1. INTRODUCTION

In the classical paper of Landau and Lifshitz\(^1\), which initiated the modern theory of ferromagnetic resonance, one topic considered in particular was the motion of the magnetic moment \(\mathbf{\mu}\) of a uniformly magnetized domain in the internal anisotropy field and an external radiofrequency field. When \(\mathbf{\mu} = M \mathbf{v}\), where \(M\) is the magnetic-moment density and is equal to the saturation magnetization and where \(V\) is the domain volume, the Landau-Lifshitz equation can be written in the form

\[
\ddot{\mathbf{v}} = -\gamma (\mathbf{v} \mathbf{H}_A) - \gamma y H A \mathbf{v} - \gamma y H A \mathbf{v} \mathbf{H}_A. \tag{1}
\]

Here \(\gamma\) is the gyromagnetic ratio for electrons; \(\alpha\) is a dimensionless damping constant; \(H_A\) is the effective field, equal to \(-\mathbf{H} / \gamma \mu\); and \(U\) is the magnetic energy of the domain. For a uniaxial ferromagnetic crystal

\[
U = -\mu H e H - K V (en) \tag{2}
\]

where \(K\) is the constant of effective magnetic anisotropy, \(n\) is a unit vector in the direction of the axis of easiest magnetization, and \(H\) is the applied radio-frequency field.

For \(H = 0\), Eq. (1) describes the free precession of the vector \(\mathbf{v}\) in the anisotropy field \(H_A = 2K / M\) with characteristic frequency \(\omega_0 = \gamma H A\) and relaxation time

\[
t_0 = (\omega_0)^{-1} = M / 2\gamma k T. \tag{3}
\]

In a periodic field \(H\) perpendicular to \(H_A\), the motion of the magnetic moment has a typically resonant character; the real and imaginary parts of the magnetic susceptibility have a so-called Lorentz form; the dimensionless (measured in fractions of \(\omega_0\)) width of these lines along the frequency scale is of order \(\alpha\).

The goal of the present paper is to solve the following problem: how does the dimension of the particle affect the character of the motion of its magnetic moment? The dimensional effects discussed in the article should show up in very fine (~100 Å) particles, the interactions between which are here neglected. An adequate physical example is a dilute colloidal suspension of ferromagnetic particles in any nonmagnetic matrix (whether solid or liquid is immaterial, since the characteristic frequencies are so high that motion of the magnetic moment with respect to the body of the particle cannot excite any appreciable motion of the particle with respect to the liquid\(^2\)).

In the range of dimensions in which the magnetic-anisotropy energy \(K V\) is comparable with the thermal energy \(k T\), a fluctuational mechanism of reorientation of the vector \(\mathbf{\mu}\) becomes important. The motion of the magnetic moment of a single-domain particle under the influence of thermal fluctuations (first pointed out by Néel\(^3\)) is analogous to the Brownian rotation of a particle in a liquid and can be described by a Fokker-Planck equation\(^4\). Here the Landau-Lifshitz equation (1) plays the role of a dynamic equation describing the regular change of the vector \(\mathbf{\mu}\). The coefficient \(\gamma \alpha / \mu\) before the relaxation term in (1) has the meaning of a rotational mobility of the magnetic moment, so that for the rotational diffusion coefficient in Einstein's formula one obtains \(D = \gamma \alpha k T / \mu\). On comparing the characteristic time of orientational diffusion of the magnetic moment

\[
\tau = (2D)^{-1} = MV / 2\gamma k T \tag{4}
\]

with the time of rotation of a Brownian particle in a viscous liquid \(\tau_B = 3\pi \eta V / k T\), we conclude that the role of the viscosity \(\eta\) in the mechanism of magnetic diffusion is played by the quantity \(M / 6\gamma \alpha\). We note that between \(\tau\) of (4) and \(\tau_0\) of (3) there is the simple relation

\[
\tau = \tau_0, \quad \omega = KV / k T. \tag{5}
\]

We point out also the analogy between the precession of the magnetic moment in the anisotropy field and the cyclotron rotation of charged particles of a plasma in a magnetic field. In order that it may be possible to speak of precession at all, it is obviously necessary that its period be small in comparison with the rotational diffusion time (the latter here plays the same role as does the free-passage time of the particles of a plasma). In other words, over a time \(\tau\) the vector \(\mathbf{\mu}\) "forgets" about its precession produced by the magnetic torque. By (3) and (5) the condition for existence of precession \(\omega_0 > 1\) reduces to \(\sigma > \alpha\); that is, it is satisfied only for sufficiently coarse particles, \(V \gg \alpha k T / K\).

In Sec. 2, the Fokker-Planck equation for the probability \(W\) of orientations of the particle's magnetic moment is derived, and its spectral properties are analyzed. The eigenvalues \(\Lambda\) of this equation determine the frequencies \(\text{Im} \Lambda\) and damping decrements \(\text{Re} \Lambda\) of the normal modes, by superposition of which one can describe an arbitrary deviation of \(W\) from the equilibrium distribution.
and is made up of two parts—a regular part \( v_r \) and a stochastic (Brownian) part \( v_s \). The first of these, according to (1) and (2), is

\[
dv = -\gamma [eH] - \gamma V[eEH].
\]

An expression for the random velocity of wandering \( v_\infty \) can also be obtained from the Landau-Lifshitz equation, by replacing the regular field \( H_\infty \) in (8) by the stochastic field

\[
H_\infty = -kT \ln W/\mu_0
\]

gives

\[
v_s = \left[ kT \right]/\mu_0 \left[ eV \right] [W + \alpha T + N/\mu_0 eV] W/\mu_0.
\]

On substituting \( v = v_r + v_s \) in (7), we obtain, after simple transformations, the Fokker-Planck equation

\[
2 t W - \alpha [e\hbar + 2\alpha_1 (eH)] + [e\hbar + 2\alpha_1 (eH) - \alpha T] W
\]

In the absence of an external field \( (\xi = 0) \), Eq. (10) in spherical coordinates, with the polar axis along \( n \), takes the form

\[
\frac{\partial W}{\partial t} + \frac{\partial W}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial W}{\partial \varphi} = -2 \frac{\sigma}{\alpha} \cos \theta \frac{\partial W}{\partial \varphi} + 2 \sigma \sin^2 \theta \cos \theta \frac{\partial W}{\partial \varphi} + 2 \sigma (3 \cos^2 \theta - 1) W.
\]

The stationary normalized solution of this equation is the Gibb distribution

\[
W_0 = (4\pi F)^{-1/2}, \quad F(\varphi) = e^{\varphi^2} dx, \quad x = \cos \theta.
\]

An arbitrary deviation \( W - W_0 \) of the distribution function from its equilibrium value can be expanded in normal modes,

\[
W_m(\varphi, \Psi, t) = W_0(\varphi) \cos (\sigma \varphi + im \varphi - \Lambda \varphi).
\]

On substituting (13) in (11), we obtain for the amplitudes of the normal modes \( W_0 \) and for the dimensionless decrements \( \Lambda_0 = 2 \Lambda \int_0^\infty \frac{\partial W}{\partial \varphi} \) the equation

\[
\lambda = \frac{d}{dx} \left[ (1-x)^2 \frac{d}{dx} - \frac{m^2}{1-x^2} + 2ax (1-x^2) \frac{d}{dx} - 2im \frac{\sigma}{\alpha} x. \right]
\]

The solutions \( \psi(\varphi) \) of (14) are orthogonal to the solutions of the conjugate problem

\[
- \lambda \phi = \lambda \phi^* \quad \text{(15)}
\]

where \( \phi^* \) is the Hermitian conjugate operator to \( \phi \). The operator \( \hat{A} \) looks especially simple if one requires orthogonality with weights:

\[
\int_{-\infty}^{+\infty} \psi_m(\varphi) \psi_n(\varphi) e^{\mu \varphi} d\varphi = N_0 \delta_{mn}.
\]

By use of (14), this expression is easily transformed to the form

\[
\int_{-\infty}^{+\infty} \psi_m(\varphi) \psi_n(\varphi) e^{\mu \varphi} d\varphi = N_0 \delta_{mn},
\]

whence follows the equation for \( \phi(\varphi) \)

\[
- \lambda \phi = \lambda \phi^* \quad \text{(15)}
\]

On comparing this with equation (15), we conclude that \( \hat{A} = \lambda^* \); that is, the operator \( \hat{A} \) differs from \( \hat{A} \) only with respect to the sign of the last term in (14). This fact enables us to relate their eigenfunctions in a simple manner: \( \phi^* = \hat{A} \phi \). Thus the orthogonality condition (16) takes the form

\[
\int_{-\infty}^{+\infty} \psi_m(\varphi) \psi_n(\varphi) e^{\mu \varphi} d\varphi = N_0 \delta_{mn}.
\]

The nonhermicity of the operator \( \hat{A} \) indicates that its eigenvalues \( \lambda \) may be complex.

Investigation of the spectrum of \( \lambda(\varphi) \) is conveniently begun by consideration of the case of small \( \sigma \). For \( \sigma = 0 \), Eq. (14) is satisfied by the Legendre associated polynomials

\[
\psi_m(\varphi) = \left[ \frac{2l+1}{2(l+1)!} \right]^1 P_l^m(\varphi), \quad \psi_{m(0)}(\varphi) = 1(l+1).
\]

Here the normalization is so chosen that the integral \( N = +1 \) for even \( l \) and \( N = -1 \) for odd.

For nonzero but sufficiently small \( \sigma \), the solution of (14) can be constructed from series in powers of \( \sigma \).
We shall not give these simple calculations. We point out only that the decrements $\lambda$ in an arbitrary order of the perturbation theory remain real, whereas the functions $\Psi$ at $\sigma \neq 0$ cease to be even and can be expressed as the sum of a real even part (index $g$) and an imaginary odd part (index $u$):

$$\Psi = \Psi_g + i \Psi_u$$  \hspace{1cm} (19)

(for example, an expansion originating from an even level contains, at small $\sigma$, a small imaginary part, odd in $x$). It is nevertheless possible even when $\sigma \neq 0$ to speak as before of "even" and "odd" solutions, depending on which levels of the spectrum they approach at $\sigma \to 0$. We remark that the normalizing integrals (17)

$$N = \int \Psi^* e^{\sigma x} dx = \int (\Psi^*_g - i \Psi^*_u) e^{\sigma x} dx$$

of even functions are positive (since for $\sigma \to 0 \Psi_0$ vanishes in them), whereas those of odd functions are negative.

Thus as long as the series in powers of $\sigma$ converge, the perturbations (13) of the distribution function described by them decrease monotonically with time. This can also be seen directly. If one multiplies Eq. (14) by $\Psi^* \exp(\sigma x)$, integrates with respect to $x$, and subtracts from the resulting relation its complex conjugate, one obtains

$$\int \alpha \Psi^1 e^{\sigma x} dx = -2m \int \alpha \Psi^* e^{\sigma x} dx.$$  

The integral in the numerator of this formula, on substitution of $\Psi$ from (19), vanishes identically (by the parity properties).

Thus the occurrence of oscillatory perturbations is possible only for finite values of $\sigma$, larger than a certain $\sigma_0$. The latter also determines the radius of convergence of the series mentioned above. On multiplying (14) by $\Psi^* \exp(\sigma x^3)$ and operating further as in the derivation of the preceding relation, we obtain with allowance for (19)

$$\lambda - \lambda^* \int (\Psi^*_g - i \Psi^*_u) e^{\sigma x} dx = 0.$$  \hspace{1cm} (20)

Hence it is evident that the occurrence of oscillatory ($\lambda \neq \lambda^*$) perturbations for $\sigma > \sigma_0$ is preceded by the vanishing, at the point $\sigma_0$, of the normalizing integral $N$. This type of singularity in the spectra of nonhermitian operators is well known (5); they occur upon the intersection of two levels to which, in the real range of the decrements, the normalizing integrals of the amplitudes $\Psi_i$, and this is known not to take place.

Thus neighboring real decrements either undergo confluence at some point $\sigma_*$, forming a complex-conjugate pair, or do not intersect at all.

A calculation of the decrement spectrum was done by Galerkin's method. The $\Psi^1_{nm}$ of (18), the solutions of equation (14) for $\sigma = 0$, were taken as the system of basis functions. Calculation of the function $\lambda(\sigma)$ over a sufficiently wide range of values of the parameter $\sigma$, requires a large number of basis functions. In the interval investigated, $0 < \sigma < 20$, the approximation used was

$$\Psi = \sum_{m=1}^{n} c_m \Psi_{nm}^{(1)}$$

with $n = 20$, and check calculations were made with $n = 30$. Diagonalization of the characteristic determinant was accomplished on an electronic computer.

Figure 1 shows the bottom levels of the spectrum for $m = 1$ and $\alpha = 0.1$. Their form corresponds completely to the general ideas about the structure of the spectrum of eigenvalues of equation (14). Evident on the figure is the confluence of the real levels $\lambda_1$ and $\lambda_2$, with production of oscillatory modes. The dashed line shows the real part of the complex-conjugate decrements. Their imaginary part is also shown in the figure. The coordinate of the confluence point is $\sigma_* = 0.24$. This value determines the critical volume of the ferromagnetic particle, $V_0 = 0.24 kT/k$, at which the characteristic fre-
quency of precession of its magnetic moment vanishes. Hence for the dimensional decrements $A = \lambda / 2\tau$, one obtains, by (3) and (5), the asymptotic expressions (6).

3. DISPERSION OF THE MAGNETIC SUSCEPTIBILITY

The degree of orientation of the fluctuating magnetic moment $\mu = \mu e$ can be determined by averaging of the components of the vector $e$ and of their products with the distribution function satisfying equation (10). By taking into account the hermicity of the operator $L$, it is easy to write an equation for an arbitrary moment of the distribution function. One obtains, as usual, an infinite system of coupled equations.

In decoupling this system, it is necessary to keep at least two equations—for the first and second moments of the distribution function:

$$
\frac{d}{dt} e = -\left\{ e + \alpha_{0} n_{e}(e) - \frac{\alpha}{\alpha} e_{0} n_{e}(e') e \right\} - \frac{3}{2} h e + \frac{1}{2} a e + e_{0} e - 2 e_{0} e_{0} + e_{0} e_{0} - 2 h_{0} e_{0} e - 2 e_{0} e_{0}.
$$

By means of the single equation (23) alone, no satisfactory description of the motion of the magnetic moment can be obtained: for any method of closure, this equation does not contain the characteristic frequency of precession of the vector $e$ in the anisotropy field. Only the "entanglement" of the dipole ($e_{0}$) and quadrupole ($e_{0} e_{0}$) branches leads, in full agreement with the results of the preceding section, to the occurrence at finite $\tau$ of a characteristic frequency of oscillation and consequently makes possible ferromagnetic resonance in a periodic external field.

In accordance with the chosen (two moment) approximation, we shall seek a distribution function in the form

$$W = W_{0} + \alpha ee_{0} + h_{0} e_{0} e_{0},$$

where $W_{0}$ is the equilibrium function determined by formula (12), and where $a_{1}$ and $b_{1k}$ are independent of the components of the vector $e$ and are small quantities of the same order as the amplitude of the radiofrequency field $h$. On carrying out in (23) and (24) an averaging with the function (25) and on requiring only linear accuracy with respect to the quantities mentioned, we obtain for $a_{1}$ and $b_{1k}$ the equations

$$
\tau X_{a_{1}} = -\left\{ \left[ W_{0} - X_{0} + \alpha ee_{0} + h_{0} e_{0} e_{0} \right] a_{1} + \left[ \frac{d}{dt} e_{0} e_{0} \right] a_{0} n_{e}(e) e_{0} + \frac{1}{2} \left[ h_{0} e_{0} e_{0} \right] a_{0} \right\}.
$$

where we use the notation

$$X_{a_{1}} = (\alpha / 2) \int n_{e}(e) \exp (\alpha \mu) d\mu$$

for the moments of the function $W_{0}$. These quantities can be expressed in terms of derivatives of $F(0)$ by means of (12), and their tensorial structure in terms of $\delta$ symbols and components of the vector $n$. For example,

$$X_{a_{1}} = n_{0} \left( 1 - \frac{1}{F} \right) a_{0} + \frac{1}{2} \left( \frac{F}{F^{2} - 1} \right) n_{0} n_{a}.$$

We note also that the first moment of the distribution function (25) is connected with $a_{1k}$ by the relation

$$\langle e_{0} \rangle = X_{a_{1}}.$$

We first determine the longitudinal susceptibility $\chi_{L}$, supposing that the external field is parallel to the anisotropy field: $h = n = (0, 0, 1)$. In this case one obtains for $a_{22}$ from (26), the closed equation

$$\tau X_{a_{22}} = -\left\{ (1 - \alpha) X_{0} + \alpha n_{e}(e) e_{0} e_{0} + \frac{1}{2} (1 - X_{0}) \right\}.$$

By substituting $X_{ZZ} = F' / F$, $X_{ZZZZ} = F'' / F$ and using (28), we write this equation in the form

$$\frac{\alpha}{\alpha} \tau X_{a_{22}} = -\left( \frac{1}{F} + \frac{1}{F - 1} \right) X_{0} + \frac{1}{2} \left( \frac{F}{F^{2} - 1} \right) n_{0} n_{a}.$$

In the absence of an external field, Eq. (29) describes a relaxation of the projection of the magnetic moment $\mu$ on the direction of the axis of easy magnetization. Thermal fluctuations cause transitions between the states to the thermal energy $kT$; that is, it is determined by the parameter $\tau$. If at the initial instant the particle was magnetized along the $z$ axis, then the projection of the magnetic moment on this axis will decrease with time according to the expression $\exp (-t / \tau)$. It is $\tau_{1} = 1 / (\gamma T)^{-1}$. Superparamagnetism is described by the asymptotic formula of Brown (4)

$$\tau_{1} = \tau_{1} + \sigma - \sigma - e_{0},$$

whereas the dotted approaches the asymptote $2\tau_{1}$. By the Appendix and formula (5), we obtain from (29)
In a periodic field $H = H_0 e^{i\omega t}$ parallel to the anisotropy field $H_A$, we find from (29) the magnetic susceptibility of unit volume

$$\chi = -\frac{M}{H_0} (\epsilon_0) = M (1 + i \omega \tau_{||}),$$

where $\tau_{||}$ is the magnetic penetration depth.

The frequency dependence of $\chi_{||}$ is typical of relaxation systems: $\chi_{||}$ decreases monotonically with increase of $\omega$, $\chi_{||}$ has a diffuse maximum at frequency $\omega = \tau_{||}^{-1}$. In the limit of small $\sigma$, the "resonance" frequency $\omega_0 = \frac{\omega_0}{\sigma} \omega$ becomes infinite, whereas the static susceptibility is, to the second order of accuracy,

$$\chi_0 = \frac{M}{H_0} \left[ 1 + \frac{\omega_0^2}{4\pi^2} \right].$$

Thus the magnetic susceptibility of a system of fine ferromagnetic particles ($\sigma \ll 1$) is described by the Langevin formula characteristic of paramagnetic gases. The real and imaginary parts of $\chi_{||}$ for $\sigma = \alpha = 0.1$ are given in Fig. 3.

For $\sigma \gg 1$, the fluctuational mechanism of reorientation of the magnetic moment is "frozen" (according to (30) $\tau_{||} \sim \sigma^3$). Under these conditions a weak alternating field ($\epsilon \ll \sigma, \omega \tau_{||} \gg 1$) is incapable of insuring any appreciable probability of a transition between the states $\mu = \pm \mu_B$, and for $\sigma \rightarrow \infty$ the longitudinal susceptibility (31) approaches zero ($\chi_{||} \sim \sigma^2$).

We now consider the magnetic properties of subdomain particles in a transverse external field. Directing it along the $x$ axis (the $y$ axis, as before, is oriented along the anisotropy field), we obtain from (26) and (27) the following equations for the components $a_x = a$ and $b_y = b$:

$$\begin{align*}
2\sigma a &= -\omega a - 4 \frac{\alpha^2}{\sigma} F_F F_F^\prime b + F_F F_F^\prime, \\
2\sigma b &= \omega b + \frac{\alpha^2}{\sigma} a - \frac{\alpha}{\sigma} B.
\end{align*}$$

Here

$$\lambda_{\pm} = \frac{-2\sigma (1 + 3\alpha)}{\sigma} \left( F_F F_F^\prime / (F_F F_F^\prime) \right).$$

The homogeneous system of equations from (33) for $\epsilon = 0$ has the damped solutions

$$a = \exp(-\lambda_{\pm} t), \quad b = \frac{b_0}{\lambda_{\pm} t} e^{-\lambda_{\pm} t},$$

with a decrement $\lambda(\sigma)$ determined from the condition for compatibility of the system

$$\lambda = \lambda_{\pm} = (2\alpha a)^3 (F_F F_F^\prime) / (F_F F_F^\prime) = 0.$$  

(35)

The roots of the quadratic equation (35) describe well the behavior of the curves in Fig. 1. For $\sigma = 0$ we have $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = 6$, which agrees with the corresponding eigenvalues (18) of the Fokker–Planck equation. We note further that for all ferromagnets, $\sigma \ll 1$. Therefore for nonvanishing but small values of $\sigma$ it is sufficient to retain this parameter only in the combination $\sigma/\sigma$ in (35). In this approximation, the discriminant of Eq. (35) changes sign at the point $\sigma_0 = \sigma_0 / 5$. For $\sigma = 0.1$ we thus have $\sigma_0 = 0.22$, which is close to the value $\sigma_0 = 0.24$ given in Section 2. Finally, at $\sigma \gg 1$, Eq. (35) has the roots (cf. (22))

$$\lambda_{\pm} = 2\alpha (1 + \lambda_{\pm}^2).$$

The dimensional decrements corresponding to the values

$$\lambda = \lambda_{\pm} / (2\alpha) \pm \sigma_0 / \sigma_0,$$

coincide with the eigenvalues (6) of the Landau-Lifshitz equation (1).

We shall calculate the magnetic susceptibility $\chi_{||}$ in a periodic field $\epsilon$. On setting

$$(a, b, t) = (a_0, b_0 + \epsilon),$$

in (33) and solving the inhomogeneous system of equations for the amplitudes $a_0$ and $b_0$, we find

$$a_0 = \frac{\lambda_1 - \sigma}{\lambda_0} + \lambda_0 (\lambda_2 - \varepsilon_0) \xi \epsilon.$$  

(36)

In the determinant $\Delta$ of the system, $\lambda_1$ and $\lambda_2$ denote the roots of (35). By expressing $\langle e_\xi \rangle$ in terms of $\lambda_{\pm}$ by formula (28), we obtain for the magnetic susceptibility

$$\chi_0 = \frac{M \lambda_0 \lambda_2}{H_A} = \frac{2\alpha \lambda_0 \lambda_2}{H_A}.$$  

After simple transformations that use (34) and (35), the last formula can be put into the form

$$\chi_0 = \frac{M \lambda_0 \lambda_2}{H_A} = \frac{2\alpha \lambda_0 \lambda_2}{H_A}.$$  

(37)

Here the coefficients $R_1$ are functions of $\sigma$:

$$R_1 = \frac{\lambda_0 \lambda_2}{2\alpha \lambda_0 \lambda_2} \left[ \frac{2\alpha}{1 + 2\alpha / \sigma} \right] \frac{3 - \sigma + 2\alpha \omega_0}{F_F F_F^\prime}.$$  

(38)

Hence, by use of formulas given in the Appendix, we obtain in the case of small $\sigma$

$$R_1 = 3 \left[ \frac{2\alpha}{1 + 2\alpha / \sigma} \right] \frac{3 - \sigma + 2\alpha \omega_0}{F_F F_F^\prime}.$$  

(39)

the limiting values for $\sigma \rightarrow \infty$ are

$$R_1 = R_1 = 1 + \alpha, \quad R_1 = R_2 = 0.$$  

(40)

In the last case, as should be true for $KV > kT$, (37) and (40) lead to the result of Landau and Lifshitz [11].

$$\chi_0 = \chi_0 (1 + \omega_0) / (1 + \omega_0)$$

(41)

while for $KV \ll kT$, substitution of (39) in (37) gives

$$\chi_0 = \chi_0 (1 + \omega_0) / (1 + \omega_0).$$  

(42)

The real part of the susceptibility is given by formula (4). As $KV/kT \rightarrow 0$, the longitudinal and transverse static susceptibilities (32) and (43) coincide. The same is true also of the complete susceptibility: on comparing $\chi_0$ of (42) with the expression for $\chi_0$ obtained in the limit of small $\sigma$, we see that they are identical. This result is entirely natural, since in this limit the particle becomes isotropic (the magnetic anisotropy constant $K$ drops out of all the formulas).
At \( \sigma = 0.1 \), the graphs of the functions \( \chi_1(\omega) \) calculated by formulas (37) and (38) practically coincide with the graphs of \( \chi_1(\sigma) \) shown in Fig. 3. The \( \chi_1 \) lines shown in Fig. 4 are intermediate between the relaxation curves of Fig. 3 and the resonance curves of Fig. 5. At \( \sigma = 0.5 \), the dispersion of \( \chi_1 \) still has a relaxational character, although on the \( \chi_1 \) curve there is already a note at a finite value of \( \omega \), typical of resonance curves. The dispersion at \( \sigma = 1 \) must be considered rather of resonance type.

Figure 6 shows, as functions of the parameter \( \sigma \), the resonance frequency \( \omega_r \), determined by the position of the maximum on the \( \chi_1(\omega) \) curve, and the width \( \Delta \omega \) of this curve at its half-height. The function \( \omega_r(\sigma) \) has a minimum at \( \sigma_0 = 0.73 \). The value \( \sigma_0 \) may be considered a nominal boundary separating the regions of resonance \((\sigma > \sigma_0)\) and relaxational \((\sigma < \sigma_0)\) dispersion of the magnetic susceptibility. Excess of \( \sigma \) over \( \sigma_0 \) is accompanied by increase of the resonance frequency and narrowing of the absorption line; for \( \sigma \to \infty \), the limiting values \( \omega_r \approx \omega_0 \) and \( \Delta \omega \approx 2\Delta \omega_0 \) are reached (these approximate equalities are more accurate, the smaller \( \sigma \)).

We note that the value \( \sigma_0 \) is three times as large as the critical value \( \sigma_0 \) at which the characteristic frequency of precession vanishes. This is explained by the fact that in the interval \( \sigma_0 < \sigma < \sigma_0 \) the characteristic frequency, though nonzero, is small, and therefore, because of the strong damping of the precession, the character of the dispersion of the susceptibility in this range of values of the parameter is the same (relaxational) as for \( \sigma < \sigma_0 \). But even in the relaxational region, that is for \( \sigma < \sigma_0 \), there is a maximum on the \( \chi_1^* \) curves ("resonance"; see Fig. 3 for \( \chi_1^* = \chi_1 \)). With decrease of \( \sigma \) this maximum becomes smoothed out and is shifted in the direction of larger frequencies. In the case \( \sigma \ll 1 \), when formula (42) is valid, the "resonance" frequency \( \omega_r \) is equal to \( \tau^{-1} = \omega_0/\sigma \), so that \( \omega_r \) and \( \Delta \omega \) become infinite for \( \sigma \to 0 \).

We have been concerned above with the magnetic properties of an individual particle. The magnetic susceptibility of a system of noninteracting particles, whose anisotropy axes are oriented in a random fashion, is

\[
\tilde{\chi} = \frac{c}{3} (x_1 + 2x_2),
\]

where \( c = nV \) is the volume concentration of the magnetic phase and \( n \) is the number density of the particles. In the case \( \sigma \ll 1 \), we get for the static susceptibility of such a system, by (32), (43), and (44),

\[
\chi_s = \frac{m\mu^2}{3kT}.
\]

**APPENDIX**

Directly from the definition (12) of the function

\[
F(\sigma) = \int e^{\sigma x} \, dx
\]

(A.1)

follows the formula for its derivatives at zero,

\[
\left( \frac{d^sF}{d\sigma^s} \right)_{\sigma=0} = \frac{1}{2\pi + 1}.
\]

Thus in the case \( \sigma < 1 \) we have

\[
F = e^{\sigma}/2\sigma
\]

(A.3)

one obtains for the function \( f(0) \) the equation

\[
f' + (1 - 1/2\sigma) f = 0
\]

or, on transforming to the variable \( \rho = \sigma^{-1} \),

\[
-\rho^2 \frac{d^2 f}{d\rho^2} + \left( 1 - \frac{3}{2} \right) f = 0.
\]

We seek a solution of the last equation in the form of a series

\[
f = \sum a_n \rho^n.
\]

(A.5)

For the coefficients \( a_n \), we obtain from (A.4) the recurrence formula

\[
a_n = \frac{(n-1)!}{2\sigma} a_{n-1}.
\]

(A.6)

On substituting (A.5) and (A.6) in (A.3), we have

\[
F = e^{\sigma} \left( 1 + \frac{1}{2\sigma} + \frac{3}{4\sigma^2} + \ldots \right).
\]

By differentiating this formula, we find asymptotic \((\sigma \gg 1)\) expressions for the derivatives.

\[
F' = \frac{e^{\sigma}}{2\sigma} \left( 1 - \frac{1}{2\sigma} - \frac{1}{4\sigma^2} + \ldots \right),
\]

\[
F'' = \frac{e^{\sigma}}{2\sigma} \left( 1 - \frac{3}{2\sigma} + \frac{3}{4\sigma^2} + \ldots \right),
\]

\[
F''' = \frac{e^{\sigma}}{2\sigma} \left( 1 + \frac{5}{2\sigma} + \frac{15}{4\sigma^2} + \ldots \right).
\]
\[ [\mu H_c] = \mu \times H_c. \]


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8S. V. Vonsovskif, Magnetizm (Magnetism), Nauka, 1971, Chap. 23.

Translated by W. F. Brown, Jr.