

where $K_2(z_a)$ and $K_3(z_a)$ are respectively the Macdonald functions of second and third orders ($z_a = m_ac^2/T_a$). In the nonrelativistic case ($z_a \gg 1$) we have $G_a \approx 1 + 5/2z_a$ and the third integral in (1.3) yields the well-known result $3T_a/2$, while in the ultrarelativistic limit ($z_a \ll 1$) we have $G_a \approx 4/z_a$ and the integral is equal to $3T_a$.

The transition to the laboratory reference frame can be realized with the aid of a Lorentz transformation for the energy and momentum of the particles ϵ_a and $\mathbf{p}_a$[13]:

$$p_{ai}=s_{aik}p_{ak}'+c^{-2}\gamma_a u_{ai}\varepsilon_a', \quad \varepsilon_a=\gamma_a(\varepsilon_a'+\mathbf{u}_a\mathbf{p}_a');$$
$$s_{aik}=\delta_{ik}+(\gamma_a-1)u_{ai}u_{ak}/u_a^2, \quad \gamma_a=(1-u_a^2/c^2)^{-1/2} \qquad (1.4)$$

(the prime denotes the rest system of the chosen plasma component).

If we now multiply (1.1) by 1, $\mathbf{p}_a$, and $\epsilon_a - m_ac^2$ and integrate with respect to the momenta, then, using formulas (1.2)—(1.4), we can obtain the equations of continuity, motion, and thermal balance for the macroscopic parameters n_a, $\mathbf{u}_a$, and T_a:

$$\frac{\partial}{\partial t}(\gamma_a n_a)+\operatorname{div}(\gamma_a n_a\mathbf{u}_a)=0,$$
$$\gamma_a n_a\frac{d_a}{dt}(\gamma_a m_a G_a u_{ai})=-\frac{\partial P_a}{\partial x_i}-\frac{\partial}{\partial x_k}(s_{aim}s_{akn}\pi_{amn})$$
$$+\gamma_a n_a e_a\left\{\mathbf{E}+\frac{1}{c}[\mathbf{u}_a\mathbf{H}]\right\}_i+s_{aik}R_{ak}+\frac{1}{c^2}\gamma_a u_{ai}Q_a-\frac{1}{c^2}\frac{\partial}{\partial t}\Big[\gamma_a s_{aik}u_{am}\pi_{akm}$$
$$+\gamma_a\left(s_{aik}+\frac{1}{c^2}\gamma_a u_{ai}u_{ak}\right)q_{ak}\Big]-\frac{1}{c^2}\frac{\partial}{\partial x_k}[\gamma_a(s_{aim}u_{ak}+s_{akm}u_{ai})q_{am}], \qquad (1.5)^*$$
$$n_a\frac{d_a}{dt}(m_ac^2G_a-T_a)-T_a\frac{d_a n_a}{dt}=-\frac{\partial}{\partial x_k}(\gamma_a s_{akm}q_{am})$$

$$- s_{aim} s_{akn} \pi_{amn} \frac{\partial u_{ai}}{\partial x_k} + \gamma_a Q_a - \frac{1}{c^2} \frac{\partial}{\partial t} (\mathbf{q}_a \mathbf{u}_a) - \frac{1}{c^2} \Big[\gamma_a s_{aik} u_{am} \pi_{akm} + \gamma_a \Big(s_{aik} + \frac{1}{c^2} \gamma_a u_{ai} u_{ak} \Big) q_{ak} \Big] \frac{\partial u_{ai}}{\partial t} - \frac{1}{c^2} \gamma_a (s_{aim} u_{ak} + s_{akm} u_{ai}) q_{am} \frac{\partial u_{ai}}{\partial x_k},$$

where $P_a = n_a T_a$ is the partial pressure of the particles of type a, π_{aik} is the tensor of the viscous stresses, $\mathbf{q}_a$ is the heat flux density, $\mathbf{R}_a$ and Q_a are respectively the friction force and the heat release connected with the collisions, $d_a/dt \equiv \partial/\partial t + \mathbf{u}_a \nabla$ is the hydrodynamic derivative

$$P_a = \frac{c^2}{3} \int \frac{p_a^2}{\varepsilon_a} f_a \, d\mathbf{p}_a, \quad \pi_{aik} = c^2 \int \frac{1}{\varepsilon_a} \Big(p_{ai} p_{ak} - \frac{p_a^2}{3} \Big) f_a \, d\mathbf{p}_a,$$
$$\mathbf{q}_a = c^2 \int \mathbf{p}_a f_a \, d\mathbf{p}_a, \quad \mathbf{R}_a = \int \mathbf{p}_a C_{ab} \, d\mathbf{p}_a, \quad Q_a = \int (\varepsilon_a - m_a c^2) C_{ab} \, d\mathbf{p}_a \tag{1.6}$$

(all the integrals in (1.6) are taken in the rest system of the given plasma component).

In the limit of small average velocities ($u_a \ll c$) and low temperatures ($T_a \ll m_a c^2$), Eqs. (1.5) coincide with the transport equations used in[14] (this transition corresponds formally to $c \to \infty$). From the second equation of (1.5) we can see that high temperatures ($T_a \gg m_a c^2$) denote the dependence of the inertial mass of the particle on T_a. The role of the mass is now played by the quantity $m_a G(z_a)$. In the case $z_a \ll 1$, the "mass" of the particle is $m_a^* = m_a \cdot 4T_a/m_a c^2 \gg m_a$, i.e., the gas seems to become heavier.

The third equation of (1.5) is in fact the equation for the entropy. If we discard all the dissipative terms, then we obtain the adiabatic equation

$$n_a L(z_a) = \text{const}, \quad L(z_a) = \frac{z_a}{K_2(z_a)} \exp[-z_a G(z_a)]. \tag{1.7}$$

In the nonrelativistic limit, (1.7) yields the usual result for a monatomic ideal gas ($n_a/T_a^{3/2} = \text{const}$), and in the ultrarelativistic limit one obtains the adiabat for the photon gas ($n_a/T_a^3 = \text{const}$).

In order for the system (1.5) to become closed, it is necessary to connect π_{aik}, $\mathbf{q}_a$, $\mathbf{R}_a$, and Q_a with the macroscopic parameters n_a, $\mathbf{u}_a$, T_a, and their derivatives. This can be done when all the quantities vary little over distances on the order of the mean free path and over times on the order of the time of collisions between particles. The solution of the kinetic equation (1.1) can then be sought in the form $f_a = f_a^{(0)}(1 + \Phi_a)$, where Φ_a is a small correction, and $f_a^{(0)}$ is the relativistic local Maxwellian distribution function[12]:

$$f_a^{(0)} = \frac{n_a}{4\pi (m_a c)^3} \frac{z_a}{K_2(z_a)} \exp\Big[-\frac{\gamma_a}{T_a} (\varepsilon_a - \mathbf{p}_a \mathbf{u}_a) \Big]. \tag{1.8}$$

In (1.8), the quantities n_a, $\mathbf{u}_a$, and T_a are functions of the coordinates and of the time.

The correction Φ_a is proportional to those factors which cause deviations from the Maxwellian function (the gradients, the electric fields, etc.), so that Φ_a is expressed in terms of the macroscopic parameters and their derivatives, and in final analysis π_{aik}, $\mathbf{q}_a$, $\mathbf{R}_a$, and Q_a are all expressed in terms of the same quantities, after which the system (1.5) becomes closed and can be used to solve concrete problems. This program has been carried through to conclusion in[14] for a fully polarized nonrelativistic plasma. We shall show below that the problem posed can also be solved for the system (1.5), if one considers a fully ionized plasma with one sort of ions, where $u_a \ll c$, the electrons are assumed to be ultrarelativistic ($T_e \gg m_e c^2$), and the ions remain cold ($T_i \ll m_i c^2$).

2. The collision term in (1.1) is taken in the form (see[10])

$$C_{ab} = -2\pi L e_a^2 e_b^2 \frac{\partial}{\partial p_i} \int \Big(f_a \frac{\partial f_b'}{\partial p_k'} - f_b' \frac{\partial f_a}{\partial p_k} \Big) \frac{U_{ik}}{\gamma \gamma'} d\mathbf{p}';$$
$$U_{ik} = \frac{\gamma^2 \gamma'^2 (1 - \beta\beta')^2}{c[\gamma^2 \gamma'^2 (1 - \beta\beta')^2 - 1]^{3/2}} \{ [\gamma^2 \gamma'^2 (1 - \beta\beta')^2 - 1] \delta_{ik} - \gamma^2 \beta_i \beta_k - \gamma'^2 \beta_i' \beta_k' + \gamma^2 \gamma'^2 (1 - \beta\beta')(\beta_i \beta_k' + \beta_i' \beta_k) \}; \tag{2.1}$$

where $c\beta$ is the particle velocity expressed in terms of the momentum and $\gamma = (1 - \beta^2)^{-1/2}$.

In the nonrelativistic case ($p_a \ll m_a c$), U_{ik} becomes a function of only the difference between the velocities of the colliding particles, and (2.1) coincides with the expression obtained by Landau[15]. It is easy to verify that substitution of the relativistic Maxwellian distribution function (1.8) in (2.1) causes the collision terms C_{ee} and C_{ii} to vanish. This follows directly from the fact that U_{ik} possesses the property

$$(\beta_i - \beta_i') U_{ik} = (\beta_k - \beta_k') U_{ik} \equiv 0.$$

The Coulomb logarithm L is equal to the logarithm of the ratio of the characteristic maximal and minimal collision parameters, $L = \ln(b_{max}/b_{min})$. The maximum impact parameter should be taken to be the Debye screening radius $b_{max} = D = (T/4\pi e^2 n)^{1/2}$. However, at high thermal velocities v ($e^2/hv < 1$, i.e., $v/c > 1/137$, where h is Planck's constant), it is necessary to choose for the maximum impact parameter the value at which the scattering angle becomes of the same order as its quantum uncertainty. For example, when the plasma electrons are ultrarelativistic ($v_e \sim c$) we choose $b_{max} = De^2/hv$. As the lower impact parameter we substitute the value at which a deviation by an angle $\sim\pi/2$ takes place, i.e., $b_{min} = e^2/\langle pv \rangle$[13] (the angle brackets denote averaging in momentum space). In the nonrelativistic case $b_{min} = e^2/3T$, and in the ultrarelativistic limit $b_{min} = e^2/2T$.

When writing out the collision term in the form (2.1), it was assumed that the radius of curvature of the particle trajectory is much larger than the Debye length, so that the magnetic field does not influence the collision act. Of course, this statement is valid for magnetic fields that are not too strong.

The subsequent analysis is based on the fact that the crossing terms C_{ei} and C_{ie} can be greatly simplified by taking into account the large difference between the masses of the electrons and ions. For ultrarelativistic electrons, however, the role of the mass is played by the quantity m_e^* (the subsequent calculations confirm this conclusion), and the small parameter of the theory is actually the quantity m_e^*/m_i. Obviously, the electron temperature should be bounded from above by the condition $T_e \ll m_i c^2$, for otherwise the electrons will be just as "heavy" as the ions. It is easy to show that if the energies of the light and heavy particles are of the same order, then the energy exchange times between identical particles (τ_{ee}^ϵ and τ_{ii}^ϵ) are smaller than the time of energy exchange between the electrons and ions (τ_{ei}^ϵ):

$$\tau_{ee}^\varepsilon : \tau_{ii}^\varepsilon : \tau_{ei}^\varepsilon = 1 : (m_i / m_e^*)^{1/2} : m_i / m_e^*.$$

It is now clear that the equilibrium within each of the plasma components sets in earlier than the equilibrium between them, and this makes it possible for us to consider henceforth a two-temperature plasma. Expanding the tensor U_{ik} in powers of the ion velocity in the electron-ion collision term, neglecting the tensor of the viscous stresses of the ions, and integrating with respect to $d\mathbf{p}_i$, we can obtain the following expression for C_{ei} in the ion rest system:

$$C_{ei} = 2\pi e_e^2 e_i^2 L n_i \frac{\partial}{\partial p_\alpha}\left(U_{0\alpha\beta}\frac{\partial f_e}{\partial p_\beta} + \frac{\varepsilon^2}{m_i c^4}\frac{2p_\alpha}{p^3} f_e + m_i T_i \frac{\partial f_e}{\partial p_\beta} U_{1\alpha\beta}\right);$$

$$U_{0\alpha\beta} = \frac{\varepsilon}{c^2}\frac{p^2\delta_{\alpha\beta} - p_\alpha p_\beta}{p^3} \qquad U_{1\alpha\beta} = \frac{\varepsilon^3}{m_i^2 c^6}\frac{3p_\alpha p_\beta - p^2\delta_{\alpha\beta}}{p^5} + \frac{1}{m_i^2 c^2 \varepsilon}\frac{p^2\delta_{\alpha\beta} - p_\alpha p_\beta}{p}. \tag{2.2}$$

In (2.2), the first term does not depend at all on the detailed form of the distribution function of the ions (it can be designated by C'_{ei}), while the second and third terms (C''_{ei}) are small quantities $\sim m_e^*/m_i$.

With the aid of (2.2) it is easy to find the friction force exerted on the electrons by the ions, assuming that the electrons have a distribution (1.8) (where, however, $\mathbf{u}_e$ should be replaced by the relative velocity $\mathbf{V} = \mathbf{u}_e - \mathbf{u}_i$). If we assume that $\mathbf{V}$ is small in comparison with the thermal velocities of the electrons, and also neglect the terms $\sim m_e^*/m_i$ and higher, then we obtain for the friction force

$$\mathbf{R}_e^{(0)} = -m_e G_e n_e \mathbf{V}/\tau_e, \tag{2.3}$$

where τ_e is the time of scattering of the electrons by the ions:

$$\tau_e = \frac{3m_e G_e T_e c K_2(z_e)\exp(z_e)}{4\pi e_e^2 e_i^2 L n_i (1 + 2/z_e + 2/z_e^2)}. \tag{2.4}$$

In the derivation of (2.3) we used the following properties of the tensor $U_{0\alpha\beta}$:

$$\frac{\partial U_{0\alpha\beta}}{\partial p_\alpha} = -\frac{\varepsilon}{c^2}\frac{2p_\beta}{p^3}, \quad U_{0\alpha\beta}\, p_\alpha = U_{0\alpha\beta}\, p_\beta = 0.$$

For nonrelativistic temperatures, (2.4) yields a well-known result (see [14]), and in the ultrarelativistic limit we obtain

$$\tau_e = 3T_e^2 / \pi e_e^2 e_i^2 L n_i c. \tag{2.5}$$

In complete analogy with the expression for C_{ei} we can simplify the ion-electron collision term by assuming that the electron distribution function differs little from Maxwellian, and that the electron thermal velocities greatly exceed the ion velocities as well as the relative velocity $\mathbf{V}$:

$$C_{ie} = \frac{m_e G_e n_e}{n_i \tau_e}\frac{\partial}{\partial p_\alpha}\left(\frac{p_\alpha}{m_i} f_i + T_e \frac{\partial f_i}{\partial p_\alpha}\right) + \frac{\mathbf{R}_e^{(0)}}{n_i}\frac{\partial f_i}{\partial \mathbf{p}} \tag{2.6}$$

(in (2.6), the calculation is carried out in the rest system of the ions). The heat release Q_i can be obtained with the aid of (2.6) by assuming the deviation of f_i from the Maxwellian function to be small:

$$Q_i = \frac{3m_e G_e n_e}{m_i \tau_e}(T_e - T_i). \tag{2.7}$$

Using the conservation laws (1.2), we can obtain in the limit u_e, $u_i \ll c$ the relation

$$Q_e = -\mathbf{R}_e \mathbf{V} - Q_i. \tag{2.8}$$

3. The simplification of the crossing collision terms greatly facilitates the problem of finding equations for the small corrections Φ_a to the Maxwellian distribution function. It is convenient to go over first in the kinetic equation (1.1) to a new variable, namely the random momentum $\mathbf{p}'_a$. This transition can be effected in general form by turning to formulas (1.4). For our problem, however, there is no need for such a general approach; it suffices to use formulas (1.4) in an approximation linear in $\mathbf{u}_a$, and also to discard from the kinetic equation those terms that are products of two or more perturbing factors, such as the derivatives with respect to t and $\mathbf{r}$ of the macroscopic parameters, the electric field $\mathbf{E}$, and the relative velocity $\mathbf{V}$.

For electrons in the ultrarelativistic limit ($T_e \gg m_e c^2$, $p_e \gg m_e c$), Eq. (1.1) takes the form

$$C_{ee}(f_e, f_e) + C_{ei}'(f_e, f_i') - \frac{m_e^* c}{p}[\mathbf{p}\boldsymbol{\omega}_e]\frac{\partial f_e}{\partial \mathbf{p}}$$
$$= \frac{d_e f_e}{dt} + \frac{c}{p}\mathbf{p}\frac{\partial f_e}{\partial \mathbf{r}} + \left(e_e \mathbf{E}^* - \frac{p}{c}\frac{d_e \mathbf{u}_e}{dt}\right)\frac{\partial f_e}{\partial \mathbf{p}} - \frac{\partial u_{ei}}{\partial x_k} p_k \frac{\partial f_e}{\partial p_i} \tag{3.1}$$
$$- C_{ei}'(f_e, f_i - f_i') - C_{ei}''(f_e, f_i);$$
$$\boldsymbol{\omega}_e = e_e \mathbf{H} / m_e^* c, \ \mathbf{E}^* = \mathbf{E} + c^{-1}[\mathbf{u}_e \mathbf{H}],$$

where ω_e is the cyclotron frequency of the electrons, and $e_e = -e$. We have left out the primes from (3.1), and also added and subtracted the term $C'_{ei}(f_e, f'_i)$, where f'_i is the ion distribution function shifted in such a way that the average ion velocity coincides with the average electron velocity. Obviously, the term $C'_{ei}(f_e, f_i - f'_i)$ is small in comparison with $C'_{ei}(f_e, f'_i)$ in the case when the thermal velocities of the electrons exceed the relative velocity $\mathbf{V}$. Of course, the terms in the right-hand side of (3.1) are small, since we are considering the case of small gradients, electric fields, etc.

If we discard the entire right-hand side of the last equation, then the solution is an arbitrary ultrarelativistic Maxwellian distribution:

$$f_e^{(0)} = \frac{n_e}{8\pi}\left(\frac{c}{T_e}\right)^3 \exp\left(-\frac{pc}{T_e}\right). \tag{3.2}$$

Since the zeroth approximation (3.2) already gives the correct value of the parameters n_e, $\mathbf{u}_e$, and T_e, it is necessary to impose on the correction Φ_e the additional conditions:

$$\int f_e^{(0)}\Phi_e d\mathbf{p} = 0, \quad c^2\int \frac{\mathbf{p}}{\varepsilon} f_e^{(0)}\Phi_e d\mathbf{p} = 0, \quad \int (\varepsilon - m_e c^2) f_e^{(0)}\Phi_e d\mathbf{p} = 0. \tag{3.3}$$

In the next approximation, when finding the small corrections, it suffices to substitute (3.2) in the right-hand side. This gives rise to derivatives of n_e, $\mathbf{u}_e$, and T_e with respect to time; these derivatives should be replaced by their zeroth approximations. Multiplying (3.1) by 1, $\mathbf{p}$, and $(\epsilon - m_e c^2)$ and integrating with respect to the momenta of the electron, we can obtain expressions for the zeroth approximation of the derivative if we also take (3.3) into account. We now obtain for the first-approximation correction Φ_e the equation

$$I_{ee}(\Phi_e) + I_{ei}(\Phi_e) - \frac{m_e^* c}{p} f_e^{(0)}[\mathbf{p}\boldsymbol{\omega}_e]\frac{\partial \Phi_e}{\partial \mathbf{p}} = \left\{\frac{m_e^* c}{p}\left(\frac{pc}{4T_e} - 1\right)\mathbf{p}\boldsymbol{\psi}_e\right.$$
$$\left. + \frac{3m_e^{*2}}{p^2\tau_e}\left(1 - \frac{p^2c^2}{12T_e^2}\right)\mathbf{p}\mathbf{V} + \frac{1}{n_e T_e}\mathbf{p}\mathbf{R}_e^{(1)} + \frac{2}{pc}p_{\alpha\beta}W_{e\alpha\beta}\right\}\frac{f_e^{(0)}}{m_e^*}; \tag{3.4}$$
$$I_{ee}(\Phi_e) = C_{ee}(f_e^{(0)}, f_e^{(0)}\Phi_e) + C_{ee}(f_e^{(0)}\Phi_e, f_e^{(0)}), \qquad I_{ei}(\Phi_e) = C_{ei}'(f_e^{(0)}\Phi_e, f_i'),$$
$$\mathbf{R}_e^{(1)} = \int \mathbf{p} I_{ei}(\Phi_e)\, d\mathbf{p}, \quad \boldsymbol{\psi}_e = \nabla \ln\frac{T_e^3}{n_e} + \frac{e_e}{T_e}\mathbf{E}^*, \quad p_{\alpha\beta} = p_\alpha p_\beta - \frac{p^2}{3}\delta_{\alpha\beta}.$$

We have introduced here the symmetrical tensor $W_{e\alpha\beta}$, with a zero trace (the tensor of the shear velocity):

$$W_{e\alpha\beta} = \frac{\partial u_{e\alpha}}{\partial x_\beta} + \frac{\partial u_{e\beta}}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\operatorname{div}\mathbf{u}_e.$$

In the right-hand side of (3.4) we have discarded the terms $\sim m_e^*/m_i$ (for example, C''_{ei}), and we have expanded the integral $C'_{ei}(f_e^{(0)}, f_i - f'_i)$ in a series, retaining only the term linear in $\mathbf{V}$. As to the equation for the correction Φ_i, it takes the same form as given by Braginskiĭ[14], since the ions were assumed to be cold from the very outset ($T_i \ll m_i c^2$).

4. The equation for the correction (3.4) is linear, so that we can seek a solution in the form of a sum of terms, each of which is connected with one perturbing factor, namely the temperature gradient ∇T_e or the density gradient ∇n_e, the shift of the velocity **V**, the inhomogeneity of the velocity $W_{e\alpha\beta}$, or the electric field **E**.

A. From the form of Eq. (3.4) it is clear that the perturbing factors ∇T_e, ∇n_e, and $\mathbf{E}^*$ can be considered together (by introducing the vector $\boldsymbol{\psi}_e$). The solution of the equation for the correction, which is connected with the vector $\boldsymbol{\psi}_e$, is sought in the form

$$\Phi_\psi(\mathbf{p}) = A^0\mathbf{p}\psi_{e\parallel} + A'\mathbf{p}\psi_{e\perp} + A''\mathbf{p}[\omega_e\psi_e], \qquad (4.1)$$

where A^0, A', and A'' are functions of only the absolute magnitude of the momentum, and the symbols ∥ and ⊥ denote the components of the given vector respectively along and across the magnetic field. It suffices to find A' and A'', since A^0 can easily be obtained from the expression for A' by putting $\omega_e = 0$. The friction force $\mathbf{R}_\psi^{(1)}$ can be sought in the form

$$\mathbf{R}_\psi^{(1)} = n_e T_e (K^0\psi_{e\parallel} + K'\psi_{e\perp} + K''[\omega_e\psi_e]), \qquad (4.2)$$

where K^0, K', and K'' are numbers still to be determined.

Introducing the complex quantities

$$A = A' + i\omega_e hA'', \quad K = K' + i\omega_e hK'',$$

we can obtain an equation for A:

$$I_{ee}(A\mathbf{p}) + I_{ei}(A\mathbf{p}) - \frac{m_e^* c}{p} i\omega_e \mathbf{h} f_e^{(0)} A\mathbf{p} = \frac{f_e^{(0)}}{m_e^*}\left\{\frac{m_e^* c}{p}\left(\frac{pc}{4T_e} - 1\right) + K\right\}\mathbf{p} \qquad (4.3)$$

(**h** is the unit vector in the direction of the magnetic field). Following[11,14], we seek the function A in the form of a series in Sonine polynomials[16] (in this case it is convenient to use third-order polynomials):

$$A = \frac{\tau_e}{m_e^*}\sum_{m=1}^{\infty} a_m L_m^{(3)}(t_e), \quad t_e = \frac{pc}{T_e}, \quad a_m = a_m' + i\omega_e h a_m'' \qquad (4.4)$$

In (4.4), the expansion begins with m = 1, so as not to violate the condition (3.3).

Multiplying (4.3) by $L^{(3)}\mathbf{p}$ and integrating with respect to the momenta, we can obtain an infinite system of algebraic equations for the coefficients a_m:

$$\sum_{m=1}^{\infty}(\alpha_{nm} + \alpha_{nm}')a_m + i\omega_e h\tau_e \frac{(n+3)!}{6n!}a_n = \delta_{1n}\left(1 - \sum_{m=1}^{\infty} a_m\alpha_{0m}'\right), \qquad (4.5)$$

n = 1, 2, 3, ..., where we have introduced the following notation:

$$\alpha_{mn} = -\frac{\tau_e}{3n_e T_e m_e^*}\int p_k L_m^{(3)} I_{ee}(p_k L_n^{(3)})\,d\mathbf{p},$$
$$\alpha_{mn}' = -\frac{\tau_e}{3n_e T_e m_e^*}\int p_k L_m^{(3)} I_{ei}(p_k L_n^{(3)})\,d\mathbf{p}, \qquad (4.6)$$

m, n = 0, 1, 2, ... Now $\mathbf{R}_\psi^{(1)}$ and $\mathbf{q}_\psi$ are obtained from the formulas

$$\mathbf{R}_\psi^{(1)} = -n_e T_e \sum_{m=1}^{\infty}\alpha_{0m}'(a_m^0\psi_{e\parallel} + a_m'\psi_{e\perp} + a_m''[\omega_e\psi_e]),$$
$$\mathbf{q}_\psi = -\frac{4n_e\tau_e T_e^2}{m_e^*}(a_1^0\psi_{e\parallel} + a_1'\psi_{e\perp} + a_1''[\omega_e\psi_e]). \qquad (4.7)$$

In the derivation of (4.5) and (4.7) we used the orthogonality property of the Sonine polynomials[16]

$$\int_0^\infty t^k e^{-t} L_m^{(k)}(t) L_n^{(k)}(t)\,dt = \frac{(m+k)!}{m!}\delta_{mn}. \qquad (4.8)$$

In the Appendix we show how to calculate the matrix elements α_{mn} and α_{mn}'.

B. In complete analogy with the foregoing, we can investigate the term proportional to **V** in (3.4). In this case it is necessary to solve the equation

$$\sum_{m=1}^{\infty}(\alpha_{nm} + \alpha_{nm}')a_m + i\omega_e h\tau_e \frac{(n+3)!}{6n!}a_n = \alpha_{0n}' + \delta_{1n}\left(1 - \sum_{m=1}^{\infty} a_m\alpha_{0m}'\right), \qquad (4.9)$$

n = 1, 2, 3, ... The friction force and the heat flux, which are connected with **V**, are obtained from the formulas

$$\mathbf{R}_V^{(1)} = \frac{m_e^* n_e}{\tau_e}\sum_{m=1}^{\infty}\alpha_{0m}'(a_m^0\mathbf{V}_\parallel + a_m'\mathbf{V}_\perp + a_m''[\omega_e\mathbf{V}]),$$
$$\mathbf{q}_V = 4n_e T_e(a_1^0\mathbf{V}_\parallel + a_1'\mathbf{V}_\perp + a_1''[\omega_e\mathbf{V}]). \qquad (4.10)$$

The matrix elements α_{mn} and α_{mn}' are the same here as in (4.5).

C. Finally, in (3.4) it is necessary to investigate the term proportional to a shear-velocity tensor $W_{e\alpha\beta}$. It is interesting to note that when arbitrary temperatures are considered, there appears in the right-hand side of (3.4), besides the perturbing factor $\sim W_{e\alpha\beta}$, also an additional term $\sim\delta_{\alpha\beta}\,\mathrm{div}\,\mathbf{u}_e$, which corresponds to the presence of two viscosities in the dissipative liquid[17]. However, the second viscosity vanishes for both relativistic and ultrarelativistic temperatures. In the second case, the problem reduces to solution of the equation

$$I_{ee}(\Phi_e) + I_{ei}(\Phi_e) - \frac{m_e^* c}{p} f_e^{(0)}[\mathbf{p}\omega_e]\frac{\partial\Phi_e}{\partial\mathbf{p}} = \frac{2}{pm_e^* c}p_{\alpha\beta}W_{e\alpha\beta}f_e^{(0)}. \qquad (4.11)$$

If the magnetic field is directed along the z axis, then it is convenient to represent $W_{e\alpha\beta}$ as a sum of three tensors:

$$W_{e\alpha\beta} = W_{(0)\alpha\beta} + W_{(1)\alpha\beta} + W_{(2)\alpha\beta}.$$

In the chosen coordinate system, the tensor $W_{(0)\alpha\beta}$ is diagonal, and its components are

$$W_{(0)11} = W_{(0)22} = {}^1\!/_2(W_{exx} + W_{eyy}), \quad W_{(0)33} = W_{ezz}.$$

The tensors $W_{(1)\alpha\beta}$ and $W_{(2)\alpha\beta}$ have the following nonzero elements:

$$W_{(1)11} = -W_{(1)22} = {}^1\!/_2(W_{exx} - W_{eyy}), \quad W_{(1)12} = W_{(1)21} = W_{exy}$$

and

$$W_{(2)13} = W_{(2)31} = W_{exz}, \quad W_{(2)23} = W_{(2)32} = W_{eyz}.$$

It is necessary to introduce also the two tensors $W_{(3)\alpha\beta}$ and $W_{(4)\alpha\beta}$ with the following nontrivial components:

$$W_{(3)11} = -W_{(3)22} = -2\omega_e W_{exy}, \quad W_{(3)12} = W_{(3)21} = \omega_e(W_{exx} - W_{eyy})$$

and

$$W_{(4)13} = W_{(4)31} = -\omega_e W_{eyz}, \quad W_{(4)32} = W_{(4)23} = \omega_e W_{exz}.$$

These tensors have the property that the action of the operator $\mathbf{p}\times\mathbf{h}\nabla_p$ causes the term $W_{(0)\alpha\beta}p_{\alpha\beta}$ to vanish, and transforms the expressions for $W_{(1)\alpha\beta}p_{\alpha\beta}$ and $W_{(2)\alpha\beta}p_{\alpha\beta}$ into terms of the form $W_{(3)\alpha\beta}p_{\alpha\beta}$ and $W_{(4)\alpha\beta}p_{\alpha\beta}$, and vice versa.

We can now seek the solution of (4.11) in the form

$$\Phi_{W\alpha\beta} = -\frac{c^2}{T^2}\sum_{m=0}^{4} B^{(m)} W_{(m)ik} p_{ik}. \tag{4.12}$$

By introducing the complex quantities

$$B' = B^{(1)} + 2i\omega_e \mathbf{h} B^{(3)}, \quad B'' = B^{(2)} + i\omega_e \mathbf{h} B^{(4)}$$

we can obtain separately equations for the functions $B^{(0)}$, B', and B'', and it suffices to solve only the equation for B'', since then, we can find the solution for $B^{(0)}$ by putting $\omega_e = 0$, whereas the solution for the function B' can also be obtained by making the substitution $\omega_e \to 2\omega_e$:

$$I_{ee}(-B'' p_{\alpha\beta}) + I_{ei}(-B'' p_{\alpha\beta}) + \frac{m_e^* c}{p} i\omega_e \mathbf{h} f_e^{(0)} B'' p_{\alpha\beta} = \frac{1}{2}\frac{T_e}{pc} p_{\alpha\beta} f_e^{(0)}. \tag{4.13}$$

We seek again the solution of (4.13) in the form of a series in Sonine polynomials (in this case it is convenient to choose fifth-order polynomials):

$$B'' = \tau_e \sum_{m=0}^{\infty} b_m'' L_m^{(5)}. \tag{4.14}$$

The infinite system of algebraic equations for the determination of the coefficients B_m'' is

$$\sum_{m=0}^{\infty} (\beta_{nm} + \beta_{nm}') b_m'' + i\omega_e \mathbf{h}\tau_e \frac{8\cdot(n+5)!}{5!\,n!} b_n'' = \delta_{n0}, \quad n = 0, 1, 2, \ldots \tag{4.15}$$

where

$$\beta_{mn} = -\frac{\tau_e}{20 n_e}\frac{c^4}{T_e^4}\int p_{\alpha\beta} L_m^{(5)} I_{ee}(L_n^{(5)} p_{\alpha\beta})\, d\mathbf{p}, \tag{4.16}$$

and β_{mn}' is obtained from the analogous formula by replacing I_{ee} by the electron-ion collision term (the matrices β_{mn} and β_{mn}' are calculated in the Appendix). The viscous-stress tensor $\pi_{e\alpha\beta}$ is now given by

$$\pi_{e\alpha\beta} = -8\tau_e n_e T_e \sum_{m=0}^{4} b_0^{(m)} W_{(m)\alpha\beta}. \tag{4.17}$$

5. If we confine ourselves in the series (4.4) and (4.14) to the first few terms of the expansions, then we terminate in suitable fashion the infinite systems of algebraic equations (4.5), (4.9), and (4.15), which can now be solved in practice. Retaining the first two polynomials in (4.4) and (4.14), we can ultimately obtain expressions for the momentum $\mathbf{R}_e$ transferred in the collisions from the ions to the electrons, for the electron heat flux $\mathbf{q}_e$, and for the viscous-stress tensor $\pi_{e\alpha\beta}$:

$$\begin{aligned}\mathbf{R}_e = &-\alpha_\parallel \mathbf{V}_\parallel - \alpha_\perp \mathbf{V}_\perp + \alpha_\wedge[\mathbf{hV}] - \beta_\parallel \nabla_\parallel T_e - \beta_\perp \nabla_\perp T_e - \beta_\wedge[\mathbf{h}\nabla T_e] \\ &+ \frac{T_e}{3n_e}\beta_\parallel \nabla_\parallel n_e + \frac{T_e}{3n_e}\beta_\perp \nabla_\perp n_e + \frac{T_e}{3n_e}\beta_\wedge[\mathbf{h}\nabla n_e] - {}^1/_3\, e_e \beta_\parallel \mathbf{E}_\parallel - {}^1/_3\, e_e \beta_\perp \mathbf{E}_\perp \\ &- {}^1/_3\, e_e \beta_\wedge[\mathbf{hE}] - {}^1/_3\, \omega_e m_e^* \beta_\wedge \mathbf{u}_{e\perp} + {}^1/_3\, \omega_e m_e^* \beta_\perp[\mathbf{hu}_e],\end{aligned} \tag{5.1}$$

$$\begin{aligned}\mathbf{q}_e = &\lambda_\parallel \mathbf{V}_\parallel + \lambda_\perp \mathbf{V}_\perp + \lambda_\wedge[\mathbf{hV}] - \varkappa_\parallel \nabla_\parallel T_e - \varkappa_\perp \nabla_\perp T_e - \varkappa_\wedge[\mathbf{h}\nabla T_e] \\ &+ \frac{T_e}{3n_e}\varkappa_\parallel \nabla_\parallel n_e + \frac{T_e}{3n_e}\varkappa_\perp \nabla_\perp n_e + \frac{T_e}{3n_e}\varkappa_\wedge[\mathbf{h}\nabla n_e] - {}^1/_3\, e_e \varkappa_\parallel \mathbf{E}_\parallel - {}^1/_3\, e_e \varkappa_\perp \mathbf{E}_\perp \\ &- {}^1/_3\, e_e \varkappa_\wedge[\mathbf{hE}] - {}^1/_3\, \omega_e m_e^* \varkappa_\wedge \mathbf{u}_{e\perp} + {}^1/_3\, \omega_e m_e^* \varkappa_\perp[\mathbf{hu}_e],\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}&\alpha_\parallel = \frac{m_e^* n_e}{\tau_e}\alpha_0, \quad \alpha_\perp = \frac{m_e^* n_e}{\tau_e}\left(1 - \frac{\alpha_1' x_e^2 + \alpha_0'}{\Delta}\right), \\ &\alpha_\wedge = \frac{m_e^* n_e}{\tau_e}\frac{x_e(\alpha_1'' x_e^2 + \alpha_0'')}{\Delta}; \\ &\beta_\parallel = n_e\beta_0, \quad \beta_\perp = n_e\frac{\beta_1' x_e^2 + \beta_0'}{\Delta}, \quad \beta_\wedge = n_e\frac{x_e(\beta_1'' x_e^2 + \beta_0'')}{\Delta}, \\ &\lambda_\parallel = n_e T_e \lambda_0, \quad \lambda_\perp = n_e T_e\frac{\lambda_1' x_e^2 + \lambda_0'}{\Delta}, \quad \lambda_\wedge = n_e T_e\frac{x_e(\lambda_1'' x_e^2 + \lambda_0'')}{\Delta}, \\ &\varkappa_\parallel = \frac{n_e T_e \tau_e}{m_e^*}\gamma_0, \quad \varkappa_\perp = \frac{n_e T_e \tau_e}{m_e^*}\frac{(\gamma_1' x_e^2 + \gamma_0')}{\Delta} \\ &\varkappa_\wedge = \frac{n_e T_e \tau_e}{m_e^*}\frac{x_e(\gamma_1'' x_e^2 + \gamma_0'')}{\Delta};\end{aligned} \tag{5.3}$$

$$\Delta = x_e^4 + \delta_1 x_e^2 + \delta_0, \quad x_e = \omega_e \tau_e,$$

and the numerical coefficients are

$$\begin{aligned}&\alpha_0 = 0{,}8754, \quad \alpha_0' = 65{,}19, \quad \alpha_0'' = 14{,}06, \quad \alpha_1' = 3{,}000, \quad \alpha_1'' = 0{,}600, \\ &\beta_0 = 0{,}1672, \quad \beta_0' = 87{,}50, \quad \beta_0'' = 20{,}14, \quad \beta_1' = 3{,}487, \quad \beta_1'' = 0{,}750,\end{aligned} \tag{5.4}$$

$$\begin{aligned}&\gamma_0 = 0{,}7213, \quad \gamma_0' = 377{,}3, \quad \gamma_0'' = 92{,}25, \quad \gamma_1' = 12{,}75, \quad \gamma_1'' = 3{,}000, \\ &\lambda_0 = 0{,}4590, \quad \lambda_0' = 240{,}2, \quad \lambda_0'' = 56{,}62, \quad \lambda_1' = 9{,}000, \quad \lambda_1'' = 2{,}000, \\ &\delta_1 = 49{,}31, \quad \delta_0 = 523{,}3;\end{aligned} \tag{5.5}$$

$$\begin{aligned}&\pi_{e\alpha\beta} = -\eta_0^e W_{(0)\alpha\beta} - \eta_1^e W_{(1)\alpha\beta} - \eta_2^e W_{(2)\alpha\beta} + \eta_3^e (2\omega_e)^{-1} W_{(3)\alpha\beta} + \eta_4^e \omega_e^{-1} W_{(4)\alpha\beta}; \\ &\eta_0^e = 0{,}212 n_e T_e \tau_e, \quad \eta_2^e = n_e T_e \tau_e (4{,}80 x_e^2 + 187)/\Delta_1, \quad \eta_1^e = \eta_2^e(2x_e) \\ &\eta_4^e = n_e T_e \tau_e x_e (x_e^2 + 40{,}2)/\Delta_1, \quad \eta_3^e = \eta_4^e(2x_e), \quad \Delta_1 = x_e^4 + 63{,}8 x_e^2 + 882.\end{aligned} \tag{5.6}$$

All the formulas presented above were obtained for the case of singly charged ions $Z_i = 1$ (Z_i is the charge number). Of course, it is easy to consider the case $Z_i > 1$. Thus, it is seen from the formulas presented above that the dependence of the transport coefficients on the temperature becomes strongly altered in the case of ultrarelativistic plasma electrons. Finally, it is necessary to note that an essential change takes place in the system of hydrodynamic equations itself. In fact, by considering small average velocities of the plasma components (u_e, $u_i \ll c$), we can simplify the system (1.5), but we are still left with new terms, which are completely missing from the nonrelativistic theory[14]. Using Braginskiĭ's results[14], we can easily show that in the case of low temperatures these terms are indeed small (of the order of c^{-2}) and can be discarded. In the ultrarelativistic limit, however, they have the same order of magnitude as the remaining terms of the system of transport equations and must be retained (in this case the transport coefficients are estimated from formulas (5.3) and (5.6)).

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APPENDIX

It was necessary in the foregoing to calculate the integrals (4.6) and (4.16). α_{mn}' and β_{mn}' are calculated in elementary fashion, since they easily reduce to the form

$$\alpha_{mn}' = \frac{1}{2}\int_0^\infty t^2 e^{-t} L_m^{(3)} L_n^{(3)}\, dt, \quad \beta_{mn}' = \frac{3}{5}\int_0^\infty t^4 e^{-t} L_m^{(5)} L_n^{(5)}\, dt. \tag{A.1}$$

The calculation of the matrices α_{mn} and β_{mn} is somewhat more complicated. From (4.6) and (4.16) we can easily see that in the ultrarelativistic limit (p, $p' \gg m_e c$) the integrands tend to zero if the angle between the directions of the vectors $\mathbf{p}$ and $\mathbf{p}'$ is very small. Then, considering nonzero angles, the tensor U_{ik} can be greatly simplified by discarding terms of order $(m_e^* c/p)^2$ and above:

$$\frac{U_{ik}}{\gamma\gamma'} \approx \frac{1}{c}\frac{(pp' - \mathbf{pp}')\delta_{ik} + p_i p_k' + p_i' p_k}{pp'}. \tag{A.2}$$

Now the matrices α_{mn} and β_{mn} take the diagonal form

$$\begin{aligned}&\alpha_{00} = 0, \quad \alpha_{mn} = \frac{1}{4}\frac{(m+1)(m+3)!}{m!}\delta_{mn}, \\ &\beta_{00} = 24, \quad \beta_{mn} = \frac{1}{10}\frac{(m+2)(m+5)!}{m!}\delta_{mn},\end{aligned} \tag{A.3}$$

$m, n = 1, 2, 3, \ldots \quad (\alpha_{11} = 12,\ \alpha_{22} = 45,\ \alpha_{33} = 120, \ldots,\ \beta_{11} = 216, \ldots).$

Finally, we can write out the matrices α'_{mn} and β'_{mn}:

$$\alpha'_{mn}=\begin{vmatrix}1 & 1 & 1 & 1\ldots\\ 1 & 4 & 4 & 4\ldots\\ 1 & 4 & 10 & 10\ldots\\ 1 & 4 & 10 & 20\ldots\\ \ldots & & & \end{vmatrix},\quad \beta'_{mn}=\begin{vmatrix}14,4 & 14,4\ldots\\ 14,4 & 86,4\ldots\\ \ldots & \end{vmatrix}. \qquad (A.4)$$

*$[u_a H] = u_a \times H$.

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