Relativistic generalization of the Darwin Lagrangian

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A relativistic model Langrangian which formally describes charge-charge interaction to any order of approximation in v/c is obtained for an optically thin plasma in which the action of the radiated fields on the charges is negligible owing to the rapid emission of the fields.

1. The complete description of a system of interacting charges usually requires the introduction of independent generalized coordinates for the electromagnetic field. If, however, the velocities of all the particles are low, then the system can be described up to terms of order $(v/c)^2$ by the Darwin Lagrangian[1]:

$$L_D=-\sum_i m_ic^2\sqrt{1-\beta_i^2}-\frac{1}{2}\sum\sum_{i\neq k}\frac{e_ie_k}{r_{ik}}\left[1-\frac{\beta_i\beta_k}{2}-\frac{(\mathbf{n}_{ik}\beta_i)(\mathbf{n}_{ik}\beta_k)}{2}\right] \quad (1)$$

where $\mathbf{r}_{ik} = \mathbf{r}_i - \mathbf{r}_k$, $\mathbf{n}_{ik} = \mathbf{r}_{ik}/r_{ik}$, and $\beta_i = v_i/c$. Such a Lagrangian has been used in a number of papers[2-8] to construct thermodynamic and kinetic theories of weakly relativistic plasmas devoid of radiation fields. Such conditions obtain in, for example, thermonuclear installations if the walls of the chambers are not specular.

In astrophysics, however, a number of papers have been published which theoretically investigate optically thin bunches of an ultrarelativistic plasma in which the interaction of the charges with their own radiated fields is also negligible owing to the rapid emission of the fields. As an example, we can cite Bisnovatyĭ-Kogan, Zel'dovich, and Syunyaev's paper[9], in which the authors consider the production of electron-positron pairs in particle-particle collisions and their annihilation in a relativistic Maxwellian plasma that is not in thermodynamic equilibrium with the radiation, which is assumed to escape freely from the plasma.

We consider below a relativistic model Lagrangian that is formally suitable for describing a system of interacting charges to any order of approximation in v/c under conditions when the radiation can be neglected (a realistic limitation being the condition $\theta \lesssim mc^2$).

2. The required Lagrangian can be obtained in the following way. The complete Lagrangian of a system of charges and field is, as is well known[10], given by

$$L=-\sum_i m_ic^2\sqrt{1-\beta_i^2}-\sum_i e_i(\varphi-\beta_i\mathbf{A})+\frac{1}{8\pi}\int(E^2-H^2)\,dV. \quad (2)$$

The last integral, which corresponds to the Lagrangian function for the field, can, with the aid of the expressions

$$\mathbf{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\quad \mathbf{H}=\operatorname{rot}\mathbf{A},$$

be transformed into the form

$$\frac{1}{8\pi}\int\left\{-\mathbf{H}\operatorname{rot}\mathbf{A}-\mathbf{E}\left(\nabla\varphi+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)\right\}dV=-\frac{1}{8\pi}\int\operatorname{div}(\varphi\mathbf{E}+[\mathbf{AH}])\,dV$$
$$-\frac{1}{8\pi c}\frac{d}{dt}\int(\mathbf{AE})\,dV+\frac{1}{2}\int\left(\rho\varphi-\frac{1}{c}\mathbf{jA}\right)dV. \quad (3)^*$$

The first integral with the divergence can be transformed into an integral over an infinitely remote surface. It describes the system's radiation, which we shall neglect, assuming it to be of low intensity. It is known from quantum electrodynamics that in first-order perturbation theory the interaction between charges is described by a diagram corresponding to elastic particle-particle scattering in which the initial and final energies of the particles are equal. In this approximation, therefore, the energy of the system is conserved. Radiation (bremsstrahlung) emission occurs only in second-order perturbation theory. The cross section for bremsstrahlung emission is at least $e^2/\hbar c = 1/137$ times smaller than the elastic-interaction cross sections, and it is precisely to within this degree of accuracy that we shall neglect the radiation, discarding the first integral on the right-hand side of the formula (3).

The second term with the total time derivative can simply be dropped in the Lagrangian function, while the last integral in (3) is equal to one-half the second sum in (2), which sum describes the interaction of the charges with the field.

Thus, from (2) we find that approximately

$$L=-\sum_i m_ic^2\sqrt{1-\beta_i^2}-\frac{1}{2}\sum_i e_i\left[\sum_{k\neq i}\varphi_k(\mathbf{r}_i,t)-\beta_i\sum_{k\neq i}\mathbf{A}_k(\mathbf{r}_i,t)\right]. \quad (4)$$

Using the Coulomb gauge (div $\mathbf{A}$ = 0), we obtain from the Maxwell equations the equations for the potentials:

$$\Delta\varphi=-4\pi\rho,\quad \Box\mathbf{A}=-\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial}{\partial\mathbf{r}_i}\frac{\partial\varphi}{\partial t}. \quad (5)$$

If the field is produced by a system of point charges, then the scalar potential is equal to

$$\varphi(\mathbf{r}_i,t)=\int\frac{dV'}{R}\rho(\mathbf{r}',t)={\sum_k}'\frac{e_k}{r_{ik}},$$

and retardation is neglected. The right-hand side of the equation for $\mathbf{A}$ then turns out to be equal to

$$\Box\mathbf{A}=\Delta\mathbf{A}^{(0)},\quad \mathbf{A}^{(0)}=\frac{1}{2c}{\sum_k}'\frac{e_k}{r_{ik}}\mathbf{v}_k\cdot\tilde{\delta}_{ik},$$
$$\tilde{\delta}_{ik}=\hat{\delta}_{ik}+\widehat{n_{ik}n_{ik}},\quad \mathbf{r}_{ik}=\mathbf{r}_i-\mathbf{r}_k(t),\quad \mathbf{n}_{ik}=\mathbf{r}_{ik}/r_{ik}. \quad (6)$$

If we seek the solution to Eq. (6) in the form of an expansion:

$$\mathbf{A}=\mathbf{A}^{(0)}+\mathbf{A}^{(1)}+\mathbf{A}^{(2)}+\ldots, \quad (7)$$

then for $\mathbf{A}^{(n)}$ we obtain the system of equations

$$\Delta\mathbf{A}^{(n)}=\left(\frac{\partial}{\partial\tau}\right)^2\mathbf{A}^{(n-1)}$$

where $\tau = ct$, the operator Δ acts on the variable $\mathbf{r}_i$, and $\mathbf{r}_k = \mathbf{r}_k(t)$ is assumed to be a function of the time.

The solution to this system of equations are:

$$\mathbf{A}^{(n)}=\left(\frac{\partial}{\partial\tau}\right)^{2n}\hat{O}\cdot{\sum_k}' e_k\beta_k\frac{|\mathbf{r}_i-\mathbf{r}_k(t)|^{2n+1}}{(2n+2)!}, \quad (8)$$

where $\hat{O} = \hat{\delta}\Delta_{\mathbf{r}_i} - \hat{\nabla}_{\mathbf{r}_i}\hat{\nabla}_{\mathbf{r}_l}$.

Introducing the notation $\mathbf{R} = \mathbf{r}_i - \mathbf{r}_k(t)$, we can write

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$$\hat{O}\cdot\beta_k|\mathbf{R}|^{2n+1}=(2n+1)\left[\beta_k(2n+1)R^{2n-1}+\mathbf{R}\frac{\partial}{\partial\tau}R^{2n-1}\right],$$

and therefore for **A** we find

$$\mathbf{A}=\sum_{n=0}^{\infty}\mathbf{A}^{(n)}=\sum_k{}' e_k\left(\beta_k\lambda+\mathbf{r}_{ik}\frac{\partial\lambda}{\partial\tau}\right), \quad (9)$$

where

$$\lambda=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)}\frac{1}{(2n)!}\left(\frac{\partial}{\partial\tau}\right)^{2n}R^{2n-1}.$$

Since we wish to obtain a Lagrangian not containing particle accelerations, the $\tau = ct$ derivatives of $|\mathbf{R}| = |\mathbf{r}_i - \mathbf{r}_k(t)|$ should be computed, discarding all the time derivatives of the function $\mathbf{r}_k(t)$ higher than the first, We then find

$$\left(\frac{\partial}{\partial\tau}\right)^{2n}R^{2n-1}=\frac{(2n-1!!)^2}{r_{ik}}[\mathbf{n}_{ik}\beta_k]^{2n}, \quad (10)$$

while for the quantity λ and its first derivative, we have

$$\lambda=\frac{1}{2r_{ik}}\sum_{n=0}^{\infty}\frac{(2n-1!!)^2}{(n+1)}\frac{[\mathbf{n}_{ik}\beta_k]^{2n}}{(2n)!}=\frac{1}{r_{ik}(1+\varkappa_{ik}^k)},$$
$$\frac{\partial\lambda}{\partial\tau}=\frac{\partial}{\partial\tau}[r_{ik}+\{(r_{ik}\beta_k)^2+(1-\beta_k^2)r_{ik}^2\}^{1/2}]^{-1}=\frac{1}{r_{ik}^2}\frac{\mathbf{n}_{ik}\beta_k}{(1+\varkappa_{ik}^k)\varkappa_{ik}^k}, \quad (11)$$

where $\kappa_{ik}^k = \{1 - [\mathbf{n}_{ik}\beta_k]^2\}^{1/2}$.

Thus, the vector potential (9) can be written in the form

$$\mathbf{A}=\sum_k{}'\frac{e_k}{r_{ik}}\left(\frac{\beta_k}{1+\varkappa_{ik}^k}+\frac{\mathbf{n}_{ik}(\mathbf{n}_{ik}\beta_k)}{\varkappa_{ik}^k(1+\varkappa_{ik}^k)}\right), \quad (12)$$

and then for the Lagrangian (4) we find

$$L=-\sum_i m_ic^2\sqrt{1-\beta_i^2}-\frac{1}{2}\sum_{i\neq k}\sum\frac{e_ie_k}{r_{ik}}\left[1-\frac{\beta_i\beta_k}{(1+\varkappa_{ik}^k)}-\frac{(\mathbf{n}_{ik}\beta_i)(\mathbf{n}_{ik}\beta_k)}{\varkappa_{ik}^k(1+\varkappa_{ik}^k)}\right] \quad (13)$$

Notice that the Lagrangian (13) could have been obtained in another way by first writing down the Lagrangian of one particle in the field of the other charges as they move in a prescribed manner: to wit, rectilinearly.

As is well known, no radiation is emitted in such a motion. If the individual Lagrangian obtained in this way is generalized in the usual manner for the entire system of charges, then we again arrive at the expression (13).

3. From the model Lagrangian (13) follows the law of conservation of energy:

$$E=\sum_i\frac{m_ic^2}{\sqrt{1-\beta_i^2}}+\frac{1}{2}\sum_{i\neq k}\sum\frac{e_ie_k}{r_{ik}}\left[1+\frac{\beta_i\beta_k}{\varkappa_{ik}^k(1+\varkappa_{ik}^k)}+\frac{(\mathbf{n}_{ik}\beta_i)(\mathbf{n}_{ik}\beta_k)}{\varkappa_{ik}^{k3}(1+\varkappa_{ik}^k)}(1+\varkappa_{ik}^k-\varkappa_{ik}^{k2})\right]. \quad (14)$$

However, it would be difficult to derive in explicit form the Hamiltonian expressed in terms of the generalized momenta.

The free energy of the system described by the model Lagrangian (13) turns out to be equal to: $F = F_{id} + \Delta F_D + \Delta F_R$, where F_{id} is the free energy of the ideal gas, $\Delta F_D = -V\theta/12\pi d^3$ is the standard Debye correction $(d^{-2} = \Sigma_\alpha 4\pi n_\alpha e_\alpha^2/\theta)$, while $\Delta F_R = \kappa\,|\Delta F_D|$ takes into account all the relativistic corrections due to the interaction.

The factor κ is expressible in terms of a fairly unwieldy combination of Macdonald's functions $K_{1,2}(m_\alpha c^2/\theta)$ and, for example, in the weakly relativistic limit when $\nu_e = m_e c^2/\theta \gg 1$, it is equal to

$$\varkappa\approx\frac{2}{(1+z)^{3/2}\nu_e^{1/2}}\left(1-\frac{11}{2\nu_e}+\frac{261}{32}\frac{1}{\nu_e^2}+\ldots\right)\ll 1. \quad (15)$$

where $z = |e_i/e_e|$ is the ion charge. In this case ΔF_R is $(v_{Te}/c)^3$ times smaller than ΔF_D, which coincides with the results obtained by us earlier[3] with the aid of the Darwin Lagrangian.

The model Lagrangian (13) can also be applied to the case when $\theta \sim mc^2$ and, although it cannot be used to investigate an ultrarelativistic system, we nevertheless formally have for κ for the ultrarelativistic case $(\nu \ll 1)$ the expression

$$\varkappa\approx-\frac{2\cdot 5^{1/4}}{\nu^3(1+z)^{3/2}}\left(1-\frac{\nu^{3/2}}{2^{1/4}\cdot 5^{3/4}}+\ldots\right),\quad |\varkappa|\gg 1, \quad (16)$$

so that here the relativistic correction turns out to be negative and greater than the Debye correction.

In conclusion, let us note that although the considered Lagrangian formally does not contain the limitation $v \ll c$, it nevertheless cannot be used in the ultrarelativistic case, since when $\theta \gg mc^2$ the cross section for bremsstrahlung emission

$$\langle\sigma_{\text{brem}}^{\text{ultra}}\rangle\approx 8\alpha r_0^2\ln(kT/mc^2)$$

can exceed the cross section for Coulomb scattering of the particles

$$\langle\sigma_{\text{coul}}^{\text{ultra}}\rangle\approx 2\pi\Lambda r_0^2(mc^2/kT)^2,$$

and the neglect of the radiation is then inadmissible. (Here $\alpha = 1/137$, $r_0 = e^2/mc^2$, and Λ is the Coulomb logarithm.)

*[AH] ≡ A × H.

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