Effect of energy losses on bremsstrahlung of relativistic electrons

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The radiation emitted by high energy electrons traversing a matter-vacuum interface is investigated. It is shown that in the high-energy limit the theories that neglect the variation of the multiple-scattering constants and take it into account are identical. It is shown that in the limit of high electron energies the angular and spectral distributions are the same as those obtained in the coherent-length representation.

The bremsstrahlung of ultrarelativistic electrons in a medium has been widely discussed in the literature (see, e.g., [1,2] and the bibliography cited therein). In particular, Galitskiĭ and Gurevich [3], using the concept of the coherent length (see also [1,2]), obtained the spectral and angular distributions of the bremsstrahlung, with account taken of the absorption of virtual quanta in the medium. According to [3] at sufficiently high electron energies, such that the coherent length without allowance for the absorption of the quanta becomes larger than the quantum-absorption length L_c, the radiation intensity can be represented in the form

$$J=\mathrm{const}\cdot q_0\omega, \tag{1}$$

where $q_0 = (1/4)\langle\theta^2\rangle = 4\pi n(Ze^2)^2E_0^{-2}L_R$, $\langle\theta^2\rangle$ is the mean-squared angle of the multiple scattering per unit time, n is the number of nuclei per unit volume of the material, E_0 is the electron energy, L_R is the radiation logarithm, $L_c = 1/\omega\epsilon''$, ϵ'' is the imaginary part of the dielectric constant, and ω is the photon frequency. The quantum emission angle under the same conditions is

$$\vartheta\sim(\omega L_c)^{-1/2}. \tag{2}$$

Thus, at sufficiently high electron energies the radiation intensity proportional to the frequency of the emitted photons, and the radiation angle does not depend on the energy of the radiating particle. The analysis carried out by Galitskiĭ and Gurevich [3] was subsequently confirmed by Galitskiĭ and Yakimets [4] with the aid of an investigation of the formulas for the energy lost by an ultrarelativistic electron in a medium.

Varfolomeev et al. [5], on the basis of the analysis of the formulas for the energy losses, have recently called attention to the fact that owing to the energy loss by the electron it is necessary to take into account the change that takes place, over the coherent length, in the constants that characterize the multiple scattering of the radiating particle, in an emitted-quantum frequency region that expands rapidly with increasing electron energy. It is shown in the same paper that the subdivision of the energy losses in the absorbing medium into bremsstrahlung loss and into loss to pair production is to a certain degree arbitrary. It is of interest in this connection to analyze the form of the photon spectrum from a different point of view, namely, to investigate the spectrum of the photons radiated in vacuum by an ultrarelativistic electron passing through the boundary between an absorbing medium and the vacuum, for in this case the arbitrariness indicated above does not take place.

An analysis of the emission spectrum of a particle emitted from a medium into vacuum, which will be presented below, shows that in the limit of high electron energies, the energy losses over the coherent length do not influence the spectrum and the effective emission angle.

1. The most general method of finding the angular and spectral distributions of the radiation of an arbitrarily moving particle in an absorbing medium in the presence of an interface was formulated by Pafomov [6] with the aid of the method of images. He also derived the general formulas (4.14)–(4.21) which made it possible to determine the radiation intensity $W_{\mathbf{n}\omega}$ in a unit solid angle and a unit frequency interval for a particle moving along an arbitrary trajectory through the interface between a medium and vacuum. According to formula (4.21) of [6], we have (c = 1)

$$W_{\mathbf{n}\omega}=\omega^4R^2|[\mathbf{n}\mathbf{\Pi}_\omega]|^2, \tag{3*}$$

where $\mathbf{\Pi}_\omega$ is the Hertz vector and $\mathbf{n} = \mathbf{R}/R$ is a unit vector in the observation direction.

In the case of interest to us, of high γ-quantum energies, the dielectric constant ϵ is close to unity. This enables us to neglect the difference between ϵ and unity in the coefficients preceding the exponentials (but not in the phases of the exponentials!) in formulas (4.14)–(4.16) of [6], which define the Hertz vector. This neglect is equivalent to discarding the specularly reflected waves, and is permissible in our frequency region ($\mathrm{Re}(\epsilon-1)\ll 1$, $\mathrm{Im}\,\epsilon\ll 1$). As a result, the Hertz vector $\mathbf{\Pi}_{\omega 1}$ produced in vacuum by a particle moving in the medium can be written in the form

$$\Pi_{\omega 1}=\frac{ie\,e^{i\omega R}}{2\pi\,\omega R}\int\mathbf{v}(t)\,e^{i(\omega t-\mathbf{k}\mathbf{r}(t))}\,dt. \tag{4}$$

where $\mathbf{v}(t)$ is the velocity of the particle at the instant of time t, $\mathbf{k}$ is the wave vector of the photon in a medium with components $\mathbf{k}_\perp = \omega\mathbf{n}_\perp$ and $k_z = \omega\sqrt{\epsilon}n_z$, while the z axis of the coordinate system is directed from the medium to the vacuum, the origin is on the interface between the medium and vacuum, and the medium is located in the region $z < 0$.

The Hertz vector produced in the vacuum by a particle that moves on the section of the path in the vacuum is given by

$$\Pi_{\omega 2}=\frac{ie\,e^{i\omega R}}{2\pi\omega R}\int\mathbf{v}(t)\,e^{i(\omega t-\mathbf{k}_0\mathbf{r}(t))}\,dt, \tag{5}$$

where $\mathbf{k}_0 = \omega\mathbf{n}$. Substituting (4) and (5) in (3) and averaging $W_{\mathbf{n}\omega}$ over the possible trajectories, we obtain the following expression for the intensity of the radiation of the particle moving through the interface between the medium and vacuum:

$$W_{\mathbf{n}\omega}=\frac{e^2\omega^2}{4\pi^2}\int_{-t_1}^{t_2}\!\!\int[\mathbf{n}\mathbf{v}][\mathbf{n}\mathbf{v}']\psi_{\mathbf{k}_0}^{(-)*}(\mathbf{r})\,e^{i\omega t}.$$

$$\cdot\psi_{k_0}^{(-)}(\mathbf{r}')e^{-i\omega t'}w(\mathbf{r},\mathbf{v},t;\mathbf{r}',\mathbf{v}',t')d\mathbf{r}\ldots dt', \tag{6}$$

where

$$\psi_{k_0}^{(-)}=\begin{cases}\exp(i\mathbf{k}_0\mathbf{r}) & \text{if} \quad z>0\\ \exp(i\mathbf{k}'\mathbf{r}) & \text{if} \quad z<0\end{cases} \tag{7}$$

and $w(\mathbf{r}, \mathbf{v}, t; \mathbf{r}', \mathbf{v}', t') = w_1(\mathbf{r}, \mathbf{v}, t)w_2(\mathbf{r}, \mathbf{v}, t\,|\,\mathbf{r}', \mathbf{v}', t')$ is the joint probability of observing particle coordinates $\mathbf{r}$ and $\mathbf{v}$ at the instant t, and $\mathbf{r}'$ and $\mathbf{v}'$ at the instant t′; $w_1(\mathbf{r}, \mathbf{v}, t)$ is the probability of observing the coordinates $\mathbf{r}$ and $\mathbf{v}$ at the instant t; $w_2(\mathbf{r}, \mathbf{v}, t\,|\,\mathbf{r}', \mathbf{v}', t')$ is the conditional probability of observing the coordinates $\mathbf{r}'$ and $\mathbf{v}'$ at the instant t′ if their values at the instant t were $\mathbf{r}$ and $\mathbf{v}$.

2. It is of interest to note that formula (6) for $W_{n\omega}$ can be obtained with the aid of the following simple reasoning, which also explains the physical meaning of the functions $\psi_{k0}^{(-)}(\mathbf{r})$ introduced above. In fact, the vector potential $\mathbf{A}(\mathbf{r}, \omega)$ produced by the particle satisfies the equation

$$\Delta\mathbf{A}+\omega^2\varepsilon(\omega, \mathbf{r})\mathbf{A}=-4\pi j(\mathbf{r}, \omega) \tag{8}$$

with the corresponding boundary conditions; $\epsilon(\omega, \mathbf{r}) = \epsilon(\omega)$ in the medium and $\epsilon(\omega, \mathbf{r}) = 1$ outside the medium.

The solution of (8) can be obtained with the aid of the exact retarded Green's function, which is a tensor quantity because $\mathbf{A}$ is a vector field. If, however, we can neglect the specularly reflected waves (as is the situation in our case), then the solution of (8) can be written in the form

$$\mathbf{A}(\mathbf{r},\omega)=-4\pi\int G(\mathbf{r},\mathbf{r}',\omega)\mathbf{j}(\omega,\mathbf{r}')d^3r', \tag{9}$$

where $G(\mathbf{r}, \mathbf{r}', \omega)$ is the scalar retarded Green's function and satisfies the equation

$$\Delta G+\omega^2\varepsilon(\omega, \mathbf{r})G=\delta(\mathbf{r}-\mathbf{r}'). \tag{10}$$

We now use the fact that as $\mathbf{r}\to\infty$ the Green's function $G(\mathbf{r}, \mathbf{r}', \omega)$ can be represented in the form

$$\lim_{r\to\infty}G(\mathbf{r},\mathbf{r}',\omega)=-\frac{1}{4\pi}\frac{e^{ik_0r}}{r}\psi_{k_0}^{(-)*}(\mathbf{r}'), \tag{11}$$

where $\psi_{k_0}^{(-)}(\mathbf{r}')$ is the eigenfunction of the homogeneous equation

$$\Delta\psi_{k_0}^{(-)}+\omega^2\varepsilon^*(\omega,\mathbf{r})\psi_{k_0}^{(-)}=0, \tag{12}$$

and contains converging waves at infinity. Relation (11) in the particular case of a spherically symmetrical real potential (in our case, ϵ) is given in [7]. Analysis shows that it is valid in the general case of complex potentials, which need not necessarily have spherical symmetry.

Substituting (11) in (9) and applying the curl operation to the obtained expression, we have for the magnetic field of the radiated wave

$$\mathbf{H}(\mathbf{r},\omega)=\operatorname{rot}\mathbf{A}=i\left[\mathbf{k}_0r^{-1}\int\psi_{k_0}^{(-)*}(\mathbf{r}')\mathbf{j}(\omega,\mathbf{r}')d^3r'\right]. \tag{13}$$

Recognizing that for a point-like particle the current is

$$\mathbf{j}(\omega,\mathbf{r}')=\int e\mathbf{v}(t)\delta(\mathbf{r}'-\mathbf{r}(t))e^{i\omega t}dt, \tag{14}$$

we can rewrite $\mathbf{H}(\mathbf{r}, \omega)$ in the form

$$\mathbf{H}(\mathbf{r},\omega)=i\frac{e}{r}\int dt[\mathbf{k}_0\mathbf{v}(t)]\psi_{k_0}^{(-)*}(\mathbf{r}(t))e^{i\omega t}. \tag{15}$$

Using (15) and the equation

$$W_{n\omega}=r^2|\mathbf{H}(\omega, \mathbf{r})|^2/4\pi^2,$$

we obtain (7).

It must be emphasized that a relation of the type (11), which expresses the asymptotic form of the Green's function in terms of the eigenfunctions of the homogeneous Maxwell equations of the type $\mathbf{A}_{k_0}^{(-)}$, holds also in the general case. This means that the intensity $W_{n\omega}$ can likewise be expressed in the general case in terms of the eigenfunctions of the homogeneous Maxwell equations containing converging waves at infinity. The foregoing becomes obvious if we turn to the quantum-mechanical formula for $W_{n\omega}$. In this case the radiation intensity $W_{n\omega}$ is determined by the square of the modulus of a matrix element in the form $\langle\gamma e'|\hat{j}\mathbf{A}|e\rangle$. It is well known that the exact wave functions of all the particles produced in the reaction must be chosen such that their asymptotic form has converging waves at infinity [8] (compare also with the analogous requirement imposed on the wave function of an electron in the final state in the exact bremsstrahlung theory [8]). On the other hand, the exact matrix element $\langle\gamma|\hat{\mathbf{A}}|0\rangle$ satisfies the homogeneous Maxwell equations, thus proving the statement made above.

Expressing $W_{n\omega}$ in terms of the solution $\mathbf{A}_k^{(-)}$ (in the classical and quantum-mechanical cases) may offer advantages, since the solutions of the homogeneous Maxwell equations are frequently well known.

3. To obtain concrete expressions describing the distribution of the emitted photons, we must find the form of the functions w_1 and w_2. If we disregard the change of the multiple-scattering constants, then these functions satisfy the usual Fokker-Planck equation. In this case we follow Pafomov's procedure [6], namely, we solve the problem of the radiation of a particle emitted from an absorbing medium. We then obtain for $W_{n\omega}$ an expression that coincides with that obtained by Pafomov (see formulas (27.44)—(27.46) in [6]), provided we make in this expression the substitutions $\beta\to-\beta$ and $\epsilon\to\epsilon^*$ in all the functions except $\eta_1 = [4\omega\beta q\,\mathrm{Im}(\epsilon-\sin^2\vartheta)^{1/2}]^{1/2}$.

In the general case, the expression for the intensity $W_{n\omega}$ contains contributions that are connected with the transition and bremsstrahlung mechanisms of the radiation, and also with their mutual interference, and the analysis of the photon spectrum must be carried out with allowance for all the processes [1,6]. It turns out, however (see [6]), that as $q\to 0$ the radiation of the photons polarized perpendicular to the plane of their emission takes place only in the presence of multiple scattering of the electrons, and does not contain a contribution that comes from the transition radiation. It is natural to classify such photons as bremsstrahlung photons [6]. Their intensity $W_{\perp n\omega}^{b}$ is determined by formula (27.49) of [6] and in the high-energy case of interest to us ($q\sim E^{-2}\to 0$) it takes the form

$$W_{\perp n\omega}^{b}=\frac{e^2q}{2\pi^2}L_c\left[\left(1-\beta-\frac{1}{2}(\varepsilon'-1)+\frac{1}{2}\vartheta^2\right)^2+\left(\frac{1}{2}\varepsilon''\right)^2\right]^{-1}, \tag{16}$$

where $\epsilon' = \mathrm{Re}\,\epsilon$. In the region of electron energies so high that $1-\beta\ll\epsilon''$, and at γ-ray frequencies such that $\epsilon'-1\ll\epsilon''$, we obtain from (16) the effective radiation angle

$$\vartheta\sim\sqrt{\varepsilon''}=(\omega L_c)^{-1/2}. \tag{17}$$

This angle coincides with the quantum radiation angle obtained by Galitskiĭ and Gurevich [3] from an analysis of the interference conditions.

Integrating (16) over the angles, we obtain the following expression for the spectral distribution of the bremsstrahlung

$$\int W_{\perp n\omega}^{b}d\Omega=e^2qL_c^2\omega. \tag{18}$$

We see that the radiation intensity is proportional to the frequency of the emitted photons ω in accordance with the conclusion drawn in [3] with the aid of the coherent-length concept.

4. For the analysis that follows, it is convenient to carry out in expression (6) the change of variables t, t′ $\to$ −t, −t′, and $\mathbf{v}$, $\mathbf{v}'$ $\to$ $-\mathbf{v}$, $-\mathbf{v}'$, which is equivalent to the time-reversal transformation. Using the fact that by virtue of the symmetry of the exact equations of motion with respect to time reversal the exact functions w satisfies a relation of the type [9]

$$w(\mathbf{r}, -\mathbf{v}, -t|\mathbf{r}', -\mathbf{v}', -t')=w(\mathbf{r}, \mathbf{v}, t|\mathbf{r}', \mathbf{v}', t'),$$

we obtain as a result the following equation for $W_{\mathbf{n}\omega}$:

$$W_{\mathbf{n}\omega}=\frac{e^2\omega^2}{4\pi^2}\iint_{-t_2}^{t_1}[\mathbf{nv}][\mathbf{nv}']\psi_{\mathbf{k}_0}^{(-)*}(\mathbf{r})e^{-i\omega t}\psi_{\mathbf{k}_0}^{(-)}(\mathbf{r}')e^{i\omega t'}$$
$$\times w_1(\mathbf{r},\mathbf{v},t)\,w_2(\mathbf{r},\mathbf{v},t|\mathbf{r}',\mathbf{v}',t)\,d\mathbf{r}\ldots dt', \quad (19)$$

where the probability densities w satisfy the time-reversed initial conditions. It is convenient, following Pafomov [6], to formulate conditions for the w that describe the motion of the particle in the medium on the interface between the medium and the vacuum at the instant t = 0 of passage of the electron through the interface:

$$w_1(\mathbf{r},\ \mathbf{v},\ t=0)=\delta(\mathbf{r})\delta(\mathbf{v}+\mathbf{v}_1); \qquad w_2(t=t')=\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{v}-\mathbf{v}')$$

($\mathbf{v}_1$ is the velocity of the particle in vacuum). In other words, in this case the problem reduces to finding the distribution functions for a particle entering the medium at the instant t = 0 with velocity $\mathbf{v}_0 = -\mathbf{v}_1$.

We write down the velocity $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}$, where $\delta\mathbf{v}$ is the change of the electron velocity in the medium as a result of the collisions. Then the product $[\mathbf{n}\times\mathbf{v}]\cdot[\mathbf{n}\times\mathbf{v}']$ breaks up into a sum of four terms, namely

$$[\mathbf{nv}][\mathbf{nv}']=[\mathbf{nv}_0]^2+[\mathbf{nv}_0][\mathbf{n}\delta\mathbf{v}'] +[\mathbf{n}\delta\mathbf{v}][\mathbf{nv}_0]+[\mathbf{n}\delta\mathbf{v}][\mathbf{n}\delta\mathbf{v}'].$$

Accordingly, the intensity $W_{\mathbf{n}\omega}$ also breaks up into four terms. The first term, at constant electron velocity, describes the transition radiation. The fourth term, at $\epsilon'' = 0$, describes the bremsstrahlung [6]. Let us consider, for the sake of argument, the radiation to be proportional to the fourth term $[\mathbf{v}\times\delta\mathbf{v}]\cdot[\mathbf{n}\times\delta\mathbf{v}']$, which we shall henceforth call the bremsstrahlung term. The conclusions that will be drawn below, namely that the form of the spectrum and the emission angles are independent, in the limit of high electron energies, of the energy loss over the coherent length, pertain equally well also to the contributions made to the intensity $W_{\mathbf{n}\omega}$ by the first three terms.

Since the quantities $\delta\mathbf{v}$ and $\delta\mathbf{v}'$ differ from zero only in the medium, by substituting in (19) the expression for $\psi_{\mathbf{k}_0}^{(-)}$ in the medium we obtain the following expression for $W_{\mathbf{n}\omega}^{b}$:

$$W_{\mathbf{n}\omega}^{b}=\frac{e^2\omega^2}{2\pi^2}\,\mathrm{Re}\int_0^{t_1}dt\int_0^{t_1-t}\delta\mathbf{v}\,\delta\mathbf{v}'\,e^{i\omega\tau}\exp(2\,\mathrm{Im}\,k_z z+i\mathbf{k}\cdot\boldsymbol{\rho})$$
$$\times w_1(\mathbf{r}\mathbf{v}t)\,w_2(\mathbf{r}\mathbf{v}t|\mathbf{r}+\boldsymbol{\rho},\mathbf{v}',t+\tau)\,d\mathbf{r}\ldots d\tau. \quad (20)$$

In the derivation of (20) we took into account the fact that the product $[\mathbf{n}\times\delta\mathbf{v}]\cdot[\mathbf{n}\times\delta\mathbf{v}']\approx(\delta\mathbf{v}\cdot\delta\mathbf{v}')$ since the vectors $\mathbf{n}$ and $\delta\mathbf{v}$ ($\delta\mathbf{v}'$) are practically perpendicular in the region of high electron energies of interest to us.

According to Varfolomeev et al. [5], when account is taken of the electron energy losses over the coherent length the distribution functions w satisfy an equation of the Fokker-Planck type with an rms multiple-scattering angle and a velocity that depend on the time. The use of this equation leads, for example, for the function

$$u_2=\int e^{i\omega\tau}e^{i\mathbf{k}\cdot\boldsymbol{\rho}}w_2(\mathbf{r}\mathbf{v}t|\mathbf{r}+\boldsymbol{\rho},\mathbf{v}',t+\tau)\,d^3\rho,$$

which enters in (20), to an equation of the type

$$\frac{\partial u_2}{\partial\tau}=q(\tau)\Delta_\theta u_2-i\mathbf{k}_\perp\cdot\boldsymbol{\theta}v(\tau)u_2+ik_z\cdot v(\tau)\frac{\theta^2}{2}u_2+i(\omega-k_z\cdot v(\tau))u_2, \quad (21)$$

$u_2(\mathbf{r}=0)=\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')$, where, as usual, we have introduced the transverse angle vectors $\boldsymbol{\theta}$ with the aid of the equality $\delta\mathbf{v}=\mathbf{v}\boldsymbol{\theta}$, where $v(\tau)=|\mathbf{v}|$. Equation (21) is analogous to the equation for the function u_2 obtained by Pafomov [6], except that in (21) the coefficients depend on the time. In the case $\epsilon = 1$, Eq. (21) is analogous to Eq. (6) from [5].

The solution (21) can be easily written if attention is paid to the fact that it coincides in form with the Schrödinger equation for an oscillator with a time-dependent frequency and mass, and acted upon by a time-dependent force. The prescription for solving this equation is given, for example, in [7]. The solution obtained in this manner is quite cumbersome and will not be presented here. For our purposes, it suffices to find the asymptotic form of the solution as $E\to\infty$.

Introducing to this end the function

$$\psi(\tau)=\exp\{-i(\omega\tau-k_z\cdot\textstyle\int v(\tau)\,d\tau)\}u_2(\tau),$$

we obtain from (21) an equation in the form

$$\frac{\partial\psi}{\partial\tau}-q(\tau)\Delta_\theta\psi=f(\theta,\tau)\psi,$$
$$\psi(\tau=0)=\delta(\theta-\theta'),\qquad f(\theta,\ \tau)=-i\mathbf{k}_\perp\cdot\theta v(\tau)+ik_z\cdot v(\tau)\theta^2/2. \quad (22)$$

We separate in the coefficient $q(\tau)$ the parameter λ, which tends to zero as $E\to\infty$. For example, we write $q(\tau)=\lambda\alpha(\tau)$ (according to [5], $q(\tau)=q_0\exp(2(t+\tau)/L)$, where q_0 is the rms scattering angle and is proportional to $1/E^2$), and choose $\alpha(\tau)\to 1$ as $\tau\to 0$.

We make the change of variable

$$\tau_1=\int_0^\tau\alpha(\tau')\,d\tau'.$$

This change transforms (22) into

$$\frac{\partial\psi}{\partial\tau_1}-\lambda\Delta_\theta\psi=\tilde f(\theta,\tau)\psi,$$
$$\psi(\tau_1=0)=\delta(\theta-\theta'),\qquad \tilde f(\theta,\ \tau_1)=f(\theta,\ \tau)/\alpha(\tau). \quad (23)$$

We introduce the Green's function with the aid of the equation

$$\frac{\partial G}{\partial\tau_1}-\lambda\Delta_\theta G=\delta(\tau_1-\tau_1')\delta(\theta-\theta'), \quad (24)$$

$$G(\theta-\theta',\tau_1-\tau_1')=\frac{1}{4\pi\lambda(\tau_1-\tau_1')}\exp\left[-\frac{(\theta-\theta')^2}{4\lambda(\tau_1-\tau_1')}\right]\theta(\tau_1-\tau_1'), \quad (25)$$

$$\theta(x)=1\ \text{if}\ x>0,\ \theta(x)=0\ \text{if}\ x<0.$$

With the aid of the indicated Green's function, Eq. (23) can be transformed into an integral equation:

$$\psi(\theta,\tau_1)=G(\theta-\theta',\tau_1)+\int_{-\infty}^{+\infty}\int_0^{\tau_1}G(\theta-\theta'';\tau_1-\tau_1'')$$
$$\times\tilde f(\theta'',\tau_1'')\psi(\theta'',\tau_1'')\,d\theta''\,d\tau_1''. \quad (26)$$

We now use the fact that in any finite time interval, as $\lambda\to 0$, the Green's function $G\to\delta(\boldsymbol{\theta}-\boldsymbol{\theta})\theta(\tau-\tau')$. This circumstance enables us to write (26) in the form

$$\psi(\theta,\tau_1)=\delta(\theta-\theta')\theta(\tau_1)+\int_0^{\tau_1} f(\theta,\tau_1'')\psi(\theta,\tau_1'')d\tau_1'', \qquad (27)$$

which is equivalent to the following differential equation (expressed in terms of the earlier variables τ):

$$\partial\psi/\partial\tau=f(\theta,\tau)\psi,\quad \psi(\tau=0)=\delta(\theta-\theta). \qquad (28)$$

The solution of (28) can be easily obtained:

$$\psi=\delta(\theta-\theta')\exp\left\{\int_0^{\tau} f(\theta,\tau')d\tau'\right\}. \qquad (29)$$

Returning to the function u_2, we can represent the formula (20) in the form

$$W^{b}_{n\omega}=\frac{e^2\omega^2}{2\pi^2}\operatorname{Re}\int_0^{t_1} dt\,\theta^2 w_1(\tau,\theta,t)\exp(2\operatorname{Im}k_z z)$$
$$\times\int_0^{t_1-t}\exp\left\{i\left(\omega\tau-k_z\cdot\int v(\tau)d\tau\right)+\int_0^{\tau} f(\theta,\tau')d\tau'\right\}dr\,d\theta\,d\tau. \qquad (30)$$

Since the distribution function w_1 becomes an increasingly sharper function of θ near $\theta = 0$ with increasing electron energy, and the velocity $v(\tau) \to 1$ (it follows from (30) that the difference between v and unity can be neglected if $1 - v \ll \epsilon''$, and the phase f in the argument of the exponential can be neglected if $qL_c \ll \epsilon''$, i.e., $(qL_c)^{1/2} \ll (\omega L_c)^{-1/2}$), it follows that in the limit of high energies the bremsstrahlung intensity is given by

$$W^{b}_{n\omega}=\frac{e^2\omega^2}{2\pi^2}\operatorname{Re}\int_0^{t_1} dt\langle\theta^2\rangle_t\exp(-2\operatorname{Im}k_z t)\int_0^{t_1-t}\exp[i(\omega-k_z\cdot)\tau]d\tau. \qquad (31)$$

According to (31), as a result of an equation of the type (21) the possible dependence of $W^{b}_{n\omega}$ on the energy loss, in the limit of high electron energies, is connected only with the dependence of the mean-squared angle $\langle\theta^2\rangle_t = \int\theta^2 w_1(\mathbf{r}, \theta, t)d\mathbf{r}d\theta$ on the energy loss.

If the energy loss can be neglected over the length L_c, then

$$\langle\theta^2\rangle_t=\int_0^{t}\frac{E_s^2}{E^2(\tau)}\frac{d\tau}{L}=4q_0t.$$

Substituting this expression in (31), we obtain the result (1), (2), and (18), i.e., the spectrum predicted in[3].

Varfolomeev et al. chose the dependence of the energy on the path in the form $E = E_0e^{-t/L}$. This dependence, as follows from Eq. (6) of [5], leads to an exponential growth of $\langle\theta^2\rangle_t$ which is faster than the decrease of the exponential $\exp(-2\operatorname{Im}k_zt) = \exp(-t/L_c)$. According to (31), this leads to an exponential growth of $W^{b}_{n\omega}$ with increasing t_1, i.e., with increasing electron mean free path in the medium. It should be noted, however, that the relation $E = E_0e^{-t/L}$ is valid only if one neglects the influence of the medium on the energy loss. Thus, for example, the Landau-Pomeranchuk effect leads to a slower decrease of the electron energy with distance[1]. It is also well known[10] that, owing to the polarization, the ionization loss by unit length, for example in the limit of highest energies, become constant and independent of the energy of the incident particle. Since the theory of these losses actually makes no use of the concrete form of ϵ, the conclusion that these losses are constants is valid in the limit as $E \to \infty$ also in our case (although now the losses include also the pair-production processes). On the other hand, only the ionization losses remain, in the limit as $q \to 0$ ($E \to \infty$), both in the theory that does not take account of the change of the multiple-scattering constants[4], and in the theory that takes this change into account[5], and in this limit the two theories coincide.

Thus, the energy lost by the particle on its path in the medium is described, in the high-energy limit, by an essentially weaker dependence on t than exponential. This means that as $E \to \infty$ we can neglect the change of the electron energy in the medium over the quantum absorption length L_c (for example, at constant losses $E = E_0 - \text{const}\cdot t$ and for any finite t we have $E \approx E_0$ at sufficiently large E_0). Therefore $\langle\theta^2\rangle_t \sim t$ in this limit, and consequently we find that in the high-energy limit the spectral and angular distributions are described by formulas (1), (2), and (18). At lower energies or at emitted-quanta frequencies such that $1 - v(\tau) \gtrsim \epsilon''$ and $(qL_c)^{1/2} \gtrsim (\omega L_c)^{-1/2}$ (see the derivation of formula (31)), allowance for the change of the energy over the length L_c may turn out to be important. It follows from the indicated inequalities that a similar allowance may be important at electron energies $E < E_s(\omega L_c)^{1/2}$ or, if the particle energy is fixed, at quantum frequencies $\omega > E^2/L_cE_s^2$. For example, according to the first inequality, in the investigation of the spectrum of the γ quanta with energy $\sim 10^9$ eV, the electron energy should be less than 10^{14} eV. However, an analysis of the role of the losses should be carried out with the aid of the Fokker-Planck equation with coefficients that take into account the influence of the medium on the electron deceleration law. In addition, it is necessary to add to the indicated equation a term of the type $-w/\tau$, which describes the relaxation of the electron distribution function due to radiation and pair-production processes. The matter reduces in fact to the use of a kinetic equation that takes into account the polarization of the medium both in the collision term, which is connected with the electron scattering, and in the collision term connected with radiation and pair-production processes.

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*$[n\pi_\omega] \equiv n \times \pi_\omega$.

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