Electrodynamics of inhomogeneous type-II superconductors

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(Submitted June 20, 1973)

Zh. Eksp. Teor. Fiz. **65**, 1704–1714 (October 1973)

A broad range of electric and magnetic fields, in which the current is a universal function of the ratio of the electric field strength to the critical current, exists in type-II superconductors possessing weak inhomogeneities of the effective interaction between electrons, of the mean free path, and of the film thickness. The dependence of the critical current on magnetic fields strength and temperature in this region is determined.

1. INTRODUCTION

A periodic vortex structure arises in type-II superconductors in sufficiently strong magnetic fields.[1] In homogeneous superconductors, such a structure possesses a finite resistance which, in weak electric fields, does not depend on the value of the electric field. Inhomogeneities retard the motion of the vortex lattice, and this leads to a significant change in the volt-ampere characteristic.

Inhomogeneities in a superconductor are considered below in a fashion similar to what was done previously by us:[2,3] the inter-electron interaction constant and the path length of the electrons are assumed to be random functions of the coordinates. In a strong electric field, the effect of the inhomogeneities on the volt-ampere characteristics is small and is expressed in terms of the pair correlation of these random functions. In magnetic fields that are small in comparison with H_{c2}, the lattice is a set of individual vortex filaments. The inhomogeneities in this case lead to random forces acting on the vortices.[3] The volt-ampere characteristic in this case was found in the paper of Schmid and Hauger.[4]

An interesting phenomenon was observed by Fiory,[5] who passed a high-frequency current through the superconducting film in addition to the constant current. Steps were observed in the volt-ampere characteristic in this case at electric field intensities related to the frequency by $2\pi Em = B\omega na$, where a is the constant of the vortex lattice and m and n are integers. At fields close to H_{c2} these steps disappeared. The magnitude and shape of these steps in the case of weak magnetic fields were found by Schmid and Hauger.[4] The magnitude of these steps in an arbitrary magnetic field is found below.

2. EQUATION OF MOTION OF THE VORTEX LATTICE

We first consider the case in which the effective interaction between the electrons, g, is inhomogeneous. We write this relation in the form

$$g^{-1}=\langle\langle g^{-1}\rangle\rangle+g_1(\mathbf{r}). \tag{1}$$

The random quantity $g_1(\mathbf{r})$ is determined by the correlation function

$$\langle\langle g_1(\mathbf{r})g_1(\mathbf{r}_1)\rangle\rangle=\varphi(\mathbf{r}-\mathbf{r}_1). \tag{2}$$

The distance at which $\varphi(\mathbf{r})$ falls off is determined by the size of the inhomogeneities and is assumed to be large in comparison with the interatomic distance (for example, the size of the crystallites or the dimensions of inclusions of another phase).

The equations which express the order parameter and the vector potential in terms of Green's functions have the form[6]

$$\begin{gathered} g^{-1}\Delta_1(\mathbf{r},t)=F(\mathbf{r},t;\mathbf{r},t),\quad g^{-1}\Delta_2(\mathbf{r},t)=F^+(\mathbf{r},t;\mathbf{r},t),\\ (4\pi)^{-1}\operatorname{rot}\operatorname{rot}\mathbf{A}=\mathbf{j}=\frac{ie}{m}\left(\frac{\partial}{\partial\mathbf{r}'}-\frac{\partial}{\partial\mathbf{r}}\right)_{\mathbf{r}'\to\mathbf{r}}G(\mathbf{r},t;\mathbf{r}',t).\end{gathered} \tag{3}$$

In a homogeneous superconductor without an electric field, the order parameter $\Delta_1=\Delta_2^*=\Delta^0(\mathbf{r})$ and the vector potential $\mathbf{A}^0(\mathbf{r})$ do not depend on the time. Here $|\Delta^0(\mathbf{r})|$ and curl $\mathbf{A}^0(\mathbf{r})$ are periodic functions of the coordinates, forming a lattice. In a weak electric field $\mathbf{E}$ and for weak inhomogeneities, the lattice moves as a whole, in zeroth approximation, with the velocity $\mathbf{V}=(\mathbf{E}\times\mathbf{B})\mathbf{B}^{-2}$, which is a slowly changing function of the time and of the coordinates. We shall use adiabatic perturbation theory in order to derive equations for the averaged quantities which change slowly over the lattice constant.

In the first approximation, the order parameter Δ and the vector potential $\mathbf{A}$ have the form

$$\begin{gathered}\Delta_{1,2}(\mathbf{r},t)=\Delta^0_{1,2}(\mathbf{r}+\mathbf{u})e^{\pm2ie\chi},\\ \mathbf{A}=\mathbf{A}^0(\mathbf{r}+\mathbf{u})+\frac{\partial\chi}{\partial\mathbf{r}},\qquad\frac{\partial\chi}{\partial t}=-\frac{\partial\mathbf{u}}{\partial t}\mathbf{A}^0(\mathbf{r}+\mathbf{u}),\end{gathered} \tag{4}$$

where $\mathbf{u}$ is a slowly changing function of the coordinates and of the time, and Δ^0 and $\mathbf{A}^0$ depend on the local value of the magnetic induction $\mathbf{B}$ as a parameter. If the mean current is small, then the magnetic induction depends slowly on the coordinates. We shall assume that the distances at which the magnetic induction changes significantly are large in comparison not only with the dimensions of the cell, but also with the significant distances at which the effect of the inhomogeneities is important. We can therefore find the mean induction $\mathbf{B}$ by averaging the local value of the magnetic field over the dimensions of the cell and over the inhomogeneities:

$$\begin{gathered}\mathbf{B}=\langle\langle\mathbf{H}\rangle\rangle,\quad\mathbf{H}=\operatorname{rot}\mathbf{A}_0(\mathbf{r}+\mathbf{u}),\\ \operatorname{rot}\mathbf{H}=4\pi(\mathbf{j}_0(\mathbf{r}+\mathbf{u})+\langle\langle\mathbf{j}\rangle\rangle),\end{gathered} \tag{5}$$

where $\mathbf{j}_0$ depends on $\mathbf{B}$ and is found from the static equations. Our problem is to find the dependence of $\langle\langle\mathbf{j}\rangle\rangle$ on the electric field.

We substitute Eqs. (4) in the set of equations (3) and multiply this set on the left by

$$(\mathbf{e}\partial_+\Delta_0^*(\mathbf{r}+\mathbf{u}),\quad\mathbf{e}\partial_-\Delta_0(\mathbf{r}+\mathbf{u}),\quad[\mathbf{H}_0(\mathbf{r}+\mathbf{u})\mathbf{e}])e^{-i\mathbf{q}\mathbf{r}},$$

where $\mathbf{e}$ is a unit vector in one of the two directions perpendicular to the direction of the induction $\mathbf{B}$ and $\partial_\pm=\partial/\partial\mathbf{r}\pm2ie\mathbf{A}$. After this, the set (3) transforms into the equation for the quantity $\mathbf{u}_\mathbf{q}=\int\mathbf{u}(\mathbf{r},t)e^{-i\mathbf{q}\cdot\mathbf{r}}d^3\mathbf{r}$,

$$\begin{gathered}L_{\alpha\beta}u_\mathbf{q}^\beta+K_{\alpha\beta}\frac{\partial u_\mathbf{q}^\beta}{\partial t}\\ =-\nu\int d^3\mathbf{r}\,g_1(\mathbf{r})\left(\frac{\partial}{\partial r}|\Delta_0(\mathbf{r}+\mathbf{u})|^2\right)e^{-i\mathbf{q}\mathbf{r}}-(2\pi)^3\delta_\mathbf{q}[\langle\langle\mathbf{j}\rangle\rangle\mathbf{B}]_\alpha,\end{gathered} \tag{6}$$

where $\nu = mp_0/2\pi^2$ is the density of states on the Fermi surface. The last term in Eq. (6) arises from the fact that the mean transport current is taken into account in Eqs. (5) for the mean field. With this notation, no increasing terms appear in the quantity $\mathbf{u}(\mathbf{r}, t)$.

The coefficients $L_{\alpha\beta}$, $K_{\alpha\beta}$ are found from the relations

$$e_\alpha \hat{D}_{\alpha\beta} u^\beta_{\mathbf{q}} = \int d^3\mathbf{r}\, e^{-i\mathbf{qr}} \begin{pmatrix} \mathbf{e}\ \partial_-\ \Delta_0 \\ \mathbf{e}\ \partial_+\ \Delta_0^* \\ [\mathbf{He}] \end{pmatrix}^+ \hat{D} \begin{pmatrix} \mathbf{u}\ \partial_-\ \Delta_0 \\ \mathbf{u}\ \partial_+\ \Delta_0^* \\ [\mathbf{Hu}] \end{pmatrix}; \qquad (7)^*$$

$\hat{D} = \hat{L}, \hat{K}$. The operators $\hat{L}$, $\hat{K}$ can be found by linearization of the set of equations of Gor'kov and Maxwell with respect to small, slowly varying increment to the order parameter Δ and the vector potential $\mathbf{A}$:

$$\left(\hat{L} + \hat{K}\frac{\partial}{\partial t}\right)\begin{pmatrix} \delta\Delta_1 \\ \delta\Delta_2 \\ \delta\mathbf{A} \end{pmatrix} = 0. \qquad (8)$$

In the static case, the equations of Gor'kov and Maxwell can be obtained by minimizing the free energy F with respect to the order parameter Δ and the vector potential $\mathbf{A}$. Therefore, the operator $\hat{L}$ is equal to the second variational derivative of F with respect to Δ and $\mathbf{A}$. On the other hand, for a slowly varying deformation $\mathbf{u}(\mathbf{r})$, the free energy is equal to

$$\delta F_0 = \frac{1}{2}\int d^3\mathbf{r}\left\{(C_{11}-C_{66})\left(\frac{\partial \mathbf{u}}{\partial \mathbf{r}}\right)^2 + C_{66}\left(\frac{\partial u^\beta}{\partial r_\alpha}\right)^2 + C_{44}\left(\frac{\partial u^\alpha}{\partial z}\right)^2\right\}, \qquad (9)$$

where C_{11}, C_{44}, C_{66} are the elastic moduli. Near the transition temperature ($T_c - T \ll T_c$), the expressions for the elastic moduli were found by Labusch[7] for an arbitrary magnetic field. Comparing expressions (7) and (9), we get for q that are small in comparison with the reciprocal lattice dimension

$$L_{\alpha\beta} = (C_{11}-C_{66})q_\alpha q_\beta + \delta_{\alpha\beta}(C_{66}q_\gamma^2 + C_{44}q_z^2). \qquad (10)$$

The operator $\hat{K}$ has the form

$$\hat{K} = \frac{\nu}{2}\begin{pmatrix} \tau_s - \tau_s^2\Delta\hat{M}\Delta^* & \tau_s^2\Delta\hat{M}\Delta, & 2ieD\tau_s\Delta\hat{M}\partial/\partial\mathbf{r} \\ \tau_s^2\Delta^*\hat{M}\Delta^*, & \tau_s - \tau_s^2\Delta^*\hat{M}\Delta, & -2ieD\tau_s\Delta^*\hat{M}\partial/\partial\mathbf{r} \\ 2ieD\tau_s\frac{\partial}{\partial\mathbf{r}}\hat{M}\Delta^*, & -2ieD\tau_s\frac{\partial}{\partial\mathbf{r}}\hat{M}\Delta, & 4e^2D[1 + D(\partial/\partial\mathbf{r})\hat{M}\partial/\partial\mathbf{r}] \end{pmatrix},$$

$$\hat{M} = (2\tau_s|\Delta|^2 - D\partial^2/\partial\mathbf{r}^2)^{-1}, \qquad (11)$$

in the case of a superconductor with a large concentration of magnetic impurities. Here $D = vl_{tr}/3$ is the diffusion coefficient, τ_s the electron path time with spin flip. In the arbitrary case, the coefficient $K_{\alpha\beta}$ which enters into Eq. (6) can be expressed in terms of the conductivity of a superconductor without inhomogeneities, in the mixed state. Without inhomogeneities, the quantity $\mathbf{u}$ does not depend on the coordinates, $-\partial\mathbf{u}/\partial t$ is equal to the velocity of motion of the lattice $\mathbf{V}$, and Eq. (6) takes the form

$$K_{\alpha\beta}V_\beta = [\langle\langle \mathbf{j}\rangle\rangle\mathbf{B}]_\alpha. \qquad (12)$$

Assuming the Hall angle to be small, we get

$$K_{\alpha\beta} = \delta_{\alpha\beta}\mathbf{B}^2\sigma, \qquad (13)$$

where σ is the conductivity of the superconductor in the mixed state.

Returning to the coordinate representation in (6), we obtain, with account of Eqs. (10) and (13),

$$-\left[(C_{11}-C_{66})\frac{\partial}{\partial\boldsymbol{\rho}}\left(\frac{\partial\mathbf{u}}{\partial\boldsymbol{\rho}}\right) + \left(C_{66}\frac{\partial^2}{\partial\boldsymbol{\rho}^2} + C_{44}\frac{\partial^2}{\partial z^2}\right)\mathbf{u}\right] + \sigma\mathbf{B}^2\frac{\partial\mathbf{u}}{\partial t}$$
$$= -\nu\left\langle g_1(\mathbf{r})\frac{\partial}{\partial\boldsymbol{\rho}}|\Delta_0(\mathbf{r}+\mathbf{u})|^2\right\rangle - [\langle\langle\mathbf{j}\rangle\rangle\mathbf{B}], \qquad (14)$$

where ρ is a two-dimensional vector in the plane perpendicular to the magnetic field and $\langle\ldots\rangle$ denotes averaging over the cell.

We note that the partial derivative with respect to time $\partial\mathbf{u}/\partial t$ appears in Eq. (14), while the term of the form $(\partial\mathbf{u}/\partial t \cdot \partial/\partial\mathbf{r})\mathbf{u}$ is lacking. This is connected with the absence of Galilean invariance in the vortex lattice and will be important in the study of the step widths that develop in the volt-ampere characteristics in a high-frequency field.

3. CONDUCTIVITY IN A CONSTANT FIELD

It is convenient to separate the motion of the lattice as a whole (with mean velocity $\mathbf{V} = -\langle\langle\partial\mathbf{u}/\partial t\rangle\rangle$) in Eq. (14) and make the substitution

$$\mathbf{u} \to \mathbf{u} - \int \mathbf{V}\,dt.$$

Averaging Eq. (14), we obtain an expression for the mean current

$$[\langle\langle\mathbf{j}\rangle\rangle\mathbf{B}] = \sigma B^2\mathbf{V} - \nu\left\langle\!\!\left\langle g_1(\mathbf{r})\frac{\partial}{\partial\boldsymbol{\rho}}\left|\Delta_0\left(\mathbf{r}+\mathbf{u}-\int\mathbf{V}\,dt\right)\right|^2\right\rangle\!\!\right\rangle. \qquad (15)$$

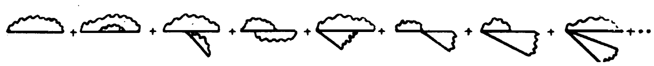
The equation which expresses the deformation $\mathbf{u}$ in terms of g_1, follows from Eqs. (14), (15):

$$\mathbf{u}(\mathbf{r}, t) = -\nu\int dt'\int d^3\mathbf{r}'\,G(\mathbf{r}-\mathbf{r}', t-t')$$
$$\times\left\{\left\langle g_1(\mathbf{r}')\frac{\partial}{\partial\boldsymbol{\rho}}\left|\Delta_0\left(\mathbf{r}'+\mathbf{u}-\int^{t'}\mathbf{V}\,dt_1\right)\right|^2\right\rangle\right. \qquad (16)$$
$$\left. - \left\langle\!\!\left\langle g_1(\mathbf{r}_1)\frac{\partial}{\partial\boldsymbol{\rho}_1}\left|\Delta_0\left(\mathbf{r}_1+\mathbf{u}-\int^{t'}\mathbf{V}\,dt_1\right)\right|^2\right\rangle\!\!\right\rangle\right\},$$

where the Green's function G satisfies the equation

$$-\left[(C_{11}-C_{66})\frac{\partial}{\partial\boldsymbol{\rho}}\left(\frac{\partial G}{\partial\boldsymbol{\rho}}\right) + \left(C_{66}\frac{\partial^2}{\partial\boldsymbol{\rho}^2} + C_{44}\frac{\partial^2}{\partial z^2}\right)G\right]$$
$$+ \sigma\mathbf{B}^2\frac{\partial G}{\partial t} = \delta(t-t')\delta(\mathbf{r}-\mathbf{r}'). \qquad (17)$$

Expanding the right side of Eq. (16) in a series in $\mathbf{u}$ and solving this equation by interaction, we get an expression for $\mathbf{u}$ in terms of g_1. Substituting this expansion in Eq. (15), we obtain a representation in the form of a series of diagrams for the mean current:



Here the straight line denotes the Green's function G, which is determined by Eq. (17); the wavy line denotes the correlation function φ, which is determined by Eq. (2). To each vertex there corresponds a factor

$$\frac{\partial^{n+1}}{\partial\rho^{n+1}}\left|\Delta_0\left(\mathbf{r}-\int\mathbf{V}\,dt\right)\right|^2,$$

where n is the number of continuous lines emerging on the right from the vertex. A single continuous line enters each vertex from the left except at the extreme left.

We first consider the case in which the electric field and, consequently, the vector $\mathbf{V}$, do not depend on the time. In this case, the correction to the current (to first order in φ), which is shown by the first diagram, is equal to

$$[\mathbf{j}^{(1)}\mathbf{B}] = \frac{i\nu^2}{2}\sum_n\int\frac{d^3\mathbf{q}}{(2\pi)^3}\varphi_n|(|\Delta|_n^2)|^2\mathbf{K}_n K_n^2\{[C_{66}q_\beta^2 + C_{44}q_z^2$$
$$+ i\sigma\mathbf{B}^2(\mathbf{K}_n\mathbf{V})]^{-1} + [C_{11}q_\beta^2 + C_{44}q_z^2 + i\sigma\mathbf{B}^2(\mathbf{K}_n\mathbf{V})]^{-1}\},$$

where $\mathbf{K}_n$ are the vectors of the reciprocal lattice,

$$|\Delta_0(\mathbf{r})|^2=\sum_n(|\Delta|_n{}^2)\exp(i\mathbf{K}_n\mathbf{r}),$$

$$\varphi_n=\int d^3\mathbf{r}\,\varphi(\mathbf{r})\exp(-i\mathbf{K}_n\mathbf{r}).$$

At small velocities $\mathbf{V}$, the small $\mathbf{q}$ are important in the integral,

$$C_{66}\mathbf{q}^2\sim\sigma\mathbf{B}^2V/a, \tag{18}$$

where a is the linear dimension of the cell. Completing the integration, we get the following expressions:

In the bulk sample:

$$[\mathbf{j}^{(1)}\mathbf{B}]=\frac{\nu^2}{8\pi\sqrt{2C_{44}}}(C_{66}{}^{-1}+C_{11}{}^{-1})\sum_n\varphi_n \times(|\Delta|_n{}^2)|^2\mathbf{K}_n\mathbf{K}_n{}^2(\sigma\mathbf{B}^2|(\mathbf{K}_n\mathbf{V})|^{1/2})\,\mathrm{sign}(\mathbf{K}_n\mathbf{V}), \tag{19}$$

analytic calculations for a film of thickness d give

$$[\mathbf{j}^{(1)}\mathbf{B}]=\frac{\nu^2}{16dC_{66}}\sum_{K_n}\varphi_n\mathbf{K}_n\mathbf{K}_n{}^2|(|\Delta|_n{}^2)|^2\,\mathrm{sign}(\mathbf{K}_n\mathbf{V}). \tag{20}$$

The sum over K_n in Eqs. (19), (20) can be found in the limiting cases of strong and weak fields. Near the critical field H_{c2}, the quantities $|\Delta|_n^2$ fall off very rapidly with increase in $|K_n|$ and we may retain in the sums of (19), (20) only the terms with the smallest (nonzero) value

$$|\mathbf{K}_n|=K=2(2\pi eB)^{1/2}3^{-1/4}.$$

As a result we obtain for the film:

$$[\mathbf{j}^{(1)}\mathbf{B}]=\frac{\nu^2}{4dC_{66}}\varphi_kK^3\exp\left(-\frac{2\pi}{\sqrt{3}}\right)\langle|\Delta|^2\rangle^2\{\mathbf{e}_\mathbf{V}\cos(\alpha-\pi/6)+\mathbf{e}_{[\mathbf{VH}]}\sin(\alpha-\pi/6)\}, \tag{21}$$

and for the bulk sample:

$$[\mathbf{j}^{(1)}\mathbf{B}]=\frac{\nu^2}{4\pi\sqrt{2C_{44}}}(C_{66}{}^{-1}+C_{11}{}^{-1})\varphi_kK^3(\sigma B^2VK)^{1/2}\langle|\Delta|^2\rangle^2\exp\left(-\frac{2\pi}{\sqrt{3}}\right)$$
$$\times\{\mathbf{e}_\mathbf{V}[\sin^{3/2}\alpha+\cos^{3/2}(\alpha-\pi/6)+\cos^{3/2}(\alpha+\pi/6)]$$
$$+\mathbf{e}_{[\mathbf{VH}]}[\sin(\alpha-\pi/6)(\cos^{1/2}(\alpha-\pi/6)+{}^1/_2(\sin^{1/2}\alpha+\cos^{1/2}(\alpha+\pi/6))$$
$$+{}^1/_2\sqrt{3}\cos(\alpha-\pi/6)(\cos^{1/2}(\alpha+\pi/6)-\sin^{1/2}\alpha)]\},$$

where α is the angle between the directions of the velocity and the vector of the unit cell ($0\le\alpha\le\pi/3$); $\mathbf{e}_\mathbf{V}$, $\mathbf{e}_{\mathbf{V}\times\mathbf{H}}$ are unit vectors along $\mathbf{V}$ and $\mathbf{V}\times\mathbf{H}$, respectively.

In the other limiting case $H\ll H_{c2}$, the lattice represents a set of individual vortices. In this case, the expressions (19), (20) undergo transition into the formulas obtained in the work of Schmid and Hauger[4] if we represent the random force acting on the vortex in terms of the inhomogeneity of the effective interaction.[3]

It follows from Eqs. (19), (20), and (21) that the conductivity depends on the direction of the electric field relative to the vectors of the elementary cell. The Hall conductivity vanishes at the extremal points of the ordinary conductivity ($\alpha=0$, $\alpha=\pi/6$). In the film, the Hall conductivity has a discontinuity at the point $\alpha=0$.

All the diagrams given in the figure (except the first) give a correction to the current of second order in φ. Expanding the expressions (15), (16) to fourth order in g_1, we obtain

$$[\mathbf{j}^{(2)}\mathbf{B}]=i\nu^4\sum_{n,m}\int\frac{d^3\mathbf{q}\,d^3\mathbf{q}_1}{(2\pi)^6}\varphi_n\varphi_m|(|\Delta|_n{}^2)|^2|(|\Delta|_m{}^2)|^2\mathbf{K}_n$$
$$\times\{G^{n,m}_{n,\mathbf{q}}G^{n,m}_{m,\mathbf{q}_1}(G^{n,m}_{n+m,\mathbf{q}-\mathbf{q}_1}-G^{n,m}_{n-m,\mathbf{q}-\mathbf{q}_1})+(G^{n,m}_{n,\mathbf{q}})^2(G^{m,m}_{n-m,\mathbf{q}_1}-G^{m,m}_{m,\mathbf{q}_1})$$
$$+|G^{n,m}_{m,\mathbf{q}_1}|^2(G^{n,n}_{n-m,\mathbf{q}}-G^{n,n}_{n,\mathbf{q}})\}, \tag{22}$$

$$G^{n,m}_{l,\mathbf{q}}=(\mathbf{K}_nG_{l,\mathbf{q}}\mathbf{K}_m),\quad G_{l,\mathbf{q}}=\int dt\int d^3\mathbf{r}\,G(\mathbf{r},t)\exp[-i\mathbf{q}\mathbf{r}-i\mathbf{K}_l\mathbf{V}t]. \tag{23}$$

For the component of the current directed along the electric field, the important region of integration over $\mathbf{q}$ and $\mathbf{q}_1$ in the integral (22) is determined by the condition (18), so that this component of the current is equal in order of magnitude to $j^{(2)}\sim j^2_{(1)}/j_0$, where $j^{(1)}$ is determined by Eqs. (19), (20); $\mathbf{j}_0=\sigma\mathbf{E}=\boldsymbol{\sigma}\mathbf{B}\times\mathbf{V}$. Thus, the expansion of the expression for the current density in a series in the inhomogeneity φ is an expansion in a parameter proportional to φ/E for the film and $\varphi E^{-1/2}$ for the bulk sample.

4. THE CRITICAL CURRENT

In accord with experiment, we shall assume that as the electric field approaches zero, the current density tends to some fixed value j_c. Then

$$j_\parallel(E)=j_{cr}f(\sigma E/j_{cr}), \tag{24}$$

where f(x) is some universal function, which is generally different for fields close to H_{c2} and small in comparison with H_{c2}. The expressions (19), (20), (22) represent the first terms in the expansion of this function for large values of x. Comparison of $j^{(1)}$ with j_0 and $j^{(2)}$ with $j^{(1)}$ allow us to obtain an estimate for the value of the critical current and to make clear its dependence on the magnetic field and the temperature:

$$j_{cr}\sim\begin{cases}\dfrac{\nu^2}{20dC_{66}B}\sum_n\varphi_nK_n{}^3|(|\Delta|_n{}^2)|^2 & \text{for the film}\\ B^{-1}\left\{\dfrac{\nu^2}{25C_{66}\sqrt{C_{44}}}\sum_n\varphi_nK_n^{7/2}|(|\Delta|_n{}^2)|^2\right\}^2 & \text{for the bulk sample}\end{cases} \tag{25}$$

We consider various limiting cases. If the size of the inhomogeneity r_c is small in comparison with the size of the pair ξ, then φ_n depends weakly on $\mathbf{K}_n$ and the inhomogeneities are characterized by a single parameter $\varphi_0\sim g_1^2r_c^3$. Estimating the sums over the vectors of the reciprocal lattice $\mathbf{K}_n$ in formulas (25), we obtain the following expressions for the critical current density:

$$j_{cr}\sim\begin{cases}\dfrac{e\nu^2\varphi_0\langle|\Delta|^2\rangle^2}{60C_{66}d\xi} & \text{for the film}\\ B\left[\dfrac{e\nu^2\varphi_0\langle|\Delta|^2\rangle^2}{70C_{66}\sqrt{C_{44}}\,\xi^{3/2}}\right]^2 & \text{for the bulk sample}\end{cases} \tag{26}$$

The elastic moduli C_{44}, C_{66} in (26) have been found for various limiting cases in the papers of Labusch:[7]

$$C_{44}=HB/4\pi,$$

$$C_{66}\sim\begin{cases}\dfrac{\sqrt{6\pi}}{64e^2\lambda^4}\left(\dfrac{\lambda}{a}\right)^{1/2}\exp\left(-\dfrac{a}{\lambda}\right), & B\ll H_{c1}\\ B\,[64\pi e\lambda^2]^{-1}, & H_{c1}\ll B\ll H_{c2},\\ \dfrac{H_c{}^2}{4\pi}\dfrac{\varkappa^2(2\varkappa^2-1)}{[1+(2\varkappa^2-1)\beta]^2}0{,}48\left(1-\dfrac{B}{H_{c2}}\right)^2, & H_{c2}-B\ll H_{c2}\cdot\end{cases} \tag{27}$$

where the induction B for the triangular lattice is connected with the distance a between the vortices by the relation

$$B=2\pi[ea^2\sqrt{3}]^{-1}. \tag{28}$$

In fields that are not close to H_{c1}, one can obtain an interpolation formula from Eqs. (26), (27) that gives the dependence of the critical current on the magnetic field and the temperature:

$$\frac{j_{cr}}{j_{c0}}\sim\begin{cases}10^{-1}\left(\dfrac{H_{c2}(T)}{B}\right)\dfrac{\varphi_0T_c}{vl_{tr}d}\dfrac{T_c}{T_c-T} & \text{for the film}\\ \dfrac{H_{c2}H_c{}^2}{B^3}\varphi_0{}^2\left(\dfrac{T_c}{vl_{tr}}\right)^3\dfrac{T_c}{T_c-T} & \text{for the bulk sample}\end{cases}, \tag{29}$$

where j_{c0} is the temperature-dependent critical pair-breaking current in zero magnetic field:

$$j_{c0}=[12\sqrt{3}\pi e\lambda^2(T)\xi(T)]^{-1}. \tag{30}$$

If the dimensions of the inhomogeneities are larger than the dimension of the vortex ξ, but much smaller than the linear dimension of the cell a then an additional factor $\ln^4(r_c/\xi)(\xi/r_c)^{11}$ appears in Eqs. (26), (29) for the bulk sample and $\ln^2(r_c/\xi)(\xi/r_c)^5$ for the film. If the dimension of the inhomogeneities exceeds the linear dimension of the cell, then the critical current depends on the form of the inhomogeneity in a significant way. For smooth inhomogeneities, when the correlation function depends analytically on r^2, the critical current falls off exponentially with increase in the parameter (r_c/a). For inhomogeneities with a sharp edge, the fall of the critical current is power-law.

Account of the inhomogeneity of the free path length leads to replacement of the correlation function φ by

$$\varphi(\mathbf{r})\to\varphi(\mathbf{r})+b(1-T/T_c)^2[\ll l(\mathbf{r})l(0)\gg/\ll l\gg^2-1]. \tag{31}$$

The coefficient in Eq. (31) depends smoothly on the temperature and the value of the magnetic field. For T close to T_c and H near H_{c2}, the coefficient $b = 4\pi^2/3$. The inhomogeneities in the film thickness lead to the addition of one more term to the correlation function:

$$(1-T/T_c)^2[\ll d(\mathbf{r})d(0)\gg/\ll d\gg^2-1].$$

Thus, the inhomogeneities of the effective interaction turn out to be very important for temperatures near critical.

5. REGION OF APPLICABILITY

Derivation of general formulas for the critical current and the volt-ampere characteristic that do not depend on the detailed form of the inhomogeneities is possible only for weak inhomogeneities. In this case, the lattice distortions are small and depend slowly on the coordinates even for small inhomogeneities. Equations (10), (14) are applicable if the important values of the momentum **q**, determined by the condition (18), are small in comparison with the reciprocal linear dimension of the cell. For currents of the order of the critical current, we get from (18)

$$C_{66}a^{-2}\gg Bj_{cr}\xi^{-1}. \tag{32}$$

Substituting expression (27) for C_{66} in the inequality (32) and expression (30) for the pair breaking current j_{c0} in fields that are not close to H_{c1}, we obtain

$$j_{cr}/j_{c0}\ll(B/H_{c2})(1-B/H_{c2})^2. \tag{33}$$

If the correlation radius of the inhomogeneities r_c is greater than the pair dimension ξ, then the right side of (33) must be multiplied by

$$\xi^{-1}\min\{r_c, a\}.$$

It follows from (33) that, in fields of the order of the critical field H_{c2}, but not close to H_{c2}, expression (25) for the critical field is valid up to its maximum value, of the order of the critical pair-breaking field. In fields that are small in comparison with H_{c2}, condition (33) cannot be satisfied for sufficiently large inhomogeneities. Here the interaction of the vortices with the inhomogeneities is stronger than their interaction with one another. In this case, as was shown earlier,[3] the critical current of thin films depends weakly on the magnetic field. At the boundary of the region, which is determined by the inequality (33), the two expressions for the critical current of thin films are identical.

The significant distances at which the inhomogeneities deform the lattice are determined from the condition (18). It was assumed above that the induction B changes little at these distances. For a current density of the order of the critical value, this condition leads to the additional restriction

$$j_{cr}/j_{c0}\ll 10(B/H_{c2})^2\varkappa^4(1-B/H_{c2})^{-2}. \tag{34}$$

At large values of the parameter κ, the condition (34) is weaker than the condition (33) for fields not very close to H_{c2}. If the condition (34) is not satisfied, then the connection of the current with the induction turns out to be nonlocal.

It was assumed above that the electric field is sufficiently small so that effects which are nonlinear in the field do not need to be taken into account in a homogeneous superconductor. Estimation of these nonlinear effects, which are connected with heating of the superconductor and with the change in the electron distribution function, requires separate treatment.

6. THE SUPERCONDUCTOR IN AN ALTERNATING FIELD

Formulas (15), (16) permit us to find the current in a superconductor located in an alternating electric field. We shall consider the case in which there is, in addition to the constant field, also an alternating field with frequency ω:

$$\mathbf{E}(t)=[\mathbf{B}\mathbf{V}(t)]=[\mathbf{B}\mathbf{V}_0]+[\mathbf{B}\mathbf{V}_1]\cos\omega t. \tag{35}$$

We substitute the expression for $\mathbf{V}(t)$ from Eq. (35) in Eq. (16). Then, in first order in the inhomogeneities, we get, after expansion of $|\Delta_0(\mathbf{r}-\mathbf{V}_0 t-\mathbf{V}_1\sin\omega t/\omega|^2$ in a Fourier series in the time,

$$\mathbf{u}(\mathbf{q},t)=-iv\sum_{l,n}g_1(-\mathbf{K}_l+\mathbf{q})(|\Delta|_l^2)J_n\left(\frac{\mathbf{K}_l\mathbf{V}_1}{\omega}\right) \times G_{-L,-\mathbf{q}}\mathbf{K}_l\exp[-i(\mathbf{K}_l\mathbf{V}_0+n\omega)t], \tag{36}$$

where the Green's function $G_{L,\mathbf{q}}$ is determined by Eq. (23), and the index $L = (\mathbf{K}_l\cdot\mathbf{V}_0) + n\omega$. Substituting this expression for $\mathbf{u}(\mathbf{q},t)$ in Eq. (15), we get

$$[\mathbf{j}^{(1)}\mathbf{B}]=iv^2\sum_{n,n_1,m}\int\frac{d^3\mathbf{q}_1}{(2\pi)^3}\varphi_m|(|\Delta|_m^2)|^2\mathbf{K}_m \times J_n\left(\frac{\mathbf{K}_m\mathbf{V}_1}{\omega}\right)J_{n_1}\left(-\frac{\mathbf{K}_m\mathbf{V}_1}{\omega}\right)G^{m,m}_{L_1\mathbf{q}}\exp[-i\omega(n+n_1)t], \tag{37}$$

$$L_1=\mathbf{K}_m\mathbf{V}_0-n_1\omega.$$

It follows from (37) that the singularity in the Green's function G has been displaced from the point $\mathbf{K}\cdot\mathbf{V}_0 = 0$ to the point

$$\mathbf{K}_m\mathbf{V}_0=n\omega. \tag{38}$$

Singularities appear in the average current at these same points.

Averaging expression (37) over the time and carrying out integration over the momenta **q**, we obtain the following expression for the current density in the film:

$$[\mathbf{j}^{(1)}\mathbf{B}]=\frac{v^2}{16dC_{66}}\sum_{m,n}\varphi_m\mathbf{K}_m\mathbf{K}_m^2J_n^2\left(\frac{\mathbf{K}_m\mathbf{V}_1}{\omega}\right)|(|\Delta|_m^2)|^2\operatorname{sign}[\mathbf{K}_m\mathbf{V}_0-n\omega]. \tag{39}$$

Steps appear in the volt-ampere characteristic at voltages satisfying the Josephson relation (38). The magnitude of these steps is proportional to the critical current. In fields $H_{c1}\ll H\ll H_{c2}$, it falls off with the field as B^{-1}. In fields $H_{c2}-H\ll H_{c2}$, the step amplitude is slowly dependent on the value of the magnetic field as long as the condition (33) is satisfied.

Such steps have been observed experimentally by

Fiory.[5] An explanation of this phenomenon that is physically equivalent to ours was given in the work of Schmid and Hauger.[4] Their results are applicable in the range of fields $H \ll H_{c2}$. Even in this region, there is a small difference between our results and the results of Schmid and Hauger[4] that is connected with the determination of the width of the steps of the volt-ampere characteristic. Formula (39), which was obtained in first order in the inhomogeneities, gives a zero width of the steps:

$$\sigma B^2 n\, \delta\omega_n \sim \frac{v^2}{20dC_{66}} \sum_m \varphi_m K_m^4 |(|\Delta|_m^2)|^2 \ln \left| \frac{(K_n - K_m) V}{V K_m} \right|. \quad (40)$$

From Eq. (40) we obtain the result that the width of the steps is connected, in order of magnitude, with the critical current by the relation

$$n\delta\omega \sim B j_{cr} [\sigma B^2]^{-1} \min\{\xi^{-1}, r_c^{-1} + a^{-1}\}. \quad (41)$$

Schmid and Hauger[4] have shown that the steps have finite width even in zeroth order in the inhomogeneities. This difference is connected with the fact that in the derivation of the equation for the shift in **u** from the phenomenological equations[4] an additional term $(\dot{\mathbf{u}}\nabla)\mathbf{u}$ appears in them in comparison with Eq. (14), leading to a finite width of the steps in zeroth order in the inhomogeneities. This term is lacking in the microscopic derivation given above. The question of the width of the steps is still further complicated by the fact that, at least for low velocities, a "melting" of the lattice takes place.[2] Therefore the quantity $\mathbf{K}_n \cdot \mathbf{V}$ has different values at different places in the sample.

7. CONCLUSION

The value of the critical current in type-II superconductors is expressed in terms of the correlation functions that characterize the inhomogeneities. If the amplitude of the inhomogeneities is small or their size is small in comparison with the pair dimension ξ, then the critical current is small in comparison with the pair breaking current. In this case, a wide range of magnetic fields exists, not too near H_{c2} and not too small in comparison with H_{c2}, in which the interaction between the vortices is stronger than their interaction with the inhomogeneities, and the results obtained above are valid. In this region, the critical current falls off with increase in the magnetic field in proportion to B^{-1} for a film and B^{-3} for a bulk sample if B is much smaller than H_{c2}. In the region of B of the order of H_{c2}, the critical current depends weakly on the magnetic field. In the narrow range near H_{c2}, when condition (33) or (34) is violated, the critical current falls off with approach of the field to H_{c2}. Condition (33) is also violated in fields that are small compared to H_{c2}. On a further decrease in the magnetic field, the critical current again ceases to depend on the field.[3] The character of the important inhomogeneities can be deduced from the temperature dependence of the critical current for a fixed ratio $H/H_{c2}(T)$.

If the critical current is determined by the inhomogeneities of the effective interaction, the current falls off according to the law $(T_c - T)^{1/2}$ more slowly than the pair-breaking current as the temperature approaches the critical temperature. At fields of the order of H_{c2}, the critical current in films can become of the order of the pair-breaking current for $T_c - T \sim T_c^2 \varphi_0 (v l_{tr} d)^{-1}$. In this temperature region, the inhomogeneities of the effective interaction lead to a smearing out of the phase transition. If the critical current is determined by the inhomogeneities of the path length or the thickness of the film, then it falls off upon approach to T_c more rapidly than the pair-breaking current, according to the law $(T_c - T)^{5/2}$ for a film and $(T_c - T)^{9/2}$ for a bulk sample.

In the region considered, the volt-ampere characteristic is a universal function of the ratio $\sigma E/j_{cr}$. The first terms of the asymptotic expansion of this function for $\sigma E \gg j_{cr}$ were found above. In order to find the numerical coefficient in (26) for the critical current, and not only the dependence on magnetic field and temperature, it is necessary to find this function as $E \to 0$. For this purpose, it is necessary to sum the whole series of perturbation theory for the expansion of the current in powers of the inhomogeneities.

The results obtained above give the local connection of the current with the field intensity and the magnetic induction. In order to find the field and current distributions in real samples, it is necessary to solve the Maxwell equations by using the connection of the current with the magnetic and electric fields that was found above.

*[Hu] = H × u.

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Translated by R. T. Beyer
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