Group Velocity and the macrocausality conditions for the "forward" scattering amplitude

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Several new limitations on the "forward" scattering amplitude imposed by macrocausality requirements are formulated on the basis of an analysis of the propagation of a wave packet in a macroscopic medium. It is shown, in particular, that for particles with a zero rest mass the real part of the scattering amplitude at zero energy cannot be positive if the ratio $\mathrm{Im}\,a(\kappa)/\mathrm{Re}\,a(\kappa)$ tends to zero as $\kappa \to 0$.

1. STATEMENT OF THE PROBLEM

Any situations in which the propagation of signals with a velocity greater than the velocity of light in vacuum is possible must be completely excluded from a consistent, relativistic, causal theory. As is well-known, this imposes definite restrictions on the properties of the S-matrix. In particular, the dispersion relations for the two-particle scattering amplitude follow from the principle of causality.

In the present article certain additional inequalities for the "forward" scattering amplitude are derived on the basis of a causal analysis of the propagation of wave packets over macroscopic distances; these inequalities have apparently not been previously discussed. The structure of the article is as follows.

a) The passage of wave packets, corresponding to particles A, through a layer of a macroscopic medium, consisting of particles B, is investigated. The index of refraction, which is expressed in terms of the amplitude for elastic coherent scattering A + B → A + B at zero angle in the laboratory coordinate system,[1,2] is introduced in order to describe the coherent interaction of the passing wave with the rarefied medium. The index of refraction determines the dispersion law in the medium; accordingly the group velocity of the wave packet in the medium[3,4] also depends on the "forward" scattering amplitude.

b) In a weakly-absorbing rarefied macroscopic medium, composed of physical particles B, the group velocity of the wave packet A must be smaller than the velocity of light in vacuum. Specific restrictions on the amplitude for the elastic scattering of massless particles, and also for particles with nonzero mass in the ultrarelativistic limit, follow from this requirement.

In a situation when the events corresponding to the creation and detection of the particles are spatially separated, the propagation of a wave packet in the medium corresponds to the transmission of a signal.[1)] The local inequalities for the real part of the "forward" amplitude, which are derived in the present article, are related to the impossibility of transmitting such signals over macroscopic distances with a velocity faster than light. Therefore, one can regard them as macrocausality conditions. The most definite predictions pertain to the case of the low-energy scattering of particles with zero mass.

2. THE GROUP VELOCITY AND THE MOTION OF THE CENTER OF THE PACKET IN A DISPERSIVE MEDIUM

Let us consider from a formal point of view the propagation of a wave packet of arbitrary nature in the presence of dispersion. Let us denote the wave vector, corresponding to the plane wave, by **k**, and the frequency by ω. The dispersion law determines the function $\omega(k)$, where $k = |\mathbf{k}|$. We shall assume that damping of the wave doesn't occur or else it can be neglected. In this case $\mathrm{Im}\,\omega(k) = \mathrm{Im}\,k = 0$.

Let us represent the wave packet in the form of a superposition of waves with definite values of ω and **k**:

$$\psi(\mathbf{r}, t) = \int c(\mathbf{k})\, e^{i\mathbf{k}\mathbf{r}} e^{-i\omega(k)t} d^3\mathbf{k}. \tag{1}$$

Let us impose the normalization condition

$$\int |\psi(\mathbf{r}, t)|^2 d^3\mathbf{r} = (2\pi)^3 \int |c(\mathbf{k})|^2 d^3\mathbf{k} = 1. \tag{2}$$

on the function $\psi(\mathbf{r}, t)$. Let us introduce the coordinate of the center of the packet,

$$\mathbf{R} = \int |\psi(\mathbf{r}, t)|^2 \mathbf{r}\, d^3\mathbf{r}. \tag{3}$$

Substituting (1) into (3) and taking the equality

$$\mathbf{r} e^{i\mathbf{k}\mathbf{r}} = -i \frac{\partial}{\partial \mathbf{k}} e^{i\mathbf{k}\mathbf{r}},$$

into consideration, after elementary calculations we obtain

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{v}t, \tag{4}$$

where

$$\mathbf{R}_0 = (2\pi)^3\, \mathrm{Im} \int c(\mathbf{k}) \frac{\partial}{\partial \mathbf{k}} c^*(\mathbf{k})\, d^3\mathbf{k}, \tag{5a}$$

$$\mathbf{v} = (2\pi)^3 \int |c(\mathbf{k})|^2 \frac{\partial \omega(k)}{\partial \mathbf{k}} d^3\mathbf{k}. \tag{5b}$$

One can rewrite Eq. (5b) in the form

$$\mathbf{v} = \overline{(\partial\omega(k)/\partial\mathbf{k})}, \tag{6}$$

where the bar denotes averaging with respect to **k**. If we are dealing with a quasi-monochromatic packet (the effective spread $|\Delta\mathbf{k}|$ is very small in comparison with the average value of the wave number k_0), then according to Eq. (6)

$$\frac{d\mathbf{R}}{dt} = \frac{\partial\omega(k_0)}{\partial \mathbf{k}_0} = \frac{d\omega(k_0)}{dk_0}\mathbf{n} \tag{7}$$

where **n** is a unit vector in the direction of $\mathbf{k}_0$.

By definition, the quantity $d\omega(k_0)/dk_0$ is nothing other than the group velocity. Thus, the group velocity coincides with the velocity of motion of the center of the quasi-monochromatic packet. It is obvious that formulas (5) are valid even when the spreading of the packet is taken into account.[2)]

If the quantum mechanical state of particle A is described by a superposition of states with definite momentum $\mathbf{p} = \hbar\mathbf{k}$ and energy $E = \hbar\omega$, then in the coordinate representation a given particle corresponds to the probability packet $\psi(\mathbf{r}, t)$, which is defined according to Eq. (1). Here the function $\psi(\mathbf{r}, t)$ is proportional to the amplitude of the probability to detect particle A at the instant of time t by a detector located at the point $\mathbf{r}$.[3)]

In the case of the motion of A particles in the macroscopic medium, which is constructed out of B particles at rest (we assume that if the B particles exist, then in principle such a medium can be realized), the relation between the momentum $\hbar \mathbf{k}$ and the energy $\hbar\omega$ has the form

$$k=\varkappa n(\varkappa), \tag{8}$$

where $\hbar\kappa = c^{-1}\hbar[\omega^2 - (mc^2/\hbar)^2]^{1/2}$ is the momentum of particle A in vacuum, m is the mass of particle A, c is the velocity of light in vacuum, and $n(\kappa)$ is the index of refraction. For sufficiently small densities N of the B particles, the index of refraction is related to the coherent "forward" scattering amplitude $a(\kappa)$ (for the elastic scattering of particle A by particle B in the laboratory coordinate system) by the well-known relation[1,2]

$$n(\varkappa)=1+2\pi N\frac{a(\varkappa)}{\kappa^2}. \tag{9}$$

In the general case $a(\kappa)$ denotes the scattering amplitude at zero angle without any change of the internal quantum numbers of the two particles A and B.[4)]

It is significant that formula (9) is valid for arbitrary values of κ, provided that the index of refraction does not differ very much from unity. In what follows we shall regard the density N as a subsidiary parameter. One can always choose N to be small enough so that the utilization of formula (9) will be valid. Here

$$\begin{aligned}\operatorname{Re} k&=\varkappa+2\pi N\frac{\operatorname{Re}a(\varkappa)}{\varkappa},\\ \operatorname{Im} k&=2\pi N\frac{\operatorname{Im} a(\varkappa)}{\varkappa}=\frac{N\sigma(\varkappa)}{2},\end{aligned} \tag{10}$$

where $\sigma(\kappa)$ is the total cross section for the interaction of particles A and B.

If the thickness of the layer, consisting of B particles, is small in comparison with the average absorption length $L_0 = 1/N\sigma(\kappa)$ and if $\operatorname{Im} a(\kappa) \ll |\operatorname{Re} a(\kappa)|$, one can neglect the imaginary part of the wave number k, thus assuming the medium to be transparent. In this connection the group velocity (the velocity of motion of the center of the quasi-monochromatic packet in the medium) is, according to formula (7), given by

$$v=\frac{d\omega(k)}{d(\operatorname{Re}k)}=\frac{d\omega(\varkappa)}{d\varkappa}\Big/\frac{d(\operatorname{Re}k)}{d\varkappa}=v_0[1+2\pi N\mathscr{R}]^{-1}, \tag{11}$$

where $v_0 = \kappa c^2/\omega(\kappa)$ is the classical velocity of particle A; here and below we shall use the following notation in order to simplify writing down the formulas:

$$\mathscr{R}=\frac{d}{d\varkappa}\left(\frac{\operatorname{Re}a(\varkappa)}{\varkappa}\right). \tag{12}$$

For massless particles we have

$$v=c(1+2\pi N\mathscr{R})^{-1}\approx c[1-2\pi N\mathscr{R}]. \tag{13}$$

In the absence of attenuation (actually, for very small cross sections and correspondingly large values of the absorption length L_0) the group velocity of the packet in the dispersive medium must coincide with the velocity of the signal's propagation and, according to the principle of relativistic causality, cannot exceed the velocity of light in vacuum (see[3,4] and also Sec. 3). From the inequality $v \leq c$ and allowing for (13), we obtain the following restriction on the coherent "forward" scattering amplitude for particles with zero rest mass:

$$\frac{d}{d\varkappa}\left(\frac{\operatorname{Re}a(\varkappa)}{\varkappa}\right)^{\sigma\to 0}\geqslant 0. \tag{14}$$

The superscript on the left-hand side of Eq. (14) indicates that generally this inequality is valid if the total cross section $\sigma(\kappa') = 4\pi\kappa'^{-1} \operatorname{Im} a(\kappa')$ is very small in the neighborhood $\kappa - \Delta\kappa < \kappa' < \kappa + \Delta\kappa$. In the ultrarelativistic limit one can also write down this same relation in the case when the mass of particle A isn't equal to zero.

According to its derivation, inequality (14) has a limited character. The quantity R may even assume negative values for large scattering cross sections. In particular, one always has $R < 0$ in the neighborhood of a resonance peak. One can, however, anticipate that these negative values are bounded in absolute value, where the corresponding upper limit is proportional to the total scattering cross section $\sigma(\kappa)$. We shall consider this question in more detail in the next section.

3. THE MACROCAUSALITY CONDITIONS FOR MASSLESS PARTICLES

Let us consider the motion of particles with zero rest mass in a macroscopic medium from the viewpoint of the principle of relativistic causality. Let the initial longitudinal length of the packet, corresponding to this particle (the uncertainty of the coordinate along the direction of motion), be equal to Δz. According to the uncertainty relation, $\Delta z = (\overline{z^2} - (\overline{z})^2)^{1/2} \gtrsim 1/\Delta k$, where $\hbar\Delta k = (\overline{p^2} - (\overline{p})^2)^{1/2}$ is the uncertainty in the particle's momentum. In the present case the time required for the signal to propagate over the distance L is equal to the difference between the instants corresponding to the detection and the creation of the particle, and on the average this time interval is equal to L/v, where v is the velocity of the center of the packet (the group velocity). However, in principle the indicated time interval can only be determined to within an accuracy

$$\Delta t\sim\alpha\left[\left(\frac{\Delta z}{v}\right)^2+\left(\frac{L\Delta v}{v^2}\right)^2\right]^{1/2},$$

which is dictated by the uncertainty relation. Here $\Delta v \approx |d^2\omega(k)/dk^2|\Delta k$ is the uncertainty in the particle's velocity and α is a positive constant of the order of unity; the second term in the expression for Δt corresponds to the spreading of the packet during its motion in the dispersive medium. According to the principle of causality, the quantity $\tau = (L/v) - (L/c)$, having the meaning of the time delay associated with the motion of the packet in the medium in comparison with the time of flight of a massless particle in vacuum, must satisfy the inequality $\tau > -\Delta t$. With this taken into consideration, the macrocausality condition can be written in the form

$$\frac{v-c}{c}<\alpha\left[\left(\frac{\Delta z}{L}\right)^2+\left(\frac{\Delta v}{v}\right)^2\right]^{1/2}. \tag{15}$$

If we consider quasi-monochromatic packets for which

$$\frac{\Delta v}{v}\sim\left|\frac{d^2\omega(k)}{dk^2}\right|\Delta k\Big/\left|\frac{d\omega(k)}{dk}\right|\ll\left|\frac{c-v}{v}\right|,$$

the term $(\Delta v/v)^2$ appearing on the right-hand side of inequality (15), which is related to the packet's spreading, can be dropped. For completely transparent media the ratio $\Delta z/L$ can be arbitrarily small, and the result $v \leq c$ follows from (15).[5)]

The course of the discussions cited above isn't changed even in the presence of absorption if we consider the distances L to be small in comparison with the absorption length L_0. The inequality (15) is valid in any case, provided that $\Delta z \ll L \ll 1/N\sigma(\kappa)$. At the same time, if $L \gg 1/N\sigma(\kappa)$, the probability that the particle will pass through the layer of matter becomes exponentially small, and it is no longer correct to associate the transmission of some kind of physical signal with the propagation of the probability packet. Accordingly, we shall assume that in formula (15) the quantity $L \leq 1/N\sigma(\kappa)\beta$, where $\beta \sim 1$, and the quantity $\Delta z \geq 1/\Delta\kappa$, where $\Delta\kappa$ is the uncertainty in the wave number. Substituting formula (13) for the group velocity into (15), we obtain the inequality

$$\mathcal{R} > -\eta\sigma(\varkappa)/\Delta\varkappa, \qquad (16)$$

where $\eta = \alpha\beta/2\pi > 0$;[6] this inequality goes over into the relation (14) if we formally set $\sigma(\kappa) = 0$. The quantity $\Delta\kappa$, of course, takes different values for different packets. It is clear, however, that the relation between the real and imaginary parts of the scattering amplitude can be formulated independently of the type of packets under consideration. It is clear that a certain characteristic interval $\Delta\kappa_0$ of wave numbers in the neighborhood of κ must play the role of the maximum value of the quantity $\Delta\kappa$ in inequality (16); in this interval the real and imaginary parts of the scattering amplitude and their derivatives do not change in order of magnitude—provided that this interval is smaller than κ. If the indicated interval exceeds κ, one must assume $\Delta\kappa_0 \sim \kappa$.[7]

Having introduced the notation $\Delta\kappa_0 = \kappa/\gamma$, let us rewrite (16) in the form

$$\mathcal{R} > -C\sigma(\varkappa)/\varkappa, \qquad (17)$$

where $C = \eta\gamma > 0$ is a dimensionless finite number. The approach discussed here doesn't permit us to specifically indicate the lower limit of the possible values of the ratio $\mathcal{R}\kappa/\sigma(\kappa)$.

4. THE SIGN OF THE SCATTERING AMPLITUDE FOR MASSLESS PARTICLES AT ZERO ENERGY

According to (17), if

$$\varkappa|\mathcal{R}|/\sigma(\varkappa) > C, \qquad (18)$$

then inequality (14) must certainly be satisfied, that is, $\mathcal{R} > 0$ (see Eq. (12) for the definition of the symbol $\mathcal{R}$). This enables us to reach a definite conclusion about the sign of the real part of the scattering amplitude at zero energy. In fact, let the cross section $\sigma(\kappa)$ remain finite or else behave like κ^{-q} as $\kappa \to 0$, where $0 < q < 1$. It follows from the optical theorem that in this case $\mathrm{Im}\, a(0) = 0$. It is clear that if $\mathrm{Re}\, a(0) \neq 0$, then

$$\lim_{\varkappa\to 0} \varkappa|\mathcal{R}|/\sigma(\varkappa) = \infty.$$

At the same time the quantity C is a finite number (for values of κ close to zero, we find $\gamma = \kappa/\Delta\kappa_0 \sim 1$ and $C = \eta\gamma \sim 1$). Therefore, for sufficiently small values of κ the quantity R must be positive. In other words, relation (14), which follows from the fact that the group velocity in a transparent medium is bounded, is valid at energies close to zero. Hence follows the result

$$\mathrm{Re}\, a(0) \leqslant 0. \qquad (19)$$

We see that an important result follows from the principle of causality: the real part of the amplitude for the "forward" scattering of a particle with zero rest mass by any other particle cannot be positive, provided that the imaginary part of the scattering amplitude tends to zero as $\kappa \to 0$.[8]

As is well-known, in the low-frequency limit the amplitude for the "forward" scattering of a photon (without any change in its polarization) by any particle is equal to $(-e^2/mc^2)$, where e is the particle charge and m is its mass.[9] In connection with what has been said, it becomes clear that the negative sign of the Thomson amplitude is not accidental; a positive sign of the amplitude would contradict the principle of causality. From our point of view, the inequality (19) must automatically be satisfied in any internally consistent dynamical theory of massless particles (independently of its specific structure), provided that

$$\lim_{\varkappa\to 0} \frac{\mathrm{Im}\, a(\varkappa)}{\mathrm{Re}\, a(\varkappa)} = 0$$

For example, the amplitudes for the scattering of neutrino and antineutrino by an electron vanish as $\kappa \to 0$ within the framework of the universal theory of the weak interactions. As one can easily show in scalar and pseudoscalar theories with direct coupling (characterized by the interaction Lagrangians $L_{sc} = g\varphi\bar{\psi}\psi$ and $L_{ps} = g\varphi\bar{\psi}\gamma_5\psi$), the amplitudes for the zero-energy scattering of massless bosons at an angle θ are equal, respectively, to $a_{sc}(0, \theta) = -(g^2/mc^2)\cos\theta$ and $a_{ps}(0, \theta) = -g^2/mc^2$, which again agrees with (19).

Let us emphasize that we are everywhere talking about physical states B, which in principle can be realized in nature. By definition such states should be stable with respect to the creation of quanta A with zero momentum (the vacuum remains stable in the external field created by a physical particle). The indicated stability obviously also exists for a medium composed of B particles—in any event for sufficiently small densities N. The result (19) indicates that no physical, stable states B exist which would correspond to a positive scattering amplitude for massless particles at zero energy; otherwise the group velocity in a medium consisting of such particles would exceed the velocity of light in vacuum. It is significant that this conclusion follows from the principle of causality and is not related to specific field-theory concepts.

From the point of view of field theory, the positive amplitude may be the "bare" amplitude, but not the renormalized physical amplitude. One can verify this for the example of the interaction of a massless scalar particle A with a particle B whose mass is not zero. One can easily see that if $a(0) > 0$ (we omit the symbol indicating the real part of the amplitude, since it is understood that $\mathrm{Im}\, a(0) = 0$), the "bare" one-particle state B is unstable: in the field B corresponding to an effective attraction, an infinite number of quanta A with zero momenta are created, and their total energy is negative and tends to $(-\infty)$. Stability is reestablished if a nonlinear interaction of the type $V = (1/4)\lambda A^4$ is considered. In this connection the renormalized physical state $B' = B +$ condensate A appears (cf.[10]). It is clear that the renormalized physical amplitude can only be negative (or vanish in the limiting case); otherwise the state B' would again turn out to be unstable. In the theory of massless scalar particles with the additional interaction Hamiltonian $V = (1/4)\lambda A^4$, one can show that the renormalized physical amplitude associated with the "bare" amplitude $a(0) > 0$ is equal to $(-2a(0))$ to within terms of order λ.[9] The detailed discussion of this question falls outside the scope of the present article.

5. THE CASE OF NONVANISHING REST MASS

One can also apply the method expounded above to the case when the mass is not equal to zero. Since the entire investigation pertains to a rarefied medium for which $N|a(\kappa)|/\kappa^2 \ll 1$, the macrocausality condition (15) does not impose any restrictions on the behavior of the "forward" scattering amplitude for $\kappa \lesssim mc/\hbar$. In particular, in contrast to (19) the sign of Re a(0) can be arbitrary. As one can easily see, in the ultrarelativistic limit ($\kappa \gg mc/\hbar$) the restrictions on the "forward" scattering amplitude reduce to the inequalities (14) and (16). In this connection if the amplitude does not oscillate, one can assume that $\gamma = \kappa/\Delta\kappa \sim 1$. Hence follows the asymptotic inequality

$$\varkappa\mathscr{R}/\sigma(\varkappa) > -C, \tag{20}$$

where C is a positive constant of the order of unity whose value does not depend on the energy.

In connection with the inequality (20), it is interesting to discuss the situation which arises when the Pomeranchuk theorem is violated. It is known that if asymptotic equality of the total scattering cross sections for particles b and antiparticles $\bar{b}$ does not hold, i.e., $\sigma(\infty) = \sigma_b \neq \tilde{\sigma}(\infty) = \sigma_{\bar{b}}$, then it follows directly from the analytic properties of the amplitude and from the requirements of crossing symmetry (see, for example, Chap. 8 of [11]) that

$$\operatorname{Re} a_b(\varkappa) = -\operatorname{Re} a_{\bar{b}}(\varkappa) = -\frac{\sigma'}{2\pi^2}\varkappa \ln \varkappa, \tag{21}$$

as $\kappa \to \infty$, where $\sigma' = (1/2)(\sigma(\infty) - \tilde{\sigma}(\infty))$. We note that in the present case the ratio Im $a(\kappa)$/Re $a(\kappa)$ asymptotically tends to zero for both particles and antiparticles. Therefore, at first glance one can neglect absorption and use relation (14). It is obvious that one of the amplitudes (21) does not satisfy this inequality: in the limit of very large energies the group velocity (11) for b particles with $\sigma' > 0$ or for $\bar{b}$ antiparticles with $\sigma' < 0$ exceeds the velocity of light in vacuum ($v \to c(1 + N|\sigma'|/\pi\kappa)$). This result suggests that violation of the Pomeranchuk theorem contradicts the principle of causality. This conclusion would actually be valid if the use of the macrocausality condition (15) were valid up to distances L many times exceeding the average absorption length $L_0 = 1/N\sigma(\kappa)$. But for $L \gg L_0$ the analysis of signal transmission with the aid of a wave packet apparently loses its meaning (see section 3). One can easily see that the amplitudes (21) satisfy inequality (20) which is weaker than (14): as $\kappa \to \infty$ we find

$$\varkappa\mathscr{R}/\sigma(\varkappa) \to \pm\frac{1}{2\pi^2}\frac{|\sigma'|}{\sigma(\infty)} \geqslant -\frac{1}{4\pi^2}.$$

Thus, the question of a possible connection between asymptotic equality of the total cross sections for the scattering of particles and antiparticles and the principle of causality still remains open.

6. MACROCAUSALITY CONDITIONS AND DISPERSION RELATIONS

Let us show that the restrictions (14) and (16) are in agreement with the dispersion relations for the amplitudes describing the "forward" scattering of a photon by an arbitrary particle, where these dispersion relations have the form

$$\operatorname{Re} a_\lambda(\varkappa) = \operatorname{Re} a_\lambda(0) + \varkappa \operatorname{Re} a_\lambda'(0) + \frac{\varkappa^2}{4\pi^2} P\int_0^\infty \frac{\sigma_\lambda(y)}{y(y-\varkappa)}dy + \frac{\varkappa^2}{4\pi^2}\int_0^\infty \frac{\sigma_{-\lambda}(y)}{y(y+\varkappa)}dy. \tag{22}$$

Here $\sigma_\lambda(y)$ is the total cross section for the scattering of a photon with helicity λ by a target with a given polarization at an energy $\hbar c\kappa$ ($\lambda = \pm 1$); Re $a_{+1}(0)$ = Re $a_{-1}(0) = -e^2/mc^2$ is a subtraction constant which is equal to the Thomson amplitude; Re $a'_{+1}(0)$ = −Re $a'_{-1}(0)$ is a second subtraction constant which is proportional to the square of the anomalous magnetic moment of the scatterer.[9]

If we formally set $\sigma_\lambda(\kappa) = 0$, the integrand in the third term of formula (22) will not have a singularity at $y = \kappa$. In this case one can differentiate the integral (which is defined in the sense of a principal value) with respect to the parameter κ like an ordinary integral,[12] and in the approximation under consideration we obtain

$$\frac{d}{d\varkappa}\left(\frac{\operatorname{Re} a_\lambda(\varkappa)}{\varkappa}\right)^{(\sigma=0)} = -\frac{\operatorname{Re} a_\lambda(0)}{\varkappa^2} + \frac{1}{4\pi^2}\left[P\int_0^\infty \frac{\sigma_\lambda(y)\,dy}{(y-\varkappa)^2} + \int_0^\infty \frac{\sigma_{-\lambda}(y)\,dy}{(y+\varkappa)^2}\right]. \tag{23}$$

Since $\sigma(y) > 0$ and Re $a_\lambda(0) \leq 0$, the right-hand side of Eq. (23) is an intrinsically positive quantity. Thus, if one can neglect the total scattering cross section at a given energy, then the photon dispersion relations lead directly to the inequality (14).

The exact formula for the derivative

$$\mathscr{R}_\lambda = \frac{d}{d\varkappa}\left(\frac{\operatorname{Re} a_\lambda(\varkappa)}{\varkappa}\right)$$

(compare with the notation introduced in (12)), with the identity

$$P\int_{\varkappa-\Delta\varkappa_0}^{\varkappa+\Delta\varkappa_0} \frac{dy}{y-\varkappa} = 0$$

taken into account, takes the form

$$\mathscr{R}_\lambda = -\frac{\operatorname{Re} a_\lambda(0)}{\varkappa^2} + \frac{1}{4\pi^2}\int_0^{\varkappa-\Delta\varkappa_0} \frac{\sigma_\lambda(y)\,dy}{(y-\varkappa)^2}$$

$$+\frac{1}{4\pi^2}\int_{\varkappa+\Delta\varkappa_0}^\infty \frac{\sigma_\lambda(y)}{(y-\varkappa)^2}dy + \frac{1}{4\pi^2}\int_0^\infty \frac{\sigma_{-\lambda}(y)}{(y+\varkappa)^2}dy \tag{24}$$

$$+\frac{1}{4\pi^2}\int_{\varkappa-\Delta\varkappa_0}^{\varkappa+\Delta\varkappa_0} \frac{\sigma_\lambda(y) - \sigma_\lambda(\varkappa) - (y-\varkappa)\,d\sigma_\lambda(\varkappa)/d\varkappa}{(y-\varkappa)^2}dy - \frac{\sigma_\lambda(\varkappa)}{2\pi^2\Delta\varkappa_0},$$

where $\Delta\kappa_0$ is any interval which satisfies the condition $0 \leq \Delta\kappa_0 \leq \kappa$. The fourth integral in formula (24) can be represented in the form $\Delta\kappa_0 d^2\sigma(\xi)/d\xi^2$, where $\kappa - \Delta\kappa_0 \leq \xi \leq \kappa + \Delta\kappa_0$. For sufficiently small values of $\Delta\kappa_0$, the quantity

$$\left|\frac{d^2\sigma(\xi)}{d\xi^2}\right| \Delta\varkappa_0^2 < \sigma(\varkappa),$$

and it follows from Eq. (24) that

$$\mathscr{R} > -\frac{1}{2\pi^2}\frac{\sigma(\varkappa)}{\Delta\varkappa_0}\left(1 + O\left(\left|\frac{d^2\sigma(\xi)}{d\xi^2}\right|\Delta\varkappa_0^2/\sigma(\varkappa)\right)\right), \tag{25}$$

which agrees with (16) (here $\eta \approx 1/2\pi^2$). We also note the integral inequality

$$\mathscr{R}_\lambda > \frac{1}{4\pi^2}\int_0^{2\varkappa} \frac{\sigma_\lambda(y) - \sigma_\lambda(\varkappa) - (y-\varkappa)\,d\sigma_\lambda(\varkappa)/d\varkappa}{(y-\varkappa)^2}dy - \frac{\sigma_\lambda(\varkappa)}{2\pi^2\varkappa}, \tag{26}$$

which becomes obvious if one substitutes $\Delta\kappa_0 = \kappa$ into Eq. (24).

For arbitrary particles with zero mass and spin s, one should replace $\sigma_{-\lambda}(y)$ by $\tilde{\sigma}_{-\lambda}(y)$ in the dispersion relations (22) and in formulas (23) and (24), where $\tilde{\sigma}$ is the total cross section for the scattering of the antiparticle and $\lambda = \pm s$. We note that if one starts from the dispersion relations with a single subtraction:

$$\operatorname{Re} a_\lambda(\varkappa) = \operatorname{Re} a_\lambda(0) + \frac{\varkappa}{4\pi^2}P\int_0^\infty \frac{\sigma_\lambda(y)}{y-\varkappa}dy - \frac{\varkappa}{4\pi^2}\int_0^\infty \frac{\tilde{\sigma}_{-\lambda}(y)}{y+\varkappa}dy, \tag{27}$$

formulas (23) and (24) for the quantities $\mathscr{R}_\lambda$ do not undergo any changes.

As shown in Sec. 4, for a finite cross section $\sigma_\lambda(0)$ the subtraction constant Re $a_\lambda(0)$ must be negative or else equal to zero. Let us emphasize that, for Re $a_\lambda(0) > 0$, the dispersion relations (22) and (27) do not satisfy the macrocausality condition (16) for sufficiently small values of κ. Thus, the mere analyticity of the scattering amplitudes in general still doesn't imply that the theory is automatically causal.

It is interesting that the requirement Re $a_\lambda(0) \leq 0$ imposes a prohibition on the dispersion relations without subtractions. In actual fact, the opposite inequality (see[13], Chap. 10, Sec. 1) follows from the corresponding dispersion relations which are valid provided that $\lim_{\kappa\to\infty} |a_\lambda(\kappa)| = 0$:

$$\operatorname{Re} a_\lambda(0) = \frac{1}{4\pi^2}\int_0^\infty (\sigma_\lambda(y) + \sigma_{-\lambda}(y))\,dy > 0.$$

This means that in a causal theory the "forward" scattering amplitude, which is analytic in the energy, for a massless particle must satisfy the asymptotic relation

$$\lim_{\kappa\to\infty} |a_\lambda(\kappa)| \neq 0. \tag{28}$$

7. CONCLUDING REMARKS

The inequalities (14), (16), (19), and (20), which impose definite restrictions on the behavior of the "forward" scattering amplitude have been derived above on the basis of the principle of macrocausality. On the whole these restrictions turn out to be substantially weaker than those which follow from analyticity and crossing symmetry. But at the same time they are even more general, since any internally consistent theory, in particular, a nonlocal theory, must be compatible with the requirement of macrocausality.

It is significant that the indicated inequalities contain a certain amount of additional information about the properties of the scattering amplitudes. This is apparently due to the fact that the coherence effect, due to the interference of the incident and scattered waves, is investigated in the macroscopic approach; this effect doesn't appear in the usual analysis of scattered waves, corresponding to an elementary event involving the interaction of two given particles (see, for example,[11], Chap. 4).

Let us say a few words about formula (9) for the index of refraction, which is used in the present article. For arbitrary values of the parameter $N\lambda^3$ ($\lambda = 2\pi/\kappa$), the mechanism leading to the formation of a coherent wave in the medium consists in the interference of the incident wave with the secondary waves which are scattered in a "forward" direction without momentum transfer by all the particles in the medium. This implies the universal nature of formula (9) in the approximation which is linear in the "forward" scattering amplitude, which is verified by direct calculations within the framework of scattering theory (see[1,2] and Chap. 11 of[13]).10) Therefore, there is every reason to assume that the application of relation (9) and formulas (11) and (13), which are related to it, to the problem of causality is completely justified.

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1)Let us emphasize that the topic of discussion is ordinary particles with mass $m \geqslant 0$; our investigation is not applicable to the hypothetical tachyons,[5] which possess a space-like 4-momentum in vacuum (corresponding to an imaginary mass).

2)A similar treatment of the group velocity was considered earlier by Vaĭnshteĭn[6] within the framework of classical electrodynamics.

3)We note that it is theoretically impossible to localize a photon or any other particle with zero rest mass without absorption in a spatial region having linear dimensions smaller than the wavelength.[7] In this case $|\psi(\mathbf{r}, t)|^2$ determines the relative probability of detecting the absorption of particle A by the heavy particle which is located at the point $\mathbf{r}$.

4)In particular, if the spin structure of the "forward" scattering amplitud is described by the matrix $\hat{A}(\kappa)$ in the spin space of the two particles, then the eigenvalues of the matrix $\hat{a}(\kappa) = \mathrm{Tr}_B \hat{A}(\kappa)\hat{\rho}_B$ play the role of $\hat{a}(\kappa)$ in formula (9) where $\hat{\rho}_B$ is the spin density matrix for particle B (see, for example,[8]).

5)It is well-known that it is impossible to define a probability packet with a sharp space-time front by using positive-energy states, which hinders the unambiguous formulation of the concept of a signal in quantum theory. We assume that the dimensions of the packet are are determined by finite fluctuations of the coordinates, and we neglect the probability of detecting the exponentially small tail of the packet. Such an approach to the investigation of macrocausality conditions satisfies the correspondnec principle with classical theory.

6)Here we start from the relation $(v-c)/c < \alpha\Delta z/L_{max}$, neglecting (in accordance with what was said earlier) the term $(\Delta v/v)^2 \sim (2\pi N\Delta\kappa d\mathscr{R}/d\kappa)^2$ in expression (15).

7)In the case of resonance scattering $\Delta\kappa_0 \sim \Gamma$, where Γ is the width of the resonance peak.

8)In the present case the fact that formula (13) for the group velocity immediately becomes incorrect when $\kappa = 0$ is of no significance in view of the continuity of the amplitude $a(\kappa)$ and in connection with the arbitrariness in the choice of the parameter N.

9)In the case of bosons with mass, the role of the interaction $V = (1/4)\lambda A^4$ has been discussed in detail in the articles by Migdal.[10] According to[10], if $m_A \neq 0$ and $a(0) > 0$, the medium consisting of B particles will be stable for densities N below a certain critical value N_0, but for $N > 0$ it becomes unstable with respect to the creation of A particles with zero momentum. An interaction of the type $V = (1/4)\lambda A^4$ leads to the elimination of the instability; in this connection a boson condensate of the A particles is formed in the medium. If $m_A \to 0$ and $a(0) > 0$, the medium is unstable for arbitrarily small densities N (formally $N_0 \to 0$ in this case). Let us emphasize, however, that for zero rest mass the given instability of the medium is, as indicated above, simply related to the instability of the particles themselves.

10)Formula (9) lies at the foundation of the theory of the coherent regeneration of short-lived neutral K mesons at high energies.[14] See, for example,[8, 15] for a discussion of other interference phenomena in elementary particle physics which are related to the concept of the index of refraction.

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134