A modified Korteweg–de Vries equation in electrohydrodynamics

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In the framework of a modified Korteweg–de Vries (KdV) equation we study waves on the free surface of an ideally conducting liquid in the presence of transverse electric field. It is shown that the solution is a two-parameter family of solitons, Eq. (16), and a family of shock waves, Eq. (19). A qualitative study is made of the evolution in time of the modified KdV equation for local initial distributions, and it is shown that the cubic nonlinearity leads to the formation of shock waves and hinders the production of solitons. The problem of the scattering of a soliton by a shock wave is solved exactly. By means of the nonlinear Miura transformation, Eq. (38), a one-parameter family of solitons is found for the KdV equation.

For nonlinear waves in weakly dispersive media there are two essential processes; the increase of steepness of the wave profile owing to nonlinear effects, as known from gas dynamics, and the dispersion spreading of the profile.

If the amplitude is not too large the dispersion is able to compete with the nonlinearity. Through the equilibrium of these two processes it becomes possible for waves to exist which propagate with constant velocity without deformation of the profile. It would seem that with increasing amplitude the nonlinear effects should prevail and lead to the formation of shock waves. However, if we take the nonlinear and dispersion terms into account in the equations only to the first nonvanishing approximation, which leads to the Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t}\pm 6u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0,\qquad \beta>0, \tag{1}$$

then for sufficiently smooth initial conditions the Cauchy problem reduces to the soliton mode, with the number of solitons determined by the similarity parameter σ.[1] As for the averaged solutions, for large σ they indeed behave like shock waves.

There are, however, many physical situations in which the nonlinearity must be taken into account to higher order, in particular, to cubic order.[3,4] We then get the so-called modified Korteweg–de Vries (MKdV) equation

$$\frac{\partial v}{\partial t}\pm v^2\frac{\partial v}{\partial x}+\beta\frac{\partial^3 v}{\partial x^3}=0,\quad \beta>0. \tag{2}$$

A considerable part of the present paper is devoted to studying the solutions of this equation.

The first section presents a formulation of the problem of the propagation of electrohydrodynamic surface waves, leading to the MKdV equation. The second section gives stationary families of solutions of the soliton type and of the shock-wave type; a qualitative treatment of the Cauchy problem is presented in the case of local initial conditions. The exact solution of the problem of elastic scattering of a soliton by a shock wave occupies the third section. Finally, the last section is devoted to the investigation of the connection between these solutions of Eq. (2) and the solutions of Eq. (1).

1. FORMULATION OF THE PROBLEM AND DERIVATION OF EQUATIONS

We consider waves on the surface of an ideally conducting nonviscous incompressible liquid in the presence of a transverse electric field. The unperturbed surface of the liquid is in the XY plane, and the Z axis extends vertically upward. The electric field is directed perpendicular to the free surface. The scheme of flow is shown in Fig. 1. We shall write the equations of electrohydrodynamics for this case.

The velocity potential Φ satisfies the Laplace equation

$$\Delta\Phi=0 \tag{3}$$

with the following kinematic conditions at the bottom and at the free surface

$$\frac{\partial\Phi}{\partial z}=0,\quad z=-h, \tag{4}$$

$$\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial x}\frac{\partial\Phi}{\partial x}+\frac{\partial\eta}{\partial y}\frac{\partial\Phi}{\partial y}-\frac{\partial\Phi}{\partial z}=0,\quad z=\eta(x,y,t). \tag{5}$$

The potential of the electric field satisfies

$$\Delta\varphi=0 \tag{6}$$

with the conditions

$$\varphi=V_0,\quad z=h, \tag{7}$$

$$\varphi=0,\quad z=\eta(x,y,t). \tag{8}$$

Equality of the normal stress on the free surface gives the dynamic equation

$$\rho\frac{\partial\Phi}{\partial t}+\frac{1}{2}\rho(\nabla\Phi)^2+\rho g\eta+\alpha\left(\frac{1}{R_1}+\frac{1}{R_2}\right)-\frac{1}{2}\left(\frac{\partial\varphi}{\partial n}\right)^2=0,\quad z=\eta(x,y,t), \tag{9}$$

where n is the normal to the surface, α is the surface tension coefficient of the liquid, and R_1 and R_2 are the principal radii of curvature of the free surface. If η is sufficiently small, we can take

$$\frac{1}{R_1}+\frac{1}{R_2}=-\Delta\eta. \tag{10}$$

There are two small parameters in the problem,

$$\varepsilon=\frac{h}{l}\ll 1,\quad \mu=\frac{v_0}{c_0}\ll 1,\quad \varepsilon^2\sim\mu;$$

here l is the characteristic linear scale of the perturbation, v_0 is the velocity amplitude of the liquid particles, and $c_0=(gh-E_0^2/\rho)^{1/2}$ is the limiting speed of propagation of long waves in the medium.

In [5] an equation is derived which contains the nonlinear and dispersion terms in the first nonvanishing approximation

$$\frac{\partial u}{\partial t}+\frac{(\rho c_0^2-E_0^2)}{\rho c_0^2}u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0,\quad \beta=\frac{c_0}{6}\left(h^2+\frac{E_0^2h^2}{\rho c_0^2}-\frac{3\alpha h}{\rho c_0^2}\right), \tag{11}$$

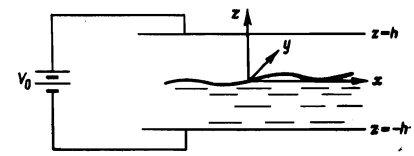

FIG. 1. Scheme of flow of an ideally conducting liquid in the presence of a transverse electric field with potential V_0.

where $u=(3/2)\partial\Psi/\partial x$, and $\Psi(x, y, t)=\Phi(x, y, \eta, t)$ is the surface potential. The soliton solutions of this equation are of the form

$$u=\pm\frac{\rho c_0^2}{|\rho c_0^2-E_0^2|}a\,\mathrm{sech}^2\sqrt{\frac{a}{12\beta}}\left(x\mp\frac{1}{3}at\right). \qquad (12)$$

Let us consider the case when the coefficient of the nonlinear term in Eq. (11) is zero. In this case the problem is to obtain equations containing nonlinear terms in the next approximation. In order not to make the analysis cumbersome, we assume that $\epsilon\sim\mu$. Using the standard method expounded in [1], we obtain the equation for the surface potential Ψ, which is written as follows:

$$\frac{\partial^2\Psi}{\partial t^2}-\Delta\Psi-\varepsilon^2\left(\frac{2}{3}-\frac{\alpha}{h\rho c_0^2}\right)\Delta^2\Psi+2\mu(\nabla\Psi)\nabla\frac{\partial\Psi}{\partial t}-2\mu\Delta\Psi\frac{\partial\Psi}{\partial t}+12\mu^2\left(\frac{\partial\Psi}{\partial t}\right)^2\Delta\Psi=0. \qquad (13)$$

Introducing a new independent variable (sic)

$$v=\frac{1}{\sqrt{c_0}}\frac{\partial\Psi}{\partial x}, \qquad (14)$$

up to small quantities of second order we can get for it the equation (2), where β is given by

$$\beta=\frac{c_0}{6}\left(2h^2-3\frac{\alpha h}{\rho c_0^2}\right).$$

The writers have made a numerical study of Eq. (2) for Gaussian initial distributions, and used it as the basis for an attempt to construct a qualitative classification of the possible types of evolution in time.

2. CLASSIFICATION OF CAUCHY PROBLEMS FOR THE EQUATION (2)

The evolution of an initial perturbation for Eq. (2) depends in an essential way on the sign of the nonlinearity [in what follows we shall distinguish Eqs. (2+) and (2-) for the corresponding signs]. For example, Eq. (2+) has a one-parameter family of solitary waves,

$$v=a\,\mathrm{sech}\frac{a}{\sqrt{\beta}}(x-a^2t) \qquad (15)$$

and the evolution goes in analogy with the KdV equation (1), i.e., any local initial distribution will at large times break up into a set of separate solitons and a spreading oscillating tail.

The behavior of an initial distribution will be qualitatively different for Eq. (2-), which allows as stationary solutions a two-parameter family of solitary waves with profiles described by the expression

$$v=\alpha\left(1-\frac{4-\varkappa_c}{1+\dfrac{2(2-\varkappa_c)}{\varkappa_c}\mathrm{ch}^2 z}\right), \qquad (16)$$

where

$$z=\frac{\alpha(4\varkappa_c-\varkappa_c^2)^{1/2}(x-ct)}{2\sqrt{\beta}}$$

and with velocity and amplitude given by the respective formulas

$$c=-\alpha^2(6-4\varkappa_c+\varkappa_c^2), \qquad (17)$$

$$a=\alpha\varkappa_c,\quad 0<\varkappa_c<2. \qquad (18)$$

Besides the solution (16) there is a one-parameter family of shock waves with profile

$$v=\alpha\,\mathrm{th}\frac{\alpha}{\sqrt{\beta}}(x-ct), \qquad (19)$$

discontinuity value 2α, and width $(\beta/\alpha)^{1/2}$ of the transition region. The speed is given by Eq. (18) with $\kappa_c=2$.

The meaning of the parameter κ_c can be most simply explained if in Eqs. (2-) and (16) we make the transformation $v\to v+\alpha$; then Eq. (2-) can be rewritten:

$$v_t-6v^2v_x-12\alpha vv_x+\beta v_{xxx}=0. \qquad (20)$$

Equation (20) possesses a two-parameter family of solitons which go to zero at infinity

$$v=-\frac{\alpha(4-\varkappa_c)}{1+\dfrac{2(2-\varkappa_c)}{\varkappa_c}\mathrm{ch}^2 z} \qquad (21)$$

and propagate with the velocity $c=\alpha^2(4\kappa_c-\kappa_c^2)$. From an estimate of the orders of magnitude of the nonlinear terms it can be seen that κ_c characterizes the ratio of the cubic nonlinear term to the quadratic term. Since for $\kappa_c\to 0$ ($a=\mathrm{const}$) the cubic nonlinearity is vanishingly small in comparison with the quadratic, in this case Eq. (20) can be reduced by the substitution $v\to u/2\alpha$ to the ordinary KdV equation (1-), and the family (21) naturally goes over into the one-parameter family of KdV solitons:

$$v=-a\,\mathrm{sech}^2\sqrt{\frac{a}{2\beta}}(x-2at). \qquad (22)$$

To examine the other limiting case $\kappa_c\to 2$ we put Eq. (21) in the form

$$v=-\alpha\nu(\mathrm{th}\,\xi_+-\mathrm{th}\,\xi_-), \qquad (23)$$

$$\nu=\frac{(4\varkappa_c-\varkappa_c^2)^{1/2}}{2},\quad 0<\nu<1,\quad \xi_\pm=\frac{\alpha}{\sqrt{\beta}}\nu(x-ct)\pm\Delta,$$

$$c=4\alpha^2\nu^2 \qquad (24)$$

It is obvious that for $\kappa_c\to 2$ ($\nu\to 1$) the family (21) goes over into a "limiting soliton," which is a wave with two shock transitions (19) of different signs, and with the discontinuities separated by the distance 2Δ, where

$$\Delta=\frac{1}{4}\ln\frac{1+\nu}{1-\nu}. \qquad (25)$$

Let us consider the solution of Eq. (20) with the initial condition

$$v(x, 0)=v_0f(x/l), \qquad (26)$$

where $f(\xi)$ is a dimensionless function characterizing the initial profile.

We rewrite Eq. (20) in dimensionless form

$$v_t-12\varkappa^{-1}vv_x-6v^2v_x+\varkappa^{-1}\sigma^{-2}v_{xxx}=0, \qquad (27)$$

where $\kappa=v_0/\alpha$, $\sigma^2=l^2v_0\alpha/\beta$. It is found that the Cauchy problem for Eq. (27) with the initial condition (26), where $f(\xi)$ is a local distribution with zero asymptotes at $\pm\infty$, can be classified in terms of the value of the parameter κ.

Numerical calculation shows that for $0<\kappa<2$ an initial Gaussian perturbation is changed into a sequence of solitons and a spreading tail. Although the number of solitons does increase with increase of the nonlinearity parameter σ (for fixed κ), this does not occur in as simple a way as in the case of the KdV equation.[6] This is due to the fact that the nonlinearity parameter σ_c for the solitons (21) depends in its turn on κ. We define σ_c in the following way:

$$\sigma_c=\frac{1}{(\beta\varkappa_c)^{1/2}}\int_{-\infty}^{\infty} v\,dx=\frac{1}{\varkappa_c^{1/2}}\ln\frac{2+(4\varkappa_c-\varkappa_c^2)^{1/2}}{2-(4\varkappa_c-\varkappa_c^2)^{1/2}}. \qquad (28)$$

It is obvious that σ_c increases with increasing κ. It was further found from the numerical solution that for $\kappa > 2$ and for arbitrarily large σ the evolution reduces to the formation from the initial perturbation of a single "limiting soliton" and a spreading tail.

In view of all this we can assert that the cubic nonlinearity, in contrast with the quadratic, hinders the formation of solitons and favors the formation of shock waves.

Besides studying local perturbations, the writers have made a numerical study of Eq. (2-) for initial distributions with the asymptotic value

$$u(+\infty)=-u(-\infty)=\alpha.$$

The result found was that the solution for large times can be represented in the form of a superposition of a shock wave (19), solitons (21), and a spreading tail.

Accordingly, unlike the KdV equation, for which the solitons (22) play the single main role, the MKdV equation has two stationary families of solutions (16) and (19), which completely determine the evolution in time. This makes the question of the stability of these solutions extremely important. Machine calculations have shown that solitons are stable with respect to collisions with each other, i.e., they do not change their amplitudes and velocities, and merely acquire additional phases. Moreover, it was found that the process of collision of a soliton with a shock wave is also elastic.

It is remarkable that, in contrast with the case of collision of two solitons, by knowing from a numerical experiment the qualitative picture of the process of collision of a soliton and a shock wave we can obtain exact expressions for the phase shifts of the soliton and the shock wave, using only conservation laws.

3. SCATTERING OF A SOLITON BY A SHOCK WAVE

Let us consider the interaction of a soliton with a shock wave as a scattering process. It is not hard to show that for large negative t the superposition of a soliton with a shock wave, i.e.,

$$u=\alpha\,\mathrm{th}\frac{\alpha}{\sqrt{\beta}}x-\alpha\nu(\mathrm{th}\,\xi_+-\mathrm{th}\,\xi_-) \qquad (29)$$

is an asymptotic ($x-ct\to-\infty$) solution of Eq. (2-). The expression (29) is written in the reference system in which the shock wave is at rest and the soliton moves relative to it with velocity $c=-\alpha^2(4-4\nu^2)$.

A numerical investigation of Eq. (2-) with the asymptotic condition (29) showed (see Fig. 2) that for large positive t the solution can be represented in the form

$$u=\alpha\,\mathrm{th}\left(\frac{\alpha}{\sqrt{\beta}}x-2\delta_y\right)+\alpha\nu[\mathrm{th}(\xi_+-\delta_c)-\mathrm{th}(\xi_--\delta_c)], \qquad (30)$$

where

$$x-ct-\delta_c\to+\infty.$$

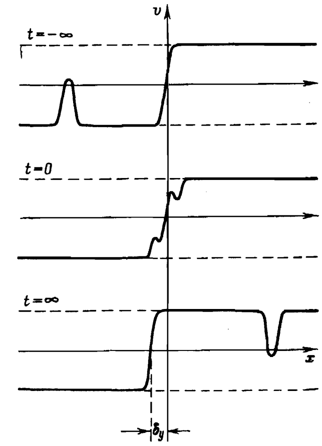

FIG. 2. Scattering of a soliton by a shock wave for the MKdV equation.

Accordingly, the result of scattering of a soliton is shifts of the phases of the soliton and the shock wave, and inversion of the soliton. The phase shifts can be found by starting from the conservation laws.

We introduce the function

$$\frac{\partial F}{\partial x}=\frac{v^2-\alpha^2}{2}. \qquad (31)$$

Using the fact that the quantity

$$\int_{-\infty}^{\infty}\left(\frac{v^4-\alpha^4}{4}+\frac{3}{2}v_x^2\right)dx,$$

is conserved, we can get from Eq. (2-)

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}F\,dx=\mathrm{const}. \qquad (32)$$

Also, it follows from (2-) and (31) that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}F\,dx=-\frac{\partial}{\partial t}\int_{-\infty}^{\infty}x\frac{v^2-\alpha^2}{2}dx, \qquad (33)$$

from which we get

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}x\frac{v^2-\alpha^2}{2}dx=\mathrm{const}. \qquad (34)$$

From Eqs. (29), (30) and (34) it follows that the total phase shift of the soliton and the shock wave is zero:

$$\delta_c+\delta_y=0. \qquad (35)$$

On the other hand, from the conservation law

$$\int_{-\infty}^{\infty}v\,dx=\mathrm{const} \qquad (36)$$

we can calculate the phase shift of the shock wave

$$\delta_y=-\frac{s}{2}=-\frac{1}{2}\ln\frac{1+\nu}{1-\nu}, \qquad (37)$$

where s is the area of the soliton. Thus in a collision the soliton receives a positive phase shift, and the shock wave a negative shift. By the way, this process can be interpreted as the interaction of three shock waves, from which indeed follows the double phase shift in the expression (30).

4. THE ONE-PARAMETER FAMILY OF SOLITONS FOR THE KdV EQUATION

Miura[7] has found a nonlinear transformation connecting a solution of the MKdV equation with solutions of the KdV equation. If v satisfies Eq. (2-), then

$$u=v^2\pm v_x\sqrt{\beta} \qquad (38)$$

is a solution of Eq. (1). The shock wave (19) is transformed by the relation (38-) into the soliton (22). This same relation allows us to find a one-parameter family of solitons analogous to (22). We substitute for v the solution (16), and using the fact that Eq. (1) is invariant under Galilei transformations, obtain a one-parameter family of solitary waves which go to zero at infinity, with velocities and amplitudes given respectively by

$$c=4\alpha^2\nu^2, \quad a=-2\alpha^2\nu^2.$$

The profile is described by the expression

$$u=-2\alpha^2\nu^2 \operatorname{sech}^2 \xi_{\pm}. \tag{39}$$

Accordingly, the nonlinear transformation of Miura, Eq. (38), changes both shock-wave type solutions (16) and also soliton solutions (19) of the MKdV equation into solitons of the KdV equation.

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