Nonlinear theory of excitation of electromagnetic waves in a dielectric medium by a nonrelativistic electron beam

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The nonlinear interaction of a nonrelativistic, monoenergetic electron beam of low density with a transverse monochromatic electromagnetic wave in a medium with dielectric permittivity $\epsilon_0(\omega)$ is investigated. A nonlinear dispersion equation is derived. In the linear approximation for the wave amplitudes this equation leads to beam instability that is characteristic for the given system. For sufficiently high wave amplitudes, however, the instability is stabilized as a result of the slowing down of the electron beam in the transverse wave field. The dispersion equation in this case has a frequency spectrum that is dependent on the wave amplitude. The maximum amplitude of the nonlinear stationary wave is estimated; the Poynting vector and the efficiency of conversion of the electron-beam energy into electromagnetic-radiation energy are found.

1. INTRODUCTION

It is known that when an electron beam passes through a medium at a velocity greater than the phase velocity of electromagnetic waves, such waves are excited. In particular, an electron beam in a plasma with a dielectric constant $\epsilon_0(\omega) = 1-\omega_{Le}^2/\omega^2$ excites so-called plasma waves, which are potential waves to within a high degree of accuracy.[1] The nonlinear stage of development of two-stream instability has been investigated previously in some detail in [2,3], and it was shown that the growth in amplitude of a monoenergetic plasma wave saturates because of the capture of beam electrons in the potential of the wave field. In the paper of Akhiezer and Polovin[4], these results of [2,3] are generalized to the case of excitation of periodic charge-density waves of the electron beam in an arbitrary dielectric medium.

In the present paper we consider the nonlinear interaction of a nonrelativistic monoenergetic electron beam with a dielectric medium under conditions such that $\epsilon_0(\omega) \gg 1$ and the wave excited by the beam is almost transverse, and therefore nonpotential to a high degree. The growth of wave amplitude in this case, as will be shown below, is no longer limited by capture of beam electrons in the potential of the wave field but by their retardation by the large transverse component of the field. The steady-state amplitude and the Umov-Poynting vector are found at the saturation stage. The latter permits us to estimate the efficiency of transformation of the energy flux of the beam into a flux of electromagnetic radiation.

The dispersion equation for the excitation of arbitrary nonpotential waves by a monoenergetic beam in a dielectric medium with $\epsilon_0(\omega)$ is easily found in the linear approximation by the method proposed in [5]; it is written in the form

$$\varepsilon_0(\omega)\left[\varepsilon_0(\omega)-\frac{k^2c^2}{\omega^2}\right]-\frac{\omega_{Le}^2}{(\omega-ku_x)^2}\left[\varepsilon_0(\omega)\left(1+\frac{k^2u_y^2}{\omega^2}\right)-\frac{k^2c^2}{\omega^2}\right]=0, \quad (1.1)$$

where ω_{Le} is the Langmuir frequency of the electron beam and u_x and u_y are the components of the directional velocity of the beam; the wave is propagated along the x axis.

It is seen from this equation that the strongest interaction of the beam electrons with the medium takes place under conditions of Cerenkov resonance, when $\omega \approx ku_x$, and for $\epsilon_0(\omega) < 1$ excitation of almost longitudinal (potential) oscillations takes place in the medium, the frequency and growth increment of which are determined from the relations

$$\omega=\omega_0+i\delta=ku_x+i\delta, \quad \varepsilon_0(\omega_0)=0,$$
$$\delta=\frac{-i+\sqrt{3}}{2}\left(\omega_{Le}^2\Big/\frac{\partial\varepsilon_0}{\partial\omega_0}\right)^{1/3}. \quad (1.2)$$

These oscillations correspond to space-charge waves, the nonlinear development of which has been studied in [2–4]. We therefore do not consider them here.

If $\epsilon_0(\omega) \gg 1$, then, in accord with Eq. (1.1), Cerenkov excitation of almost transverse electromagnetic waves will occur in the system, with the spectrum

$$\varepsilon_0(\omega_0)\omega_0^2=k^2c^2,$$
$$\delta=\frac{-i+\sqrt{3}}{2}\left[\frac{\omega_{Le}^2u_y^2\varepsilon_0\omega_0^2}{c^2\partial(\varepsilon_0\omega_0^2)/\partial\omega_0}\right]^{1/3}. \quad (1.3)$$

The growth increment of the wave differs from zero only for the condition $u_y \neq 0$. This is natural, inasmuch as the considered wave is transverse in zeroth approximation (i.e., in the absence of the beam), and the component of the electric field E_y in it differs from zero. Therefore, the beam can excite such a wave and transfer energy to it if only the work done by the beam on the field is different from zero, i.e., $E \cdot u = E_y u_y \neq 0$. In the presence of the beam, not only does excitation of this wave take place, but there is also a small longitudinal (potential) component of the wave $E_x = \alpha E_y$,

$$\alpha=\frac{\omega_{Le}^2}{(\omega-ku_x)^2}\frac{ku_y}{\omega_0}\frac{1}{\varepsilon_0}\approx\frac{u_y}{u_x}\frac{\delta}{\omega_0}\ll 1. \quad (1.4)$$

2. NONLINEAR DISPERSION RELATION

We now proceed to a description of the nonlinear stage of development of the instability considered above. To describe our system, we start out from the following equations:

$$\frac{\partial n}{\partial t}+\operatorname{div} n\mathbf{v}=0, \quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},$$
$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=\frac{e}{m}\left\{\mathbf{E}+\frac{1}{c}[\mathbf{v}\times\mathbf{B}]\right\}, \quad (2.1)$$
$$\operatorname{rot}\mathbf{B}=\frac{1}{c}\frac{\partial\hat{\varepsilon}_0\mathbf{E}}{\partial t}+\frac{4\pi}{c}(en\mathbf{v}-\mathbf{j}_0).$$

Here n and **v** are the density and total velocity of the beam electrons, $\mathbf{j}_0$ is the current in the system, which compensates the unperturbed current of the beam, and $\hat{\epsilon}_0$ is the operator which gives the dielectric constant of the medium $\epsilon_0(\omega)$ for a monoenergetic field. In this notation of the equations of the system, only the nonlinear effect of the fields on the motion of the beam electrons is taken into account; the medium is still regarded as linear. This is natural, since the beam introduces a small perturbation into the medium and, at the same time, the beam itself, under the action of the fields perturbed by it, can change the character of its motion significantly.

Assuming that a monoenergetic, almost transverse electromagnetic wave is excited in the medium with the spectrum (1.3), we can regard all the quantities in Eqs. (2.1) to be dependent on $\xi = t - kx/\omega$, where $k = (k, 0, 0)$. In this case we easily find the following integrals of motion:

$$n=n_0\frac{1-ku_x/\omega}{1-kv_x/\omega}, \tag{2.2}$$

$$\frac{k}{2\omega}(v_x^2+v_y^2)-v_x+\frac{ek}{m\omega}\Phi=\frac{k}{2\omega}(u_x^2+u_y^2)-u_x;$$
$$E_x=-\frac{\partial\Phi}{\partial\xi},\quad E_y=\frac{m}{e}\frac{\partial v_y}{\partial\xi}, \tag{2.3}$$

where n_0, u_x and u_y are the unperturbed density and velocity of the beam electrons (i.e., at points where $E_x = E_y = 0$). Taking (2.2) into account, we obtain from Eqs. (2.1)

$$\frac{ke}{m\omega}\varepsilon_0\frac{\partial^2\Phi}{\partial\xi^2}=-\omega_{Le}^2\frac{\omega(v_x-u_x)}{\omega-kv_x}$$
$$\left(\varepsilon_0-\frac{k^2c^2}{\omega^2}\right)\frac{\partial^2 v_y}{\partial\xi^2}=-\omega_{Le}^2\frac{\omega(v_y-u_y)+k(v_xu_y-u_xv_y)}{\omega-kv_x}. \tag{2.4}$$

To solve this nonlinear set of equations, it is convenient to make the substitutions

$$v_x=\omega/k+v_{1x},\qquad v_y=u_y+v_{1y}, \tag{2.5}$$

where $v_{1y} \ll u_y$. This allows us to find a simple connection between v_{1y} and Φ:

$$\left(\varepsilon_0-\frac{k^2c^2}{\omega^2}\right)v_{1y}=\varepsilon_0\frac{ku_y}{\omega}\frac{ke}{m\omega}\Phi, \tag{2.6}$$

and to obtain a nonlinear equation for v_{1y} [1)]

$$\left(\varepsilon_0-\frac{k^2c^2}{\omega^2}\right)\frac{\partial^2 v_{1y}}{\partial\xi^2}=-\omega_{Le}^2u_y\left\{\left(u_x-\frac{\omega}{k}\right)\left(\left(u_x-\frac{\omega}{k}\right)^2\right.\right.$$
$$\left.\left.-2v_{1y}u_y\left[1+\frac{\omega^2(\varepsilon_0-k^2c^2/\omega^2)}{k^2u_y^2\varepsilon_0}\right]\right)^{-1/2}-1\right\}. \tag{2.7}$$

In the linear approximation (i.e., when $v_{1y} \to 0$), we obtain the dispersion equation (1.1) from (2.7).

We now take into account the fact that $\epsilon_0 - k^2c^2/\omega^2 \to 0$ in the absence of the beam, and neglect terms on the right-hand side of (2.7) that contain such a factor. As a result, we get

$$\left(\varepsilon_0-\frac{k^2c^2}{\omega^2}\right)\frac{\partial^2 v_{1y}}{\partial\xi^2}=-\omega_{Le}^2u_y\left\{\left(u_x-\frac{\omega}{k}\right)\left[\left(u_x-\frac{\omega}{k}\right)-2v_{1y}u_y\right]^{-1/2}-1\right\}. \tag{2.8}$$

This equation is entirely analogous to that studied previously in [2-4], and for its solution, as in those papers, we can make use of the asymptotic method of Krylov-Bogolyubov.[6] Omitting the calculations, which are just the same as those given in [2-4], we shall give only the final results.

Under the conditions that $(u_x - \omega/k)^2 > 2v_{1y0}u_y$, where $v_{1y} = v_{1y0}\cos\omega\xi$, we get the following nonlinear dispersion relation from (2.8):

$$\varepsilon_0-\frac{k^2c^2}{\omega^2}-\frac{\omega_{Le}^2}{\omega^2}\frac{k^2u_y^2}{(\omega-ku_x)^2+2k^2u_yv_{1y0}}\frac{8C(\eta_1)}{K(\eta_1)}=0, \tag{2.9}$$

where

$$C(\eta_1)=\eta_1^{-4}[(2-\eta_1^2)K(\eta_1)-2E(\eta_1)],$$

and $K(\eta_1)$ and $E(\eta_1)$ are elliptic functions with modulus

$$\eta_1=\left\{\frac{2u_yv_{1y0}}{{}^1/_2(u_x-\omega/k)^2+u_yv_{1y0}}\right\}^{1/2}\leqslant1. \tag{2.10}$$

On variation of η_1 from 0 to 1, the quantity $8C(\eta_1)/K(\eta_1)$ changes from 1 to 8.

If we take into account that, in accord with (2.3), the quantity v_{1y0} is connected with the amplitude of the field oscillations by the relation $E_{y0} = m\omega v_{1y0}/e$, then Eq. (2.9) can be regarded as a nonlinear dispersion relation that connects the frequency ω and the wave vector k with the amplitude of the wave E_{y0} in the absence of significant retardation of the particles, when

$$\frac{m}{2}\left(u_x-\frac{\omega}{k}\right)^2\gg\frac{eE_{y0}u_y}{\omega},$$

i.e., when the kinetic energy of the beam electrons in the coordinates of the wave is greater than the work of the field on the beam over the period of the oscillations. Naturally, the relative motion of the beam electrons is preserved under these conditions, and therefore Eq. (2.9) admits of solutions with $\mathrm{Im}\,\omega > 0$, which correspond to oscillations that increase with time.

But if

$$\frac{m}{2}\left(u_x-\frac{\omega}{k}\right)^2\leqslant\frac{eE_{y0}u_y}{\omega}$$

we then get the following nonlinear dispersion relation from (2.8):

$$\varepsilon_0-\frac{k^2c^2}{\omega^2}-2\frac{\omega_{Le}}{\omega^2}\frac{u_y}{v_{1y0}}\left[1-2\frac{E(\eta_2)}{K(\eta_2)}\right]=0, \tag{2.11}$$

$$\eta_2=\left\{\frac{{}^1/_2(u_x-\omega/k)^2+u_yv_{1y0}}{2u_yv_{1y0}}\right\}^{1/2}\leqslant1. \tag{2.12}$$

Equation (2.11), unlike (2.9), has only real solutions ω^2, which is also natural, since the work of the field of the wave on the beam during the period of the oscillations is greater under these conditions than the kinetic energy of the beam electrons in the coordinates of the wave, and therefore the wave entirely damps the beam; the relative motion of the electron beam ceases, and at the same time the beam instability is stabilized.[2)]

3. ENERGY OF NONLINEAR WAVE AND EFFICIENCY OF RADIATOR

It follows from the above analysis that the almost transverse electromagnetic wave excited by the beam can become stationary in the system that we have considered when its amplitude reaches a value satisfying the condition

$$\frac{m}{2}\left(u_x-\frac{\omega}{k}\right)^2\leqslant\frac{eE_{y0}u_y}{\omega}.$$

Starting out from this, we can estimate the amplitude of the steady nonlinear wave, substituting for the quantity $u_x - \omega/k$ the value determined from the linear theory, $\omega - ku_x = \delta$. As a result, we get

$$E_{y0}\approx\frac{m}{2}\frac{\omega_0}{e}\frac{|\delta^2|}{k^2u_y}=\frac{m\omega_0}{2e}\frac{|\delta^2|c^2}{u_y\varepsilon_0(\omega_0)\omega_0^2}, \tag{3.1}$$

where ω_0 and δ are determined by Eqs. (1.3). The field E_{x0} is connected with E_{y0} by the relation (1.4).

We can now calculate the Umov-Poynting vector $\vec{\mathscr{P}}$, which is chiefly directed along the x axis (the departure $\mathscr{P}_y/\mathscr{P}_x \sim \alpha \ll 1$) and is equal to

$$\mathscr{P}_x = \frac{c}{4\pi}[\mathbf{E}\times\mathbf{B}]_x = \frac{k}{\omega_0}\frac{c^2}{8\pi}E_{y0}^2 = \frac{c\varepsilon_0^{1/2}}{8\pi}E_{y0}^2. \qquad (3.2)$$

The ratio of the Poynting vector to the kinetic energy flux vector of the beam, $\frac{1}{2}n_0mu^2u_x$, gives the efficiency of beam-energy conversion to radiation

$$\eta = \frac{\mathscr{P}_x}{{}^1/_2 n_0 m u^2 u_x} = \frac{1}{4}\frac{|\delta^2|^2}{\omega_{Le}^2\omega_0^2}\frac{c^5}{u_y^2u^2u_x}\frac{1}{\varepsilon_0^{3/2}} = \frac{1}{4}\frac{c^3}{u^2u_x}\frac{|\delta|}{\varepsilon_0^{1/2}\partial(\varepsilon_0\omega_0^2)/\partial\omega_0} \sim \frac{|\delta|}{\omega_0}. \qquad (3.3)$$

Thus the efficiency of conversion of the energy of the nonrelativistic beam into radiation is found to be a quantity of the order of $|\delta|/\omega_0$. We recall that on excitation by the electron beam of almost longitudinal waves in plasma, the efficiency is of the order of $(|\delta|/\omega_0)^2u^2/c^2$,[7] i.e., much smaller.

Finally, optimizing expression (3.3) in u_y, we obtain the maximum efficiency of the radiator

$$\eta_{\max} = \frac{1}{12\cdot 2^{1/2}}\left(\frac{\omega_{Le}^2}{\omega_0^2}\right)^{1/2}\frac{\varepsilon_0^{1/2}}{(\partial\varepsilon_0\omega_0^2/\partial\omega_0^2)^{1/2}} \approx \frac{1}{12\cdot 2^{1/2}}\left(\frac{\omega_{Le}^2}{\omega_0^2\varepsilon_0^{1/2}}\right)^{1/2}, \qquad (3.4)$$

which is achieved at $u_y^2 = 0.5u_x^2 = 0.5c^2/\epsilon_0$.

In conclusion, we note that in a real system (and a retarding system in the form of a corrugated waveguide with an effective $\epsilon_0(\omega_0) \gg 1$ can serve as such a system), some dissipation always takes place, and is determined by the value of $\mathrm{Im}\,\epsilon_0(\omega)$. Above we have neglected such dissipation everywhere, which is valid for sufficiently weak dissipation, when $\mathrm{Im}\,\epsilon_0 \ll \mathrm{Re}\,\epsilon_0$. In this case, we should replace $\epsilon_0(\omega)$ by $\mathrm{Re}\,\epsilon_0(\omega)$ in all the relations given above. Moreover, we should take into account that the above picture of the development of the instability will hold if the increment δ is larger than the damping decrement of the waves due to dissipation in the dielectric, i.e.,

$$\mathrm{Re}\,\delta > \frac{\omega_0^2\,\mathrm{Im}\,\varepsilon_0(\omega_0)}{\partial[\omega_0^2\,\mathrm{Re}\,\varepsilon_0(\omega_0)]/\partial\omega_0}. \qquad (3.5)$$

Excitation of a transverse wave in a dielectric medium with $\epsilon_0(\omega) \gg 1$ is possible only upon satisfaction of this condition.

[1)]Expressing v_{1y} in (2.7) in terms of Φ and assuming $\epsilon_0 < 1$, we obtain an equation which was studied in [2-4] in a description of two-stream instability for longitudinal (potential) waves.

[2)]The quantity $eE_{0y}u_y/\omega$ can also be treated as the work of the retarding Lorentz force

$$F_x = \frac{e}{c}\cdot u_yH_{0z} = \frac{eE_{0y}u_y}{\omega k^{-1}}$$

at the wavelength k^{-1} of the oscillations.

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