A model of turbulence

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The mechanism of the onset of turbulence in active and nonequilibrium media (for which the absence of an inertial interval is characteristic and, consequently, the turbulence cannot be of the Kolmogorov type) is discussed and, in the case of a one-dimensional medium, investigated experimentally. It is shown that the onset and development of turbulence in such media can be induced by a cascade-like increase in the number of parametrically coupled waves with incommensurable frequencies, which build up as a result of an explosive instability. The boundary for the onset of turbulence is determined.

1. It is well known that the onset of turbulence—the disordered motion of a nonlinear nonequilibrium medium—is connected with instability of the laminar flow, as a result of which rapidly oscillating perturbations arise, with a continuously broadening spectrum of frequencies and arbitrary phases[1-4]. The intensity of this disordered motion is limited by the linear or nonlinear dissipation. In many cases (e.g., for gravitational and capillary waves, acoustic turbulence, etc.[5]), the regions in ω-space corresponding to dissipation and instability do not overlap and are separated by a large region of "transparency" (the inertial interval[3,4]). Such turbulence, known as Kolmogorov turbulence, is characterized in the steady-state regime by a stationary energy flux through the inertial interval; its spectrum is determined by the nonlinear and dispersive characteristics of the medium in the transparency region, and, in the framework of the weak-turbulence approximation, is found from the solution of the corresponding kinetic equation for the waves, which describes a conservative system[5,6].

There exists, however, a broad class of nonequilibrium (including active) nonlinear media for which the absence of an inertial interval is characteristic, and, consequently, the turbulence cannot be of the Kolmogorov type. Belonging to this class are media with nonlinear dissipation (semiconductors with hot electrons, piezo-semiconductors, a plasma with collisions, p-n tunnel junctions, etc.), and also media in which the development of the instability is associated with the interaction of waves with opposite signs of the energy[7]. The mechanism of the establishment of turbulence in such media depends on their concrete dispersive, dissipative and nonlinear characteristics. For example, in active media with cubic nonlinearity (the imaginary part of the dielectric constant is proportional to the square of the field), under certain conditions generation of a large number of waves (modes) with incommensurable frequencies is possible. Even such a process is disordered—its correlation function, according to a theorem of Kac[8,9], is, for all x and t except for isolated points, small and proportional to $n^{-1/2}$, where n is the number of incommensurable frequencies represented in the spectrum[1)]. Such a situation, however, is not typical for the majority of active and nonequilibrium media. This is connected with the fact that, in contrast to reactive cubic (and, in general, odd) nonlinearity, which, as is well known, leads to overlap of the resonances and to the establishment of stochasticity in the dynamical system[11], dissipative nonlinearity that is odd in the field hinders the establishment of disordered motion. When the dispersion is strong, such nonlinearity usually leads to competition between the individual modes and to the establishment of a dynamical regime with a narrow spectrum, and, when the dispersion is weak, leads to synchronization of the frequencies of some modes with the harmonics of others and to the establishment of an ordered regime with a broad spectrum (of, in particular, nonlinear stationary waves[12]).

Below we discuss and investigate experimentally the mechanism of the onset of turbulence in nonequilibrium media whose nonlinearity is determined by the viscosity (conductivity) and contains components that are both odd and even in the field. It is shown that the onset of turbulence is facilitated by quadratic nonlinearity and hindered by cubic nonlinearity. The possible spectra of non-Kolmogorov turbulence are discussed.

2. As a simple example, we shall consider a wave medium with weak dispersion and noninertial nonlinearity, the medium being described by a nonlinear current $j(u) = \sigma(u, u_0)u$. In order that the dissipative medium can be nonequilibrium, the dependence $j(u)$ should have a decreasing part (like, e.g., the volt-ampere characteristic of a tunnel junction or of a Gunn semiconductor). The concrete form of the conductivity $\sigma(u, u_0)$ of such a nonequilibrium medium will be determined by the magnitude of the constant bias field u_0. When u_0 is greater than a certain critical value, the dependence $\sigma(u)$ can be represented in the form

$$\sigma(u,\ u_0)=-\gamma_1-\gamma_2 u+\gamma_3 u^2+\gamma_4 u^3+\gamma_5 u^4, \tag{1}$$

where γ_1 and γ_2 are responsible for the buildup of the oscillations and γ_3, γ_4 and γ_5 are responsible for the nonlinear damping; the values of γ_j depend on u_0: in particular, for $u_0 = u_0'$ we have $\gamma_{2,4,5} = 0$, $\gamma_{1,3} \neq 0$, and for $u_0 = u_0'' < u_0'$, on the other hand, $\gamma_1 = 0$ and $\gamma_2 \neq 0$.

Suppose that in such a medium the self-excitation conditions for a small number of quasi-harmonic perturbations with incommensurable frequencies ω_j are fulfilled. As a result of the action of the quadratic nonlinearity (we assume that $\gamma_{4,5} = 0$), perturbations arise in the medium at frequencies[2)] $\omega_l = \omega_i + \omega_j$, the number of which can be extremely large for weak dispersion and is determined by the ratio between the quadratic and cubic nonlinearities. The equations for the complex amplitude of these perturbations can be written in the form (we assume the increments to be small)

$$\partial a_j/\partial t=\zeta\left[\bar\gamma_1(\omega_j)a_j+\bar\gamma_2\left(\sum_{i,l<j}a_i a_l+\sum_{i>j,l<j}a_i a_l^*\right)-\bar\gamma_3 a_j\left(|a_j|^2+2\sum_i|a_i|^2\right)\right],$$
$$j=1,2,\ldots,n; \tag{2}$$

here a_j and t are dimensionless, $\bar\gamma_2 = \gamma_2/u^0$, $\bar\gamma_3 = \gamma_3/(u^0)^2$, u^0 is the mean amplitude of the pulsations $((u^0)^2 \sim \gamma_1(\bar u_0)/\gamma_3(\bar u_0))$, and $\zeta > 0$ has the meaning of the wave impedance of the medium; $\bar\gamma_1 = \gamma_1 - \gamma(\omega)$, where $\gamma(\omega)$ characterizes the linear losses in the medium.

The most fundamental feature of the system (2) is the equal signs of the real coefficients $\overline{\gamma}_2$ of the terms $a_i a_1$ and $a_i a_1^*$. If $\gamma_3 = 0$, then, according to these equations, perturbations which existed at the initial moment of time and satisfy the parametric-coupling condition increase explosively[13], and, in addition, new modes with incommensurable frequencies are generated, the number of which increases according to the law $\dot{n} \sim n^2$. Thus, in the absence of nonlinear absorption, the interaction of perturbations with incommensurable frequencies in the dispersionless nonequilibrium medium under consideration would lead to the vanishing of the correlation function $R(t) \sim (n(t))^{-1/2}$ in a finite time $t_0 \sim 1/n_0$ (n_0 is the number of initially excited modes with incommensurable frequencies), i.e., in this time the motion would become completely randomized. The dissipative cubic nonlinearity hinders this process by limiting the number of perturbations of different scales and incommensurable frequencies. However, if the cubic nonlinearity begins to take effect substantially later than the quadratic nonlinearity, the motion of the medium has time to become turbulent, and the cubic nonlinearity limits only the intensity of the disordered pulsations. The spectrum of the pulsations in this case can be both discrete (quasi-periodic turbulence) and continuous (developed turbulence).

We shall trace the evolution of the stationary spectrum of pulsations when the ratio between the quadratic ($\overline{\gamma}_2$) and cubic ($\overline{\gamma}_3$) nonlinearities of the medium is changed. According to (2), for $\overline{\gamma}_2 = 0$ ($u_0 = u_0'$), for arbitrary initial conditions a single-frequency regime is established in the medium as a result of competition of the modes (cf., e.g.,[14]); with increase of $\overline{\gamma}_2$ ($u_0 \lesssim u_0'$) stable many-frequency regimes appear, for which the amplitude of one of the spectral components is substantially greater than the other amplitudes. As one can convince oneself, further increase of $\overline{\gamma}_2$ leads to the appearance of stable regimes with a large number of intensely excited modes. For $\overline{\gamma}_2 = \overline{\gamma}_{2,cr}$ all the quasi-single-frequency regimes become unstable (for a two-frequency degenerate interaction, $\overline{\gamma}_{2,cr} = (\overline{\gamma}_1\overline{\gamma}_3)^{1/2}/\sqrt{2}$). Evidently, it is precisely this value of $\overline{\gamma}_2$ that must be regarded as the boundary for the onset of turbulence. In order of magnitude, $\overline{\gamma}_{2,cr} \sim (2\overline{\gamma}_1\overline{\gamma}_3)^{1/2}/n^{1/2}$, where n is the number of intensely excited modes.

The shape of the stationary-turbulence spectrum can be determined from the kinetic equation for the waves, which follows from (2) after averaging over the phases:

$$\partial N_k/\partial t = \zeta\Big[\bar{\gamma}_1(k)N_k + \sum_{k'k''}\bar{\gamma}_2^2(k)(N_{k'}N_{k''} + N_{k'}N_k + N_kN_{k''})\delta_{kk'k''} - \sum_{k'}\bar{\gamma}_3(k')N_k(2-\delta_{kk'})N_{k'}\Big]. \quad (3)$$

For example, in the case when both the linear and the nonlinear losses do not depend on the frequency, all the $\overline{\gamma}$ = const and pulsations are established in the medium, with a uniform distribution of intensity over the spectrum in the entire region of the interaction[3]:

$$N_k = N = \frac{\bar{\gamma}_1/n}{2\bar{\gamma}_3 - 3\bar{\gamma}_2^2}. \quad (4)$$

From (3) and (4) it follows, in particular, that in the weak-turbulence approximation the nonresonant absorption associated with the cubic nonlinearity does not always stabilize the exposive instability (in contrast to the case of dynamical phases), and it is necessary to take into account either the more rapid absorption (e.g., γ_5 in (1)) or the resonant four-wave interactions in the dissipative cubic nonlinearity:

$$\omega_{k'} + \omega_{k''} + \omega_{k'''} = \omega_k,$$

which lead to the appearance in (3) of the term

$$-\bar{\gamma}_3^2 \sum_{k'k''k'''}(N_{k'}N_{k''}N_{k'''} + N_{k'}N_kN_{k'''} + N_kN_{k''}N_{k'''} + N_{k'}N_{k''}N_k)\delta_{kk'k''k'''}.$$

Then, in the stationary case, the excitation intensity will be

$$N_k = N = \frac{3\bar{\gamma}_2 - 2\bar{\gamma}_3}{8\bar{\gamma}_3^2 n} + \left[\left(\frac{3\bar{\gamma}_2^2 - 2\bar{\gamma}_3}{8\bar{\gamma}_3^2 n}\right)^2 + \frac{\bar{\gamma}_1}{4\bar{\gamma}_3^2 n^2}\right]^{1/2}. \quad (5)$$

If linear high-frequency viscosity is present in a medium with a nonlinear current (1), $\overline{\gamma}_{2,3}$ will remain constant and $\overline{\gamma}_1$ will depend on k: $\overline{\gamma}_1(k) = \overline{\gamma}_1 - \nu k^2$, and as a result N_k will be a decreasing function of k. It is important, however, that the effect of $\overline{\gamma}_1$ becomes insignificant on decrease of the magnitude of the cubic nonlinearity ($\overline{\gamma}_3$) (see (5)), and the equilibrium spectrum will approach N = const.

3. The mechanism we have discussed for the onset of turbulence has been observed and studied experimentally for radio-frequency electromagnetic waves in a one-dimensional active medium—a wave guide with a nonlinear loss of the form (1). Tunnel diodes were used as the elements of the nonlinear loss, and the relationship between the coefficients $\gamma_{2,3,4,5}$ was varied by changing the position of the operating point on the characteristic of the diodes.

Two different experiments were performed. In the first the onset of turbulence was associated with the interaction of low-frequency and high-frequency waves, and in the second, with the interaction of waves of the same type, the frequencies of which occupied the interval from 0 to ω_{max}. In a finite line with weak dispersion, the conditions were fulfilled for self-excitation of low-frequency (Ω_0, q_0) and a few high-frequency (ω_i, k_i) waves, satisfying the dispersion equation[4]

$$\omega = \pm(ck + \omega_0), \quad \omega > \omega_0,$$
$$\Omega_0 = \pm c q_0, \quad \Omega_0 \ll \omega_0. \quad (6)$$

The synchronism condition

$$\omega_i \pm \Omega_0 = \omega_{i\pm1}, \quad k_i \pm q_0 = k_{i\pm1} \quad (7)$$

is clearly fulfilled here for any i = 2, 3, . . . , m, with $p\Omega_0 \neq s\omega_i$, where p and s are integers.

For small quadratic nonlinearity[5], $\overline{\gamma}_2 \ll \overline{\gamma}_3$, $\overline{\gamma}_5$ (the operating point is at the center of the decreasing part of the diodes' characteristic, u_0 = 0.3 V), a purely dynamical regime, the spectrum of which is represented in Fig. 1, is established in the system. The comparatively small number of components in the spectrum is explained by the strong competition of waves of different frequencies in an active medium with cubic nonlinearity (cf., e.g.,[12]). With increase of the quadratic nonlinearity, the spectrum of the absorbed oscillations was

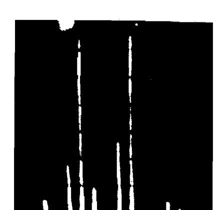

FIG. 1. Spectrum of the HF oscillations of the dynamical process established in the absence of quadratic nonlinearity ($u_0 \sim 0.3$ V).

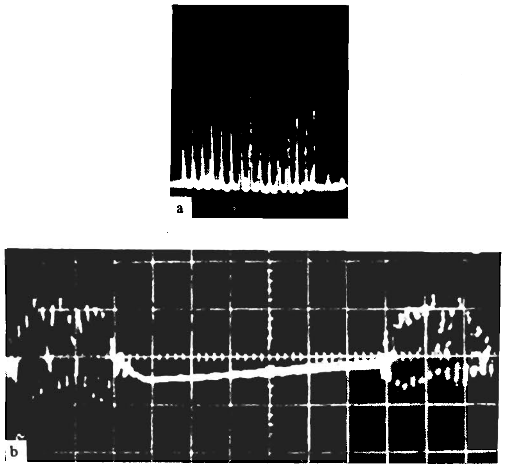

FIG. 2. Spectrum (a) and oscillogram (b) of the regime of quasi-periodic turbulence ($u_0 \sim 0.2$ V).

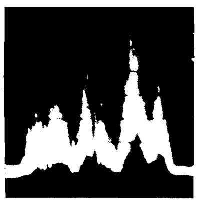

FIG. 3

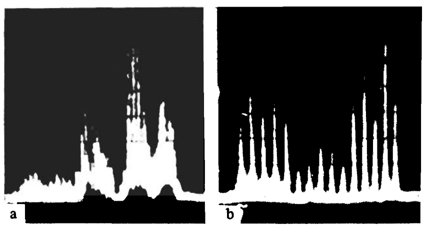

FIG. 4

FIG. 3. Spectrum of developed turbulence ($u_0 < 0.2$ V).

FIG. 4. Spectra of the developed (a) and quasi-periodic (b) turbulences established for the same bias ($u_0 \sim 0.15$ V) but different initial conditions.

continuously broadened, the connection between the phases of the oscillations at different points of the line and at different moments of time was lost, and for $\overline{\gamma}_2 \sim \overline{\gamma}_3, \overline{\gamma}_5$ ($u_0 \sim 0.2$ V) a regime of disordered pulsations with a discrete spectrum (quasi-periodic turbulence) was established in the system. The spectrum of this regime and its oscillogram are presented in Fig. 2. On the oscillogram, obtained by means of a high-speed oscillograph with a memory, the phases of the quasi-periodic oscillations are clearly visible. A further increase in the role of the quadratic nonlinearity ($u_0 \lesssim 0.15$ V) led to the establishment of developed turbulence, the spectrum of which is presented in Fig. 3. We call attention to the fact that the character of the turbulent regime depended essentially on the initial conditions, i.e., on the intensity of the initial excitation of the different components of the spectrum. For the same system parameters, both developed turbulence and turbulence with a discrete spectrum were established, depending on the previous history. Both these regimes were observed, in particular, for $u_0 = 0.15$ V (see Fig. 4)—the first when the bias was increased from 0.13 V and the second when it was decreased from 0.17 V.

Thus, there exists a region of the parameters $\overline{\gamma}_2$ and $\overline{\gamma}_3$ (the values of $\overline{\gamma}_1$, $\overline{\gamma}_4$ and $\overline{\gamma}_5$ in this region of biases practically do not change) with soft excitation of quasi-periodic turbulence and hard excitation of developed turbulence. This result is explained by the fact that the boundary of the onset of turbulence (the value $\overline{\gamma}_{2,cr}$) depends strongly on the number of intensely excited modes at the initial moment of time: if this number is sufficiently large the boundary of existence of the turbulence is lowered, and in the opposite case an increase of $\overline{\gamma}_{2,cr}$ occurs (see Sec. 2). We note that a similar situation also occurs in hydrodynamics, where, for the same Reynolds numbers, both laminar and turbulent flow can be established, depending on the magnitude of the initial perturbations, and the boundary for the onset of turbulence is also raised for small initial perturbations.

The spectrum of the HF turbulence can be found from the kinetic equation for the waves. We shall write this out for the case when $\overline{\gamma}_2^2 \sim \overline{\gamma}_{1,3,5}$, and $\overline{\gamma}_3^2$, $\overline{\gamma}_4 \ll \overline{\gamma}_5$. It has the form

$$\dot{N}_k=\sigma_k\Big\{\bar{\gamma}_1 N_k+\bar{\gamma}_2^2(N_{k-q}+N_{k+q})(\sigma_q N_k+\sigma_k N_q)-\bar{\gamma}_3 N_k\Big[2N_q+\sum_{k'}(2-\delta_{kk'})N_{k'}\Big] -\bar{\gamma}_5 N_k\Big[3N_q^2+\sum_{k'}(3-2\delta_{kk'})N_{k'}^2+6N_k\Big(N_q+\sum_{k'}(1-\delta_{kk'})N_{k'}\Big) +6\Big(N_q+\sum_{k'\neq k}N_{k'}\Big)\sum_{k''\neq k}N_{k''}\Big]\Big\}, \quad \sigma_{k,q}=\sigma_{k,q}^*>0, \tag{8}$$

where N_q is the intensity of the low-frequency wave, determined by the equation

$$\dot{N}_q=\sigma_q\Big\{\bar{\gamma}_1 N_q+\bar{\gamma}_2^2\sum_k(N_{k-q}+N_{k+q})(\sigma_q N_k+\sigma_k N_q)-\bar{\gamma}_3 N_q\Big(N_q+2\sum_k N_k\Big) -\bar{\gamma}_5 N_q\Big[N_q^2+3\sum_k N_k^2+6N_q\sum_k N_k+6\sum_k N_k\sum_{k'\neq k}N_{k'}\Big]\Big\}. \tag{9}$$

As noted in Sec. 2, the shape of the equilibrium spectrum is determined by the spectral properties of the coefficients $\overline{\gamma}_j$, which are connected with the frequency dependence of the absorption.

To conclude this section, we add that the simultaneous growth of a large number of waves with comparatively close frequencies and unsynchronized phases in a nonequilibrium medium, leading to the establishment of turbulence with a discrete spectrum, is also possible when an external signal interacts with the characteristic high-frequency waves. Such an effect has been observed by Janus and Meyer in a CdS acoustic-electric generator[16].

4. Onset of turbulence as a result of explosive interaction of waves with frequencies of the same order was observed in a system of distributively coupled segments of identical active lines, with slightly different lengths. The spectrum of the linearly excited waves in such a system is the sum of two equidistant spectra with several different spacings between the modes. Clearly, in such a spectrum there are always incommensurable frequencies, which, in view of the absence of dispersion in the system, increase continuously in number in the presence of resonance nonlinear interaction.

For $\overline{\gamma}_2 \ll \overline{\gamma}_{3,5}$ a single-frequency regime was observed in the system; with increase of the quadratic nonlinearity the spectrum of the oscillations was broadened (see Fig. 5b) and for $\overline{\gamma}_2 \lesssim \overline{\gamma}_3$ already occupied the whole transmission band of the system (Fig. 5c). Further increase of $\overline{\gamma}_2$ led to the establishment of a regime of developed turbulence with a constant distribution of spectral intensity over the whole band (Fig. 5d). From the spectra given it can be seen that, on decrease of the cubic nonlinearity $\overline{\gamma}_3$, the fall in the intensity as a function of frequency, associated with the presence of linear high-frequency losses in the system, disappears. This is in agreement with the result obtained in Sec. 2. The oscillogram of the turbulent pulsations that were observed for $u_0 = 0.15$ V is shown in Fig. 6. The charac-

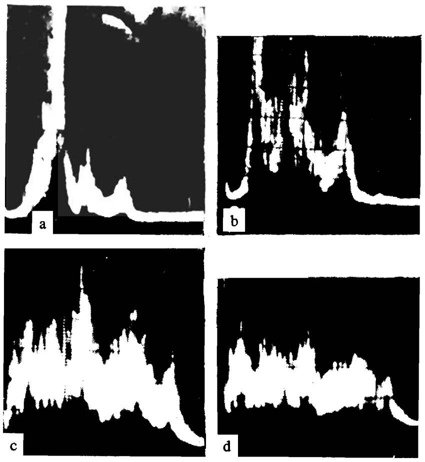

FIG. 5. Spectra of the developed turbulence in a system of coupled active wave guides ($a-u_0 = 0.3$ V, b–0.2 V, c–0.15 V, d–0.1 V).

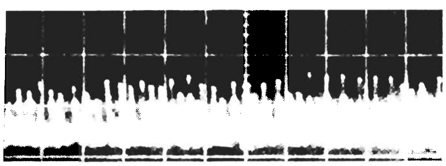

FIG. 6. Oscillogram of the turbulent regime in a system of coupled wave guides ($u_0 \sim 0.15$ V).

teristic period of the pulsations corresponds to that frequency in the spectrum at which the intensity is a maximum (see Fig. 5c).

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1)Simultaneous generation of a large number of waves with independent phases occurs in weak coupling between modes, which is possible, e.g., in an optically active medium with an inhomogeneously broadened emission line [10].

2)For the moment we neglect four-particle (sic–"four-frequency" is probably intended (transl. note)) interactions of the type $\omega_i + \omega_j + \omega_l = \omega_1$, which are associated with nonlinear absorption.

3)For the interaction of waves belonging to the same branch of the dispersion equation, the interaction region coincides with the region of no dispersion.

4)Here a simplified dispersion equation is given, in which the dispersion in the region $\omega \sim \omega_0$ and the fact that the transparency band for high-frequency waves is finite have not been taken into account. We add that, in the real system, in view of the fact that it was finite, the excitation had a resonance character; however, the Q of the system was made extremely low and the boundedness of the turbulence spectrum was associated only with the finiteness of the transmission band of the system.

5)In the region of biases $u_0 < 0.2$ V the quantity $\bar{\gamma}_4$ is negligible.

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