The propagation of a nonlinear wave in a stochastic medium

G. M. Zaslavskiĭ

Physics Institute, Siberian Division, USSR Academy of Sciences
(Submitted October 11, 1973)
Zh. Eksp. Teor. Fiz. **66**, 1632–1637 (May 1974)

The propagation of a strongly nonlinear wave (soliton) in a medium with small-scale random fluctuations is considered. A nonlinear dissipative term is found for the averaged characteristic of the wave, leading to a growth or decay of the wave on account of the fluctuations. The change of structure of the nonlinear term of the nondissipative part of the equation of motion has been derived, change which can lead to a substantial transformation of the wave profile.

1. The solution of the problem of propagation of a linear wave in a stochastic (random) medium has until now run into serious difficulties. If, however, the fluctuating part of the medium is small, there exist sufficiently good methods of analysis (cf., e.g.,[2-3]) which are analogous to the Born approximation in quantum mechanics. The same methods can also be used in the case of weak nonlinearities[4], where one can neglect the nonlinearity in the computation of corrections to the wave on account of the fluctuations of the medium.

If the nonlinearity of the wave is not small, then the linear technique is not valid in the usual form. For definiteness we shall talk about a powerful wave pulse—a soliton—propagating in a stochastic medium. One can distinguish two limiting cases: (1) the size of the soliton is much smaller than the characteristic scale of the fluctuations, and (2) the size of the soliton is large compared to the scale of the fluctuations. In the first case the fluctuations are adiabatic, and the analysis of the propagation of the nonlinear wave can be carried out, e.g., by means of the method of Whitham[5]. Essentially this type of analysis is contained in refs.[6,7] treating the propagation of a soliton in shallow water with a randomly variable depth. The method for the investigation of the second case for the propagation of a soliton in the region of small-scale fluctuations is discussed below. It is based on taking into account the nonlinear terms in the equation without any assumption on their smallness. This leads, e.g., to the fact that the dissipative term stops being linear and acquires a sufficiently complicated structure.

2. Let the motion of the nonlinear wave be described by the partial differential equation

$$v_t+c_0v_x+{}^1/_2(v^2)_x+c_0h^2v_{xxx}=\xi Q[v], \tag{1}$$

the left-hand side of which is the Korteweg-de Vries equation, and the right-hand side takes into account the inhomogeneity of the medium with $\xi = \xi(x)$. Here c_0 is the speed of sound, and h is the scale of dispersion (e.g., the depth of the "shallow water").

All the reasoning and calculations in the sequel will not depend on the functional form of $Q[v]$. We call, however, attention only to two different kinds: $Q_1[v] = v$ and $Q_2[v] = v_x$. To the first case corresponds a dissipative term in the equation (1), leading to a damping of the wave with time for $\xi < 0$ and to growing of the wave for $\xi > 0$. Such a term appears, e.g., in a chain of oscillators where the friction force is proportional to the velocity. Indeed, the linear part of the equation (1) is obtained in the long-wavelength approximation from the system

$$\ddot{y}_n-\xi\dot{y}_n=c_0^2h^{-2}(y_{n+1}-2y_n+y_{n-1}),$$

where y_n is the displacement of the n-th atom and $v = \dot{y}$. The term $\xi\dot{y}_n \equiv \xi v$ describes the friction (positive or negative, depending on the sign of ξf) due to the interaction with the medium and for a random function $\xi(x)$ it immitates the interaction with a turbulent medium.

Another interesting case where a term of the type Q_1 appears corresponds to the propagation of a surface wave in "shallow water" with a variable depth $h \equiv h_0 + h_1(x)$ in those cases when the derivative dh_1/dx is sufficiently large[8,9]. Then $\xi \sim dh_1/dx$ describes the accidental random roughness of the bottom.

As can be seen from (1), the case $Q = Q_2 = v_x$ corresponds to a random addition $\xi(x)$ to the speed of sound c_0 and also appears on account of inhomogeneities of the medium. However, the right-hand side of (1) has here a nondissipative structure.

In the sequel we shall write for convenience the expression Q_1 and at the end of the paper we shall list the results for Q_2. Regarding the quantity ξ, which has a random dependence on x we shall assume that it is small as well as its gradients

$$\xi\ll v, \qquad \frac{d}{dx}\ln\xi \ll \frac{\partial}{\partial x}\ln v. \tag{2}$$

In addition we assume that

$$\langle\xi\rangle=0, \qquad \langle\xi(x)\xi(y)\rangle=R(x-y).$$

Setting $v = \langle v\rangle + u$, $\langle u\rangle = 0$ and taking into account the smallness of $u \ll \langle v\rangle$, we obtain from (1):

$$\langle v\rangle_t+c_0\langle v\rangle_x+{}^1/_2\langle v\rangle_x{}^2+c_0h^2\langle v\rangle_{xxx}=\langle\xi u\rangle-{}^1/_2\langle u^2\rangle_x, \tag{3}$$

$$u_t+(u(\langle v\rangle+c_0))_x+u_{xxx}=\xi\langle v\rangle. \tag{4}$$

In the zeroth approximation the averaged wave satisfies the equation

$$\langle v\rangle_t+{}^1/_2(\langle v\rangle^2)_x+c_0\langle v\rangle_x+c_0h^2\langle v\rangle_{xxx}=0,$$

for which the solution is known. Assume, e.g., that this is a soliton moving with speed $c > c_0$

$$\langle v(x,\ t)\rangle=v_0(x-ct).$$

This form of the solution can be substituted into Eq. (4), since taking into account a change of the parameters (in particular the speed c) due to the fluctuations of ξ would lead to the appearance of terms of higher order of smallness in the equation (4) for the quantity u.

Taking this into account, we transform (4) to a new reference frame, moving with velocity c:

$$t=t, \qquad y=x-ct.$$

This leads to the equation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial y}\hat{H}u=\xi(y+ct)u, \tag{5}$$

where $u = u(y, t)$ and we have introduced the operator

$$\hat{H}=c_0h^2\frac{\partial^2}{\partial y^2}+v_0(y)-c+c_0. \tag{6}$$

A formal solution of (5) can be represented in the form

$$u(y,t)=\int_{-\infty}^{t} d\tau \exp\left\{-(t-\tau)\frac{\partial}{\partial y}\hat{H}(y)\right\}\xi(y+c\tau)\langle v(y,t)\rangle$$
$$=\int_0^\infty d\tau \exp\left\{-\tau\frac{\partial}{\partial y}\hat{H}(y)\right\}\xi(y+c(t-\tau))\langle v(y,t)\rangle. \tag{7}$$

Further we take into account that in view of the second inequality in (2), the fluctuations are small-scale compared to the changes in v, and one may neglect the action of the operator on $\langle v\rangle$ under the exponential sign, up to terms of higher order of smallness in ξ. For the same reason one may in the first approximation neglect the noncommutativity of the operators $\partial/\partial y$ and $\hat{H}(y)$ which figure in the exponential and act on ξ.

We consider the eigenvalue problem for the operator $\hat{H}$ defined by (6):

$$\hat{H}\psi=E\psi,$$

and expand the function $\xi(x)$ into a complete set of eigenfunctions $\psi(E;x)$ of the operator $\hat{H}$:

$$\xi(x)=\sum_E C(E)\psi(E;x), \tag{8}$$

where the sign Σ_E denotes summation over the discrete part of the spectrum and integration over the continuous part.

Substituting (8) into (7) and taking into account the approximations made above, we obtain

$$u(y,t)=\sum_E C(E)\int_0^\infty d\tau \exp\left\{-\tau\frac{\partial}{\partial y}\hat{H}(y)\right\}\psi(E;y+c(t-\tau))\langle v(y,t)\rangle$$
$$=\sum_E C(E)\int_0^\infty d\tau \exp\left\{-\tau E\frac{\partial}{\partial y}\right\}\psi(E;y+c(t-\tau))\langle v(y,t)\rangle$$
$$=\sum_E C(E)\int_0^\infty d\tau\psi[E;y-\tau(E+c)+ct]\langle v(y,t)\rangle.$$

Hence

$$u(x,t)=\sum_E C(E)\int_0^\infty d\tau\psi[E;x-\tau(E+c)]\langle v(x,t)\rangle. \tag{9}$$

With the help of the expression (9) one can now compute the right-hand side of (3). We note that in the absence of nonlinearity and dispersion $\hat{H} = (c - c_0)\partial/\partial x$, and we arrive at the case considered by Howe[2]. In the case under discussion the $\langle v\rangle$-dependence of the corrections $u(x, t)$ to the averaged wave envelope $\langle v(x, t)\rangle$ is determined by the $\langle v\rangle$-dependence of the eigenvalues E and eigenfunctions $\psi(E;x)$ of the stationary Schrödinger equation with potential $v_0 = v$.

3. For the first term in the right-hand side of Eq. (3) we have

$$I_1=\langle\xi(x,t)u(x,t)\rangle=\langle v(x,t)\rangle\left\langle\xi(x,t)\sum_E C(E)\int_0^\infty d\tau\psi(E;x-\tau(c+E))\right\rangle$$
$$=\langle v(x,t)\rangle\int_0^\infty d\tau\sum_E\int dzR(x-z)\psi^*(E;z)\psi(E;x-\tau(c+E)), \tag{10}$$

where we have used the orthogonality of the eigenfunctions $\psi(E;x)$ in order to obtain the coefficients $C(E)$ from (8). After a change of variables in (10) we have

$$I_1=\langle v(x,t)\rangle\int_0^\infty dr\int dzR(x-z)\sum_E\frac{\psi^*(E;z)\psi(E;x-r)}{c+E}$$
$$=-\langle v(x,t)\rangle\int_0^\infty dr\int dzR(x-z)G(-c;z,x-r), \tag{11}$$

where $G(E;x_1,x_2)$ is the Green's function of the operator $\hat{H}$. The equation (11) expresses the final result for the first term of the right-hand side of Eq. (3), where the dependence on $\langle v\rangle$ enters also into G, since in the approximation under consideration $v_0 = \langle v\rangle$.

We now go over to a computation of the expression

$$I_2=\langle u^2\rangle=(\langle v(x,t)\rangle)^2\left\langle\sum_{E_1}C(E_1)\sum_{E_2}C(E_2)\int_0^\infty d\tau_1\int_0^\infty d\tau_2\right.$$
$$\left.\times\psi(E_1;x-\tau_1(c+E_1))\psi(E_2;x-\tau_2(c+E_2))\right\rangle.$$

In the same manner as in the derivation of (10), (11), we express $C(E)$ from (8) and make a change of the integrations variable in the integrals with respect to τ_1 and τ_2:

$$I_2=\langle v\rangle^2\int_0^\infty dr_1\int_0^\infty dr_2\int dx_1\int dx_2R(x_1-x_2)\sum_{E_1}\frac{\psi^*(E_1;x_1)\psi(E_1;x-r_1)}{c+E_1}$$
$$\times\sum_{E_2}\frac{\psi^*(E_2;x_2)\psi(E_2;x-r_2)}{c+E_2}=\langle v\rangle^2\int_0^\infty dr_1\int_0^\infty dr_2\int dx_1\int dx_2R(x_1-x_2)$$
$$\times G(-c;x_1,x-r_1)G(-c;x_2,x-r_2). \tag{12}$$

The expression (12) is final and represents the correction to the nonlinear term on account of the random fluctuations of the medium. This is easily seen if one considers the case of a weak nonlinearity. Then the expression (12) equals

$$I_2=\text{const}\,\langle v\rangle^2.$$

Similarly for a weak nonlinearity

$$I_1=\text{const}\,\langle v\rangle>0,$$

i.e., the expression (11) describes the effective increase of the amplitude of the wave, analogous to the increase in velocity of a Brownian particle.

In the general case Eq. (3) for the averaged part of the wave acquires the form

$$\langle v\rangle_t+c_0\langle v\rangle_x-I_1+{}^1/_2(\langle v\rangle^2+I_2)_x+\langle v\rangle_{xxx}=0. \tag{13}$$

4. Let us go over to an estimate of the structure of the dissipative and nonlinear corrections I_1 and I_2. As was already noted, according to the second inequality in (2) the fluctuations of the medium, $\xi(x)$, are small-scale compared to $\langle v\rangle$. Therefore in the expansion (8) must be present essentially eigenfunctions $\psi(E;x)$ with such values of E for which they oscillate rapidly, i.e.,

$$\frac{d}{dx}\ln\psi\gg\frac{\partial}{\partial x}\ln\langle v\rangle. \tag{14}$$

But since $\langle v\rangle$ plays the role of potential in the equation $\hat{H}\psi = E\psi$, the condition (14) signifies the possibility of using the WKB-approximation. Hence

$$\psi\sim Ap^{-1/2}\exp\left\{\pm i\int^x p\,dx\right\},$$
$$p=h^{-1}(c/c_0-1-\langle v\rangle/c_0+E/c_0)^{1/2}. \tag{15}$$

Let R have the form

$$R(x)=R_0\gamma e^{-\gamma x} \tag{16}$$

and let γ be sufficiently large. Then substituting (15), (16) into (11) we find

$$I_1 \approx A_1 R_0 \langle v\rangle / (c_0 h^2 \gamma^2 + c_0 + \langle v\rangle), \tag{17}$$

where A_1 is a constant of the order of one. The expression (17) for the nonlinear negative friction goes over for small $\langle v\rangle$ into linear friction with the coefficient (increment)

$$|I_1 / \langle v\rangle| = A_1 R_0 / (c_0 h^2 \gamma^2 + c_0).$$

As the amplitude $\langle v\rangle$ of the wave increases, the effective increment decreases. Indeed, the soliton becomes narrower, since the width of the soliton is of the order

$$\Delta \sim h\left(\frac{c_0}{c-c_0}\right)^{1/2}, \tag{18}$$

and the amplitude is proportional to $(c - c_0)$. The narrower the soliton, the smoother will the background fluctuations be for it, which leads to a decrease of the viscosity. This circumstance also follows from the condition of small-scaleness in (2), fact which can be seen by substituting (18) into (2).

We now go over to an analysis of the expression for I_2. From the equation (12), with the help of the expressions (15), (16) we estimate in analogy with what was done before

$$I_2 \approx \frac{A_2 R_0}{\gamma}\left(\frac{\langle v\rangle}{c_0 h^2 + c_0 + \langle v\rangle}\right)^2, \tag{19}$$

where A_2 is a constant of the order of unity. Thus, we obtain the final form of the equation (3) for $\langle v\rangle$:

$$\langle v\rangle_t + c_0\langle v\rangle_x + A_1 R_0 \langle v\rangle / (c_0 h^2 \gamma^2 + c_0 + \langle v\rangle) + \frac{\partial}{\partial x}\left\{\left[1 + \frac{A_2 R_0}{\gamma} \Big/ (c_0 h^2 \gamma^2 + c_0 + \langle v\rangle)^2\right]\langle v\rangle^2\right\} + c_0 h^2 v_{xxx} = 0. \tag{20}$$

The expression for I_2 in (19) and (20) takes into account the change in the form of the nonlinearity in a medium with fluctuations. This effect leads to a substantial restructuring of the wave form and disappears for small $\langle v\rangle$, or in the adiabatic limit.

5. The main effects obtained above—the nonlinear structure of the "viscous" term and the change in the type of nonlinearity of the wave—appear in the propagation of a strongly nonlinear wave in a medium with fluctuations and is not, naturally, a peculiarity of the concrete model under consideration. It is not hard to see from the scheme of derivation of the corrections I_1 and I_2, that without any changes one can consider any other form of the term in Eq. (1) which is related to random inhomogeneities. A similar remark can also be made relative to Eq. (1). The consideration of another form of nonlinear wave process leads to a change of the form of the operator $\hat{H}$, but all that is needed is a knowledge of the asymptotic behavior of its eigenfunctions.

It is not hard to see that, in particular, if the right hand side of Eq. (1) contains the term ξv_x (corresponding to density fluctuations of the medium), then the dissipative term in the equation for the averaged background $\langle v\rangle$ has the form

$$I_1 = \langle \xi v_x\rangle = \langle \xi u_x\rangle \sim R_0 \langle v\rangle_{xx} / (c_0 h^2 \gamma^2 + c_0 + \langle v\rangle),$$

corresponding to a standard viscous term.

The presence of the denominator depending on $\langle v\rangle$ distinguishes essentially the equation for $\langle v\rangle$ from an equation of the Burgers type for a weak nonlinearity.

In conclusion I would like to thank A. G. Litvak for useful remarks.

[1] V. I. Klyatskin and V. I. Tatarskiĭ, Usp. Fiz. Nauk 110, 499 (1973) [Sov. Phys.-Uspekhi 16, 494 (1973)] Izv. vuzov, Radiofizika, 15, 1433 (1972)].
[2] M. S. Howe, J. Fluid. Mech. 45, 785 (1971).
[3] J. B. Keller, transl. in: Gidrodinamicheskaya neustoichivost' (Hydrodynamic instability), Mir, 1964, p. 265.
[4] V. V. Tamoĭkin and S. M. Faĭnshtein, Zh. Eksp. Teor. Fiz. 62, 213 (1972); 64, 505 (1973)] Sov. Phys.-JETP 35, 115 (1972); 37, 257 (1973)].
[5] G. B. Whitham, Proc. Roy. Soc. A283, 268 (1965).
[6] E. N. Pelinovskiĭ, Fizika atmosfery i okeana 7, 1226 (1971).
[7] N. Yajima, J. Phys. Soc. Japan 33, 1471 (1971).
[8] L. D. Landau and E. M. Lifshits, Mekhanika sploshnykh sred, Gostekhizdat, 1954, p. 62 [Engl. transl. as Fluid dynamics, Addison-Wesley and Pergamon, 1958)].
[9] L. A. Ostrovskiĭ and E. N. Pelinovskiĭ, Fizika atmosfery i okeana 6, 934 (1970).

Translated by M. E. Mayer
169