Dispersion equation for electromagnetic waves in a metal in a magnetic field

E. P. Vol'skiĭ

Institute of Solid State Physics, USSR Academy of Sciences
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The propagation of electromagnetic waves in a metal in a uniform external magnetic field is considered for the general case in which the waves are inhomogeneous (i.e., the real and imaginary parts of the complex wave vector are not parallel) and the magnetic field is arbitrarily oriented with respect to the plane surface of the metal. It is shown that for an uncompensated metal in which certain types of local coupling between the field and current vectors obtain, the dispersion equation defines a specific transformation of the plane containing the vectors representing the real and imaginary parts of the complex wave vector. A method is proposed for obtaining a general solution of the dispersion equation in the form of an inhomogeneous plane wave. The case of an isotropic metal for which the diagonal components of the magnetoresistivity tensor are equal and $\mathbf{h}=\mathbf{b}$ is investigated in detail. An approximate relation is obtained which connects the lengths and mutual orientation of the real and imaginary parts of the wave vector and is valid in the region in which helicons (weakly attenuated waves) exist.

In this paper we consider the propagation in an uncompensated metal located in a constant magnetic field of electromagnetic waves $\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})]$ excited by an external field of fixed frequency ω. The metal is assumed to be isotropic in the absence of an external magnetic field. The entire treatment is conducted in the local limit.

Under these conditions the electromagnetic field of the wave is entirely determined by two vectors: the field vector $\mathbf{B}$ of the constant external magnetic field, and the wave vector $\mathbf{k}$, which is found by solving the dispersion equation. Here we propose a method for constructing a general solution of the dispersion equation for electromagnetic waves in a metal in the presence of a magnetic field that will enable one to investigate waves of any type that can arise under those conditions.

The dispersion equation is easily solved[1] under the assumption that the real and imaginary parts of the complex wave vector $\mathbf{k} = \mathbf{k}_r + i\mathbf{k}_i$ are parallel. However, it is not difficult to show[2] that even the simplest boundary problem—that of the excitation of waves in a half-space bounded by a plane—requires the discussion of waves for which $\mathbf{k}_r$ and $\mathbf{k}_i$ are not parallel. Indeed, suppose the exciting field depends on the position in the boundary plane via the factor $\exp(i\boldsymbol{\kappa}\mathbf{r})$, where $\boldsymbol{\kappa}$ is a vector parallel to the surface. Then since the amplitude of the exciting field is constant on the surface, it follows that the imaginary parts of the wave vectors of all the excited waves, which determine their attenuations, are normal to the surface. At the same time, the boundary conditions require that the projections onto the boundary plane of the real parts of all the wave vectors be different from zero and equal to $\boldsymbol{\kappa}$. Thus, the real and imaginary parts of the wave vectors are not, in general, parallel.

Legendy[2] showed that in order correctly to solve plane boundary problems for helicons in a metal it is essential to consider waves for which the real and imaginary parts of the wave vector are not parallel.

Legendy also called attention[2] to the fact that the dispersion equation for such waves cannot be solved unambiguously without taking the boundary conditions into account since it contains the six unknown components of the real and imaginary parts of the wave vector. However, by fixing the boundary plane in space (with respect to the magnetic field) and specifying on it the magnitude and direction of the component $\boldsymbol{\kappa}$ of the wave vector of the exciting wave, we reduce the number of unknowns to two: the components normal to the surface of the metal of the real and imaginary parts of the wave vector.

To determine these two unknowns in the general case, i.e., for arbitrary orientation of the magnetic field with respect to the metal surface, one must solve a complete algebraic equation of the fourth degree with complex coefficients. This problem presents no serious mathematical difficulties, but in the general case the solution is extremely cumbersome and its physical implications are difficult to see. Only the special cases in which the magnetic field is parallel or perpendicular to the metal surface were discussed in[2].

In this paper we develop an approach that makes it possible to investigate in a rather lucid manner the behavior of all possible solutions of the dispersion equation for plane electromagnetic waves in a metal in an external magnetic field. Thus, what we do is essentially to establish the relation between the real and imaginary parts of the wave vector of a wave at a fixed frequency.

GENERAL SOLUTION OF THE DISPERSION EQUATION

With each complex wave vector $\mathbf{k} = \mathbf{k}_r + i\mathbf{k}_i$ we associate the plane in real three-dimensional space that contains the real vectors $\mathbf{k}_r$ and $\mathbf{k}_i$; we shall call it the $\mathbf{k}$ plane.

The boundary conditions described in the introductory part determine the solutions belonging to one and the same $\mathbf{k}$ plane, which passes through the normal to the surface and the vector $\boldsymbol{\kappa}$. By varying the direction of the magnetic field or the quantity $\boldsymbol{\kappa}$ we obtain all possible solutions belonging to a given $\mathbf{k}$ plane. The problem thus reduces to that of finding the law for transforming some solutions on the given **k** plane into others.

Since the wave field is determined by just the two vectors **B** and **k**, the dispersion equation must always have the form

$$\Phi(k^2, \mathbf{kB}, \omega)=0, \quad k^2=k_r^2-k_i^2+2i\mathbf{k}_r\mathbf{k}_i. \tag{1}$$

Expressing k^2 explicitly in terms of $\mathbf{k}\cdot\mathbf{B}$ and introducing dimensionless quantities, we obtain

$$Q^2=F(z)=R^2(z)\exp\{2i\alpha(z)\}, \tag{2}$$

where

$$z=\mathbf{qn}_B, \quad \mathbf{q}=\mathbf{k}/k=\mathbf{q}_r+i\mathbf{q}_i, \quad \mathbf{n}_B=\mathbf{B}/B, \quad \mathbf{Q}=\mathbf{k}/\sqrt{E}, \quad E=4\pi\omega/c^2R_HB, \tag{3}$$

and R_H is the Hall constant.

The specific form of the dispersion equation, i.e., the forms of the functions Φ and F in Eqs. (1) and (2), depends on the relation between the vectors **e**, **j**, **h**, and b characterizing the fields and currents in the metal. A few particular cases are considered and dispersion equations of the form (2) are derived in the Appendix.

For a fixed direction of the magnetic field, the dispersion equation (2) describes the relation between the complex vectors q and **Q**, i.e., it determines the transformation of the k plane that takes **q** into **Q**:

$$\mathbf{Q}=\mathbf{q}Q=\pm\mathbf{q}R(z)\exp\{i\alpha(z)\}. \tag{4}$$

It is known from the theory of complex vectors[3] that Eq. (4) means that for any value of α the ends of the real vectors representing the real and imaginary parts of **Q**/R and q drawn from the same origin lie on a single ellipse and form its pairs of conjugate radii. It is easily seen that the vectors $\mathbf{q}_r$ and $\mathbf{q}_i$ are the principal semiaxes of this ellipse. Indeed, it follows from the definitions of (3) that

$$q^2=q_r^2-q_i^2+2i\mathbf{q}_r\mathbf{q}_i=1, \quad \mathbf{q}_r\mathbf{q}_i=0, \quad q_r^2-q_i^2=1. \tag{5}$$

Thus, the dispersion equation (2) essentially determines the transformation of the **k** plane that takes the vector **k** into the principal axes.

Let us choose a Cartesian coordinate system in the **k** plane, taking the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_i$ along the axes parallel to $\mathbf{q}_r$ and $\mathbf{q}_i$, respectively. It follows from the last of Eqs. (5) that the lengths of $\mathbf{q}_r$ and $\mathbf{q}_i$ can be specified by a single parameter, so that the complex vector **q** can be written in the form

$$\mathbf{q}_{1,2}=\pm\mathbf{e}_r \operatorname{ch}\mu+i\mathbf{e}_i \operatorname{sh}\mu. \tag{6}$$

We shall specify the direction of the magnetic field by two angles: the angle ψ between the field vector **B** and the **k** plane, and the angle φ between $\mathbf{e}_i$ and the projection of **B** onto the **k** plane. Then

$$z=x+iy=\mathbf{qn}_B=\sin(\pm\varphi+i\mu)\cos\psi. \tag{7}$$

The sign before φ in (7) is determined by the choice of the ambiguous sign in (6). The correspondence between the points xy and the points $\varphi\mu$ will be one to one provided we allow only values of φ between $\pi/2$ and $-\pi/2$, as we shall do in all that follows.

Now we can use Eq. (4) and the parameters just introduced to write the following general expressions for $\mathbf{Q}_r$ and $\mathbf{Q}_i$:

$$\mathbf{Q}_r=\pm R\{\pm\mathbf{e}_r \operatorname{ch}\mu\cos\alpha-\mathbf{e}_i \operatorname{sh}\mu\sin\alpha\}, \tag{8}$$

$$\mathbf{Q}_i=\pm R\{\pm\mathbf{e}_r \operatorname{ch}\mu\sin\alpha+\mathbf{e}_i \operatorname{sh}\mu\cos\alpha\}. \tag{9}$$

The quantities R and α are functions of the parameters ψ, φ, and μ, waves 1 and 2 differing in the sign of φ in the argument of these functions, which is the same as the sign before $\mathbf{e}_r$ in the braces in Eqs. (8) and (9). The sign before R is chosen independently, but is the same in both of Eqs. (8) and (9). Thus, each point in the $\varphi\mu$ plane corresponds to four different solutions for the complex wave vector, and therefore to four different waves.

The functions $R(\psi, \varphi, \mu)$ and $\alpha(\psi, \varphi, \mu)$ are determined by the characteristics of the material medium, i.e., by the type of the local coupling between the vectors e, j, h, and b of the wave. Here, in order to illustrate the possibilities of the general approach presented above, we shall consider the case of an isotropic metal for which all the diagonal components of the magnetoresistivity tensor are equal and h = **b**.

Before considering this example, however, we note that some of the properties of solution (8), (9) are independent of the specific form of the dispersion equation (2). To show this we use Eqs. (8) and (9), and write the formulas for the lengths of the vectors $\mathbf{Q}_r$ and $\mathbf{Q}_i$ and the angle γ between them:

$$Q_r=R(\cos^2\alpha+\operatorname{sh}^2\mu)^{1/2}, \tag{10}$$

$$Q_i=R(\sin^2\alpha+\operatorname{sh}^2\mu)^{1/2}, \tag{11}$$

$$\operatorname{tg}\gamma_{1,2}=\mp\operatorname{sh}2\mu/\sin 2\alpha. \tag{12}$$

It is evident from (12) that $\mathbf{Q}_r$ and $\mathbf{Q}_i$ are parallel only when $\mu = 0$, and from (10) and (11) that $\gamma \to 90°$ and $Q_r/Q_i \to 1$ as $\mu \to \infty$. As follows from (10) and (11), a weakly damped wave with $Q_r \gg Q_i$ is possible only when $\mu \ll 1$.

SOLUTION FOR AN ISOTROPIC METAL

In this case (see the Appendix) the dispersion equation reduces to the form

$$Q^2=1/(\pm z+ig), \tag{13}$$

which is consistent with Eq. (2). A dispersion equation for helicons in a metal having virtually the same form as (13) was written in[4].

We express z in terms of ψ, φ, and μ via Eq. (7), and then, putting

$$ig/\cos\psi\equiv i\operatorname{sh}\mu_0, \quad E_1\equiv E/\cos\psi, \tag{14}$$

we rewrite (13) in the form

$$Q^2=[\sin\varphi\operatorname{ch}\mu + i(\operatorname{sh}\mu_0+\cos\varphi\operatorname{sh}\mu)]^{-1}. \tag{15}$$

It is evident from (15) that Q^2, regarded as a function on the complex $\varphi\mu$ plane, has a single simple pole on the strip $-\pi/2 \le \varphi \le \pi/2$ at the point $\varphi = 0$, $\mu = -\mu_0$, through which passes the surface-wave curve (Fig. 1):

$$\operatorname{sh}\mu_0+\cos\varphi\operatorname{sh}\mu=0. \tag{16}$$

The points on this curve correspond to solutions for which the vectors $\mathbf{Q}_r$ and $\mathbf{Q}_i$ are perpendicular to one another.

From Eq. (15) we easily obtain

$$R=[\sin^2\varphi\operatorname{ch}^2\mu+(\operatorname{sh}\mu_0+\cos\varphi\operatorname{sh}\mu)^2]^{-1/4}, \tag{17}$$

$$\sin\alpha=\{{}^1/_2(1-R^2\sin\varphi\operatorname{ch}\mu)\}^{1/2}, \tag{18}$$

$$\cos\alpha=\mp\{{}^1/_2(1+R^2\sin\varphi\operatorname{ch}\mu)\}^{1/2}. \tag{19}$$

The upper sign in (19) corresponds to points on the $\varphi\mu$ plane lying above the locus of Eq. (16) on Fig. 1.

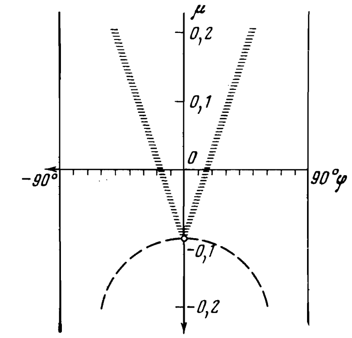

FIG. 1. Regions above the surface-wave curve (the dashed curve) in the $-90° \leqslant \varphi \leqslant 90°$ strip of the $\varphi\mu$ plane corresponding to waves of different types. The two straight lines issuing from the pole at $\varphi = 0$, $\mu = -\mu_0$ ($\mu_0 = 0.1$) approximately separate the region in which $k_r \geqslant k_i$ or $k_i \geqslant k_r$ (outside the acute angle) from the region in which $k_r \approx k_i$.

From Eq. (18) and (19), together with (8) and (9), it follows that at a given point $\varphi\mu$ the waves 1 and 2 differ by an exchange of $\mathbf{Q}_r$ and $\mathbf{Q}_i$. First we shall consider only the part of the strip $-\pi/2 \leq \varphi \leq \pi/2$ on the $\varphi\mu$ plane whose points correspond to weakly attenuated waves. But since in the example under consideration each point corresponds to two types of waves differing by an exchange of the real and imaginary parts of the wave vector, it is sufficient to require that one of the vectors $\mathbf{Q}_r$, $\mathbf{Q}_i$ be much larger than the other. For this to be the case it is necessary that $\mu \ll 1$, as has already been pointed out on the basis of Eqs. (10) and (11). Moreover, it is also evident from Eqs. (10) and (11) that one of the two quantities $|\sin\alpha|$ and $|\cos\alpha|$ must be larger than the other. This last requirement together with Eqs. (17)–(19) and the fact that μ is small leads to the inequality

$$\sin^2\varphi \gg (\mu_0+\mu\cos\varphi)^2, \qquad (20)$$

from which it also follows that the inequality

$$\mu_0 \ll 1. \qquad (21)$$

must obtain. Now let us write approximate expressions for the vectors $\mathbf{Q}_r$ and $\mathbf{Q}_i$ for the points $\varphi\mu$ that satisfy conditions (20) and (21).

Retaining only terms of the first order or lower in μ in Eqs. (17)–(19) and substituting into Eqs. (8) and (9), we obtain

$$\mathbf{Q}_A=\mathbf{Q}_{r1}=\mathbf{Q}_{i2}=\pm\frac{1}{\sqrt{|\sin\varphi|}}(-\mathbf{e}_r), \qquad (22)$$

$$\mathbf{Q}_a=\mathbf{Q}_{i1}=\mathbf{Q}_{r2}=\pm\frac{1}{\sqrt{|\sin\varphi|}}\left[\mathbf{e}_r\frac{\mu_0+\mu\cos\varphi}{2|\sin\varphi|}-\mathbf{e}_i\mu\right], \qquad (23)$$

for $\mu > 0$, and

$$\mathbf{Q}_a=\mathbf{Q}_{r1}=\mathbf{Q}_{i2}=\pm\frac{1}{\sqrt{|\sin\varphi|}}\left[-\mathbf{e}_r\frac{\mu_0+\mu\cos\varphi}{2|\sin\varphi|}-\mathbf{e}_i\mu\right], \qquad (24)$$

$$\mathbf{Q}_A=\mathbf{Q}_{i1}=\mathbf{Q}_{r2}=\pm\frac{1}{\sqrt{|\sin\varphi|}}\mathbf{e}_r, \qquad (25)$$

for $\mu < 0$.

The angle θ formed by the vectors $\mathbf{Q}_A$ defined by Eqs. (22) and (25) with the projection of the magnetic field vector onto the $\mathbf{Q}$ plane satisfies the relation

$$|\cos\theta| = |\sin\varphi|. \qquad (26)$$

Using this relation together with Eqs. (3), (A.3), and (14), and setting $\mu = 0$ in (22)–(25), we obtain

$$k^2=\frac{4\pi\omega}{c^2R_HB\cos\theta\cos\psi}\left(\pm1-\frac{i\rho}{R_HB\cos\theta\cos\psi}\right). \qquad (27)$$

Equation (27) is the ordinary dispersion equation for helicon waves in the local limit[1] as derived on the assumption that the real and imaginary parts of the wave vector are parallel.

Let us return, however, to Eqs. (22)–(25), which determine parametrically the relation between the real and imaginary parts of the wave vector in the general case in which they are not parallel. These equations determine two types of vectors: "large" vectors $\mathbf{Q}_A$, given by Eqs. (22) and (25), and "small" vectors $\mathbf{Q}_a$, given by Eqs. (23) and (24), each of which can be either the real or the imaginary part of the complex wave vector, depending on which wave (1 or 2) we are considering. The length of the vectors $\mathbf{Q}_A$ depends only on φ or, according to Eq. (26), on the angle θ between these vectors and the magnetic field vector. If this angle is fixed, the magnitude and direction of the corresponding vector $\mathbf{Q}_a$ are functions of μ.

It turns out that when μ is varied the end of the vector $\mathbf{Q}_a$ moves along a straight line that is perpendicular to a certain direction that depends only on φ and θ. This direction may be specified by the unit vector $\mathbf{p}_0$:

$$\mathbf{p}_0=\pm\left[\mathbf{e}_r\frac{2\sin\varphi}{\sqrt{1+3\sin^2\varphi}}+\mathbf{e}_i\frac{\cos\varphi}{\sqrt{1+3\sin^2\varphi}}\right]. \qquad (28)$$

The scalar product of $\mathbf{p}_0$ with the corresponding vector $\mathbf{Q}_a$ given by Eq. (23) or (24) is independent of μ and determines the minimum value p of $\mathbf{Q}_a$:

$$p=\frac{\mu_0}{\sqrt{|\sin\varphi|(1+3\sin^2\varphi)}}. \qquad (29)$$

The relation established between the vectors $\mathbf{Q}_A$ and $\mathbf{Q}_a$, i.e., essentially between the real and imaginary parts of the wave vector, is illustrated in Fig. 2 by a polar diagram. This relation can be written analytically as follows:

$$Q_a=\frac{\mu_0 Q_A}{\cos(\gamma-\gamma_0)\sqrt{1+3\cos^2\theta}}. \qquad (30)$$

Here γ and γ_0 are the angles formed by the vector $\mathbf{Q}_A$ with $\mathbf{Q}_a$ and $\mathbf{p}_0$ (Fig. 2).

In order to complete the classification of the solutions of the dispersion equation (13) we must still consider the region of small values of φ:

$$\varphi \lesssim \mu_0 \ll 1, \qquad (31)$$

i.e., essentially the vicinity of the pole of expression (15) at the point $\varphi = 0$, $\mu = -\mu_0$.

It follows from Eqs. (17), (8), and (9) that the lengths

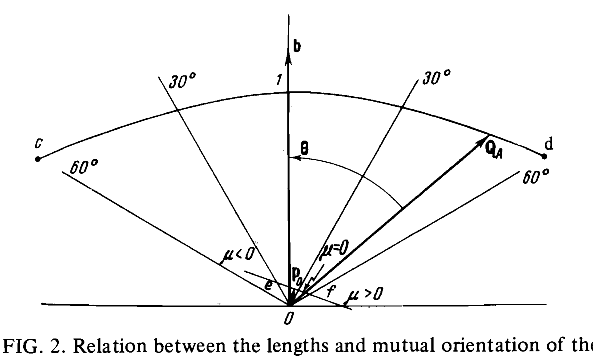

FIG. 2. Relation between the lengths and mutual orientation of the real and imaginary parts of the wave vector for values of φ and μ that satisfy condition (20). Curve cd is the locus (in polar coordinates) of the equation $\rho = 1/\sqrt{|\cos\theta|}$. As μ is varied while θ is held fixed, the end of the vector Q_a [Eqs. (23), (24)] moves along the line ef.

of the vectors $\mathbf{Q}_r$ and $\mathbf{Q}_i$ increase without limit as the pole is approached (this increase at fixed frequency is naturally limited by the condition that the local theory is applicable); however, the angle between these vectors and the ratio of their lengths depend on the direction from which the pole is approached.

Let us first consider points close to the pole and lying on the line $\mu = -\mu_0$, which, in view of (31), essentially coincides with the surface-wave curve (16). Since inequality (20) is also satisfied in this case, we can use Eqs. (22)–(25), from which it is evident that the vectors $\mathbf{Q}_r$ and $\mathbf{Q}_i$ are actually perpendicular to one another and that one of them is μ_0 times shorter than the other. By virtue of Eq. (26), the "large" vector $\mathbf{Q}_A$ is almost perpendicular to the magnetic field. If this vector is the imaginary part of the complex wave vector (wave 2 for $\varphi > 0$ and wave 1 for $\varphi < 0$) we have the surface wave first considered by Legendy[2].

If the point is moved from somewhere on the line $\mu = -\mu_0$ to the line $\mu = 0$, passing around the pole, the vector $\mathbf{Q}_a$ will increase and rotate in such a manner that on the line $\varphi = 0$ the two vectors $\mathbf{Q}_r$ and $\mathbf{Q}_i$ become equal in magnitude and almost perpendicular to the magnetic field (deviating symmetrically on opposite sides from a direction perpendicular to $\mathbf{B}$ by an angle equal to $\mu \ll 1$). This situation corresponds to the normal skin effect in a strong magnetic field.

SOLUTION FOR AN ISOTROPIC METAL WITH BOUNDARY CONDITION

The material presented in the preceding section makes it possible to obtain a set of complex wave vectors satisfying the dispersion equation (13). The waves that are actually excited in a specific case are selected from this set with the aid of boundary conditions; moreover, to solve any problem it is sufficient to determine only the normalized vectors $\mathbf{Q}_r$ and $\mathbf{Q}_i$, since the normalizing factor $E = 4\pi\omega/c^2R_HB$ (see Eq. (3)), like μ_0 (see Eqs. (14) and (A.3)), is always fixed by the experimental conditions.

As an example let us consider the excitation of waves on the plane interface between the metal and the vacuum. As we already said in the introduction, the normal to the surface is parallel to the imaginary parts of the wave vectors of the waves being excited, while the real parts of these wave vectors all have the same component κ parallel to the interface. We shall assume that the vector κ, like the vectors $\mathbf{Q}_r$ and $\mathbf{Q}_i$, are normalized to $\sqrt{E_1}$ (see Eq. (14)). The normal to the metal surface and the vector κ define the previously introduced plane of the complex vector $\mathbf{Q}$. To characterize the direction of the magnetic field we shall use the same parameters as we have used up to now, namely, the angle ψ between the field vector $\mathbf{B}$ and the $\mathbf{Q}$ plane, and the angle θ_i between the normal and the projection of $\mathbf{B}$ onto the $\mathbf{Q}$ plane, i.e., the angle between the imaginary components of the wave vector of the excited waves and this projection of $\mathbf{B}$.

Two waves (types 1 and 2) are excited in the case under consideration, and if conditions (20) and (21) are satisfied, one of them will be weakly attenuated. Let the type 1 wave be the weakly attenuated one. For both waves we can write the obvious relation

$$Q_r^2 \sin^2\gamma = \varkappa^2. \tag{32}$$

Using this relation together with Eqs. (22), (26), and (30), we obtain

$$Q_{i2} = 1/\sqrt{|\cos\theta_i|}, \tag{33}$$

$$Q_{r2} = \frac{\mu_0 Q_{i2}}{2\cos\theta_i\cos\gamma_2 + \sin\theta_i\sin\gamma_2}, \tag{34}$$

$$\operatorname{tg}\gamma_2 = -\frac{2(\varkappa/Q_{i2})\cos\theta_i}{\mu_0 - (\varkappa\sin\theta_i/Q_{i2})} \tag{35}$$

for the strongly attenuated wave, and

$$Q_{r1} = 1/\sqrt{|\cos(\theta_i - \gamma_1)|}, \tag{36}$$

$$Q_{i1} = \frac{2\mu_0 Q_{r1}}{\cos\theta_i + 3\cos(\theta_i - 2\gamma_1)}, \tag{37}$$

$$\sin^2\gamma_1 = \varkappa^2|\cos(\theta_i - \gamma_1)| \tag{38}$$

for the weakly attenuated one. If $|\kappa| \ll 1$, then, as is evident from (38), $|\gamma_1| \ll 1$, and in this case (38) gives

$$\gamma_1 \simeq \frac{1}{2}\frac{\varkappa^2}{Q_{i2}^2}\operatorname{tg}\theta_i + \frac{\varkappa}{Q_{i2}}. \tag{39}$$

The angle θ_i should not be very close to 90°, so that the condition $\mathbf{Q}_{i1} \ll \mathbf{Q}_{r1}$ may be satisfied. Let us now consider the case

$$|\varkappa| \leqslant 1, \quad \cos\theta_i \approx \mu_0^2/\varkappa^2 \ll 1. \tag{40}$$

In this case

$$\sin\gamma_1 \approx \varkappa^2, \quad Q_{r1} = \frac{1}{\varkappa}, \quad Q_{i1} = \frac{\mu_0 Q_{r1}}{3\varkappa^2\sqrt{1-\varkappa^4}}, \tag{41}$$

$$\operatorname{tg}\gamma_2 = -\frac{4\varkappa^2}{\mu_0^2} \gg 1, \quad Q_{i2} = \frac{\varkappa}{\mu_0}, \quad Q_{r2} = \varkappa. \tag{42}$$

The relations (42) describe the surface wave discussed in[2] provided $\psi = 0$.

Solutions (33)–(42) can be easily obtained by means of the graphical construction illustrated in the polar diagram of Fig. 3, where μ_0 is taken as 0.1. The curves cd and c′d′ on Fig. 3, like the curve cd on Fig. 2, represent the equation $\rho = 1/\sqrt{|\cos\theta|}$.

The vector $\mathbf{Q}_{r3}$, which represents the real part of the wave that arises on reflection of wave 1 from the metal-vacuum interface (e.g., from the opposite side of a plane parallel metal slab) is also included in the diagram of Fig. 3. The direction of the reflected wave is described by the expression (39) provided the sign of κ be taken opposite to that of the incident wave and π be added. It follows from (39) that the angle of incidence is not equal to the angle of reflection when $\theta_i \neq 0$, i.e., when the field is not normal to the surface. It is also evident from Fig. 3 that the length of the wave vector changes on reflection.

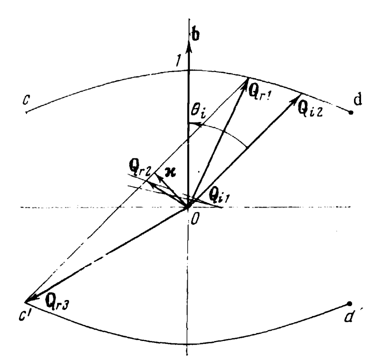

FIG. 3. Graphical method for solving the dispersion equation with allowance for the boundary conditions. The vector κ is parallel and the vectors Q_{i1} and Q_{i2} are perpendicular to the plane metal-vacuum interface.

Let us make one more remark concerning the choice of the set of basis vectors on the $\mathbf{Q}$ plane (see (6)). It follows from (22) and (25) that $\mathbf{e}_r$ is parallel to $\mathbf{Q}_A$. The vectors $\mathbf{Q}_A$ for the two waves excited in the problem discussed in this section are not parallel, as can be seen from Fig. 3. The angle between them is γ_1. Hence the basis sets $\mathbf{e}_r$ and $\mathbf{e}_i$ for these waves are also turned through an angle γ_1 with respect to one another. This proves to be important in calculating the wave fields.

WAVE FIELDS IN AN ISOTROPIC METAL

If as a result of solving the dispersion equation (13) under boundary conditions, as was done in the preceding section, we obtain a complex wave vector $\mathbf{k} = \mathbf{Q}\sqrt{E_1}$, the magnetic field of the corresponding wave will have the form given by Eq. (A.5) of the Appendix, in which the signs ± refer to waves of types 1 and 2, respectively.

In order to make use of the results derived above we must express the field (A.5) in terms of the previously introduced parameters and the wave-vector coordinate system $\mathbf{e}_\perp$, $\mathbf{e}_r$, $\mathbf{e}_i$, where $\mathbf{e}_\perp$ is the unit vector perpendicular to the $\mathbf{Q}$ plane. To do this we factor the quantity k^2 out of expression (A.5) for the field, incorporating it in the complex amplitude, and then express the field in terms of the vector q (Eqs. (3) and (6)) and the vector n_B expressed in the same coordinate system via the parameters ψ and φ. Then after some algebra we obtain

$$\mathbf{h}_m = h_m\left(\pm\frac{1}{\operatorname{ch}\mu}\mathbf{e}_\perp + \operatorname{th}\mu\,\mathbf{e}_r + i\mathbf{e}_i\right). \quad (43)$$

The ambiguous sign in the parentheses refers to waves 1 and 2. We shall always drop the factor $\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})]$; here we can regard it as part of the complex amplitude h_m.

The field (43) represents a circularly polarized wave, the field vector $\mathbf{h}_m$ rotating in the plane defined by the complex zero-length vector in the parentheses in expression (43). When $\mu \ll 1$, this plane is almost perpendicular to the plane of the wave vector $\mathbf{k}$, which contains the vectors $\mathbf{e}_r$ and $\mathbf{e}_i$.

Now let us consider the problem of the excitation of waves on the plane metal-vacuum interface. In order to make use of Eq. (43) to write the expressions for the fields corresponding to the two waves found in the preceding section, we must find the values of μ for these waves and take into account the differences in the directions of the vectors $\mathbf{e}_r$ and $\mathbf{e}_i$ for the two waves.

For the attenuated wave, as follows from (22) or (25), the vector $\mathbf{e}_{r2}$ is perpendicular to the surface, while $\mathbf{e}_{i2}$ is parallel to $\boldsymbol{\kappa}$. It follows at once from (23) that

$$\mu_2 = -\varkappa\sqrt{|\cos\theta_i|}. \quad (44)$$

The vector $\mathbf{e}_{r1}$ for the other wave is turned through an angle γ_1 with respect to the normal, as can be seen from Fig. 3. On the one hand, this angle can be related to the parameter μ_1 via the equation

$$\operatorname{tg}\gamma_1 = \frac{2\mu_1|\cos(\theta_i-\gamma_1)|}{\mu_0+\mu_1|\sin(\theta_i-\gamma_1)|}, \quad (45)$$

which follows from (22) and (26).

On the other hand, the angle γ_1 can be expressed by means of the boundary conditions, i.e., in terms of the parameters κ and θ_i. When $|\kappa| \ll 1$ we can use Eq. (39) to do this. In view of the fact that in this case $|\gamma_1| \ll 1$, and hence $\mu_1 \ll \mu_0$ (see Eq. (45)), we obtain

$$\mu_1 = \mu_0\varkappa/2\sqrt{|\cos\theta_i|} \quad (46)$$

with the aid of (39) and (45).

The vectors $\mathbf{e}_{r1}$ and $\mathbf{e}_{i1}$ can be expressed in terms of $\mathbf{e}_{r2}$ and $\mathbf{e}_{i2}$; the result, valid when γ_1 is small, is

$$\mathbf{e}_{r1} = \mathbf{e}_{r2} + \gamma_1\mathbf{e}_{i2}, \quad \mathbf{e}_{i1} = -\gamma_1\mathbf{e}_{r2} + \mathbf{e}_{i2}. \quad (47)$$

As a result we obtain the following expressions for the magnetic fields of waves 1 and 2, which are accurate to linear terms in μ_0 and κ:

$$\mathbf{h}_1 = h_1[\mathbf{e}_\perp + i(\mathbf{e}_{i2} - \gamma_1\mathbf{e}_{r2})], \quad (48)$$

$$\mathbf{h}_2 = h_2[-\mathbf{e}_\perp + \mu_2\mathbf{e}_{r2} + i\mathbf{e}_{i2}], \quad (49)$$

with

$$\gamma_1 \approx -\mu_2 = \varkappa\sqrt{|\cos\theta_i|}. \quad (50)$$

The amplitudes h_1 and h_2 can be found from the condition that all components of h be continuous at the interface[2] (we are assuming that h = b). Following Legendy[2], we take the field in vacuo as the superposition of two surface waves–the excited and reflected ones:

$$\mathbf{h}_0 = h_0(\mathbf{e}_{r2} + is\mathbf{e}_{i2})\exp\{-ik_t x - |k_t|z\}, \quad (51)$$

$$\mathbf{h}_r = h_r(-\mathbf{e}_{r2} + is\mathbf{e}_{i2})\exp\{-ik_t x + |k_t|z\}, \quad (52)$$

where $k_t = \kappa\sqrt{E_1}$, $s = \kappa/|\kappa|$, and the z axis is normal to the surface, i.e., is parallel to $\mathbf{e}_{r2}$. Then we obtain the following expressions for the amplitudes h_1, h_2, and h_r:

$$h_1 = h_2 = \frac{h_0}{1 - {}^1/_2\varkappa(1+i)\sqrt{|\cos\theta_i|}}, \quad (53)$$

$$h_r = h_0\frac{1 + {}^1/_2\varkappa(1+i)\sqrt{|\cos\theta_i|}}{1 - {}^1/_2\varkappa(1+i)\sqrt{|\cos\theta_i|}}. \quad (54)$$

If both κ and μ_0 are smaller than about 0.1, Eqs. (53) and (54), as well as the other equations derived for this case, can be used for values of θ_i not exceeding 60–70°.

If θ_i is close to 90°, i.e., if the magnetic field is almost parallel to the surface and conditions (40) are satisfied, we have

$$\mu_2 = -\mu_0, \quad (55)$$

and with the aid of Eq. (45) we can obtain the equation

$$\mu_1 = \mu_0/\sqrt{1-\varkappa^4}. \quad (56)$$

Making the appropriate substitutions into Eq. (43) and taking into account the remarks made at the end of the preceding section, we obtain

$$\mathbf{h}_1 = h_1\left[\mathbf{e}_\perp + \frac{\mu_0\varkappa^2}{\sqrt{1-\varkappa^4}}\mathbf{e}_{i2} + \mu_0\mathbf{e}_{r2} + i(\mathbf{e}_{i2}\sqrt{1-\varkappa^4} - \varkappa^2\mathbf{e}_{r2})\right], \quad (57)$$

$$\mathbf{h}_2 = h_2[-\mathbf{e}_\perp - \mu_0\mathbf{e}_{r2} + i\mathbf{e}_{i2}]. \quad (58)$$

If the field in the vacuum is described by Eqs. (51) and (52), we have

$$h_1 = h_2 = \frac{2h_0}{1+\sqrt{1-\varkappa^4} - i\varkappa^2 - i\mu_0\varkappa^2/\sqrt{1-\varkappa^4}}, \quad (59)$$

$$h_r = h_0\frac{1+\sqrt{1-\varkappa^4} + i\varkappa^2 - i\mu_0\varkappa^2/\sqrt{1-\varkappa^4}}{1+\sqrt{1-\varkappa^4} - i\varkappa^2 - i\mu_0\varkappa^2/\sqrt{1-\varkappa^4}}. \quad (60)$$

The range of κ values for which the solution (41), (42), (57)–(60) is fairly accurate can be determined with the aid of Eqs. (41) and the condition $Q_{i1} \ll Q_{r1}$. If $\mu_0 = 0.1$, then $|\kappa|$ should lie between 0.7 and 0.98.

Thus, the problem of the propagation of inhomogeneous plane waves can be completely solved in the case discussed in detail here—that of an isotropic metal.

The approximate relationships obtained here for an isotropic metal make possible a clear representation of the geometry of the field and the manner in which the waves propagate for an arbitrary orientation of the magnetic field with respect to the plane surface of the metal.

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APPENDIX

As was mentioned above, the form of the dispersion equation is determined by the nature of the coupling between the vectors **e**, **j**, **h**, and **b**. Let us consider a few examples, always assuming that the coupling is local.

Let us begin with the simplest case. We assume that **h** = **b** and that the following relationship obtains between the electric field and current:

$$\mathbf{e}=R_H[\mathbf{jB}]+\rho\mathbf{j}. \qquad (A.1)$$

Substituting (A.1) into Maxwell's equations and assuming the plane-wave solution $\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})]$, where $\mathbf{k}=\mathbf{k}_r+i\mathbf{k}_i$ is a complex wave vector, we obtain the wave equation

$$-i(\mathbf{kn}_B)[\mathbf{kh}]+ig[k^2\mathbf{h}-\mathbf{k}(\mathbf{kh})]=E\mathbf{h}. \qquad (A.2)$$

in which we have used the following notation:

$$\mathbf{n}_B=\mathbf{B}/B,\quad E=4\pi\omega/c^2R_HB,\quad g=\rho/R_HB. \qquad (A.3)$$

Equation (A.2) can be written as

$$\hat{D}(\mathbf{k})\mathbf{h}=E\mathbf{h}. \qquad (A.4)$$

The tensor $\hat{D}(\mathbf{k})$ has two eigenvectors with nonvanishing eigenvalues:

$$\mathbf{h}_\pm=a\{[\mathbf{k}[\mathbf{kn}_B]]\pm ik[\mathbf{kn}_B]\}, \qquad (A.5)$$

We choose a so that

$$\mathbf{h}_+\mathbf{h}_-=1. \qquad (A.6)$$

Moreover, it is evident from (A.5) that

$$\mathbf{h}_+\mathbf{h}_+=\mathbf{h}_-\mathbf{h}_-=\mathbf{h}_\pm\mathbf{k}=0. \qquad (A.7)$$

Equating the eigenvalues of $\hat{D}(\mathbf{k})$ corresponding to the eigenvectors $\mathbf{h}_\pm$, we obtain the dispersion equation

$$E=\pm k(\mathbf{kn}_B)+igk^2, \qquad (A.8)$$

which can be easily reduced to Eq. (13) in the main text.

Now we may consider a more complicated form of relation (A.1) by taking into account the difference between the longitudinal and transverse magnetoresistivity, which can be considerable even for a metal having a spherical Fermi surface[6]. Then relation (A.1) becomes

$$\mathbf{e}=R_H[\mathbf{jB}]+\rho_\perp\mathbf{j}+(\rho_\parallel-\rho_\perp)\mathbf{n}_B(\mathbf{jn}_B). \qquad (A.9)$$

In this case the wave equation is

$$-i(\mathbf{kn}_B)[\mathbf{kh}]+ig[k^2\mathbf{h}-\mathbf{k}(\mathbf{kh})]+if[\mathbf{kn}_B]([\mathbf{kn}_B]\mathbf{h})=E\mathbf{h}, \qquad (A.10)$$

in which

$$f=(\rho_\parallel-\rho_\perp)/R_HB,\quad g=\rho_\perp/R_HB.$$

A solution to this equation can be found in the form of a linear combination of the vectors $\mathbf{h}_\pm$. In the notation defined by Eqs. (3), the dispersion equation becomes

$$Q^2=\{\pm[z^2-{}^1/_4f^2(1-z^2)^2]^{1/2}+i[g+{}^1/_2f(1-z^2)]\}^{-1}. \qquad (A.11)$$

For a final example we return to the relation (A.1) but take into account the difference between **h** and **b** arising from the de Haas–van Alphen effect. In the linear approximation, the relation between **h** and **b** for a metal having a spherical Fermi surface always has the form

$$\mathbf{h}=\mathbf{b}-4\pi\chi\mathbf{n}_B(\mathbf{n}_B\mathbf{b}), \qquad (A.12)$$

in which $\chi=\partial M/\partial B$. Here we obtain the following wave equation in place of (A.2) and (A.4):

$$\hat{D}(\mathbf{k})\mathbf{b}-4\pi\chi(\mathbf{n}_B\mathbf{b})D(\mathbf{k})\mathbf{n}_B=E\mathbf{b}. \qquad (A.13)$$

The corresponding dispersion equation is

$$Q^2=(\pm[(1-4\pi\chi w)z^2-4\pi^2\chi^2g^2w^2]^{1/2}+ig(1-2\pi\chi w)\}^{-1}, \qquad (A.14)$$

with $w=1-z^2$.

Equation (A.14) is written for arbitrary values of g and χ; if both these quantities are small compared with unity, Eq. (A.14) reduces to an equation that differs from (A.8) or (13) only by an oscillating factor of the order of unity[7].

These examples show that in all cases in which the anisotropy of space is due only to a constant magnetic field, the dispersion equation reduces to one of the form of Eq. (2). Moreover, it is evident from (13), (A.11), and (A.14), that the function F(z) on the right in Eq. (2) is meromorphic over the entire complex z plane, i.e., it has no singularities except a finite number of isolated poles.

*[jB] ≡ j × B.

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