Stability of stationary waves in nonlinear weakly dispersive media

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It is demonstrated within the framework of the Korteweg–de Vries (KdV) equation that a periodic stationary wave is stable with respect to finite perturbations. Particular solutions of the KdV equation are found which are analogs of the well-known N-soliton solution, describe the interaction of solitons and a periodic wave, and constitute shear "dislocations" of the latter. The problem of interactions between the "dislocations" is studied.

As is well known,[1] the propagation of one-dimensional waves of finite amplitude in media with weak dispersion is described by the Korteweg-deVries (KdV) equation:

$$u_t+u_{xxx}+6uu_x=0. \quad (1)$$

The KdV equation was the first of a series of nonlinear equations studied by the inverse scattering problem method.[2] The scheme of application of this method (in the form described by Lax[3]) consists of the following. A pair of linear operators L and A are associated with the considered nonlinear equation, with the help of which this equation is represented in the form

$$\partial L/\partial t=i[L, A]. \quad (2)$$

Here the solution of the Cauchy problem reduces to the investigation of the direct and inverse spectral problems for the operator L. The use of such an approach allows us effectively to study the behavior of the solutions of Eq. (1) as $u \to 0$ as $|x| \to \infty$. In particular, a trivial consequence of the application of the inverse problem method is the proof of the asymptotic stability of the stationary solution of the KdV equation—the soliton. It has been shown[4] that as $t \to \infty$, a solution that is initially close to a soliton also represents a soliton that is continuously dependent on the initial perturbation.

However, extension of the Lax scheme to the case $u \to u_0(x, t)$ as $x \to \infty$ presents significant difficulties. In this case, the approach suggested by Shabat[5] is far more convenient. We shall refer to this below as the Shabat scheme.

In the present paper, use of this scheme allows us to study the behavior of solutions of the KdV equation in the case in which $u \to u_0$ as $x \to \infty$, where $u_0(x)$ is the stationary solution of (1), the so called cnoidal wave:[1]

$$u(x-vt) = -2\wp(x+i\omega'-vt)+v/6, \quad (3)$$

where $\gamma(x)$ is a Weierstrass elliptic function (see, for example,[6]), with periods 2ω, $2i\omega'$ (ω, ω' real). In particular, we shall prove the stability of the solution (3) and find particular solutions of the KdV equation that are direct analogs of the well-known N-soliton solution,[4] which in our case describes the interaction of solitons with the periodic wave (3) and which represent shear "dislocations" of the latter. We shall also prove that asymptotically (as $t \to \infty$), the general solution of (1) with periodic conditions on x at infinity is the set of noninteracting dislocations of the wave (3).

1. STATEMENT OF THE PROBLEM AND THE FUNDAMENTAL EQUATIONS

Let us consider the integral equation (the Marchenko equation):

$$K(x,y)+F(x,y)+\int_x^\infty K(x,s)F(s,y)\,ds=0. \quad (4)$$

We shall assume that the kernels K and F depend on the time t as a parameter. Following Shabat,[5] we define the differentiation operator D relative to the convolution of the two kernels $G*H = \int_{-\infty}^{\infty} G(x, s)H(s, y)ds$:

$$D(G*H)=DG*H+G*DH.$$

It is easy to see that the differentiation operators have the form

$$D_0=\frac{\partial}{\partial t}, \quad D_n=\frac{\partial^n}{\partial x^n}+(-1)^{n+1}\frac{\partial^n}{\partial y^n}, \quad D_f=f(x,t)-f(y,t).$$

These operators form an algebra, i.e., the commutator of any two operators and also their linear superposition are differentiation operators. Then the following theorem is valid: if K and F are the solutions of Eq. (4) and DF = 0, there exists an operator $\tilde{D}$ such that $\tilde{D}K = 0$. Here the principal differential parts of D and $\tilde{D}$ coincide and $\tilde{D}$ contains as coefficients in front of the differentiation operators the kernel K and its derivatives on the characteristic x = y.

We now consider the two operators

$$P=\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+u_0(x,t)-u_0(y,t),$$

$$Q=\frac{\partial}{\partial t}+4\frac{\partial^3}{\partial x^3}+4\frac{\partial^3}{\partial y^3}+6u_0\frac{\partial}{\partial x}+6u_0\frac{\partial}{\partial y}+3u_{0x}+3u_{0y}.$$

As is not difficult to see, they are differentiation operators. We require that the kernel F satisfy the following two equations simultaneously;

$$PF=0, \quad (5)$$

$$QF=0. \quad (6)$$

From the compatibility condition of these equations, it follows that [P, Q] = 0 (cf. with[7]). The latter is equivalent to u_0 satisfying the KdV equation. Furthermore, letting the operators P and Q act on Eq. (4), we find that K(x, y) satisfies the set of equations

$$\tilde{P}K=\left(P+2\frac{d}{dx}K(x,x)\right)K=0, \quad (7)$$

$$\tilde{Q}K=\left(Q+12\frac{d}{dx}\frac{\partial}{\partial x}K(x,x)+12\frac{d}{dx}K(x,x)\frac{\partial}{\partial x}+6\frac{d}{dx}K^2(x,x)\right)K=0. \quad (8)$$

From the compatability of these equations, it follows that $u = 2dK(x, x)/dx$ satisfies the equation

$$u_t+u_{xxx}+6uu_x+3(u_0u)_x=0. \quad (9)$$

This equation describes the propagation of the perturbation u against the background of the wave u_0. Thus, the Shabat method allows us to solve the Cauchy problem for Eq. (9) according to the following scheme:

$$K(x,x,0)\xrightarrow{PK=0}K(x,y,0)\xrightarrow{K+F+K*F=0}F(x,y,0)\rightarrow$$
$$\xrightarrow{QF=0}F(x,y,t)\xrightarrow{K+F+K*F=0}K(x,y,t)\rightarrow K(x,x,t).$$

We note that the nontrivial items of the given scheme are the second and the fourth—analogs of the direct and inverse Cauchy problem for the operator L in the Lax scheme (2).

The given method allows us to investigate effectively the stability of any solutions of the KdV equation to finite perturbations. The only limitation on the value of the perturbations follows from the Marchenko equation; to be precise, it is necessary that u(x) fall off at one of the infinities (in the given case, at $+\infty$). Specifically, we shall study the problem of the stability of a periodic stationary wave.

2. THE DIRECT SCATTERING PROBLEM

By the direct scattering problem we shall mean the problem of determining F(x, y, t). For this purpose, we consider Eqs. (5), (6). These equations are of the type for which the Fourier method is applicable. In this connection, we first find the particular solution of Eqs. (5), (6) of the form

$$F(x,y,t)=C(t)\psi(x)\psi(y).$$

It is easy to see then that ψ satisfies a Schrödinger equation with a potential u_0 having a period 2ω:

$$\frac{d^2\psi}{dx^2}-2\wp(x+i\omega')\psi=-E\psi.$$

Without limitation of generality, we set the velocity $v = 0$. (Actually, this corresponds to a transition to a set of coordinates moving at a velocity v.)

This equation—the Lamé equation—has been thoroughly studied in the literature (see, for example[6]). Its eigenfunctions are expressed in terms of the Weierstrass functions $\sigma(x)$ and $\zeta(x)$:

$$\psi_a(x)=\frac{\sigma(x+i\omega'+a)}{\sigma(x+i\omega')\sigma(a)}\exp[\zeta(a)x+\zeta(i\omega')a], \tag{10}$$

and the "energy" E is connected with the parameter a by the relation

$$E=-\wp(a).$$

Here $\psi_a(x)$ and $\psi_{-a}(x)$ are linearly independent. For real E—and it is just these values that are of interest to us—the parameter a is a complex number and varies along the boundary of a rectangle as shown in Fig. 1. Here the segments $(\omega, \omega + i\omega')$ and $(i\omega'', 0)$ correspond to the continuous spectrum. To each segment there corresponds an allowed band. The first band corresponds to values of the energy E between $\wp(\omega)$ and $-\wp(\omega + i\omega'')$, and the second to E from $-\wp(i\omega')$ to ∞.

It is curious to note that for the given potential there exist only two allowed bands. This phenomenon can be explained rather simply. The potential $u_0(x)$ refers to a periodic reflectionless potential for which, in particular, the representation

$$u_0(x)=2\left(\frac{\zeta(i\omega')}{i\omega'}+\sum_{n=-\infty}^{\infty}\frac{\varkappa^2}{\operatorname{ch}^2\varkappa(x+2n\omega)}\right),$$

is valid, with $\kappa = \pi/2\omega'$, i.e., the potential $u_0(x)$ is an infinite sum of reflectionless soliton potentials.[8] Here the wave functions (10), which correspond to the continuous spectrum, are Bloch wave functions with the quasimomentum

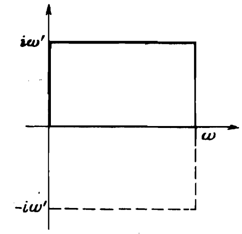

FIG. 1

$$p(a)=i\omega^{-1}(\omega\zeta(a)-\zeta(\omega)a).$$

We can establish the fact directly that the value of the quasimomentum is a continuous quantity in the transition from one allowed band to another. Here p = 0 at the point $a = \omega$; at the points $a = \pm i\omega'$ and $a = \omega \pm i\omega'$ the values of p are identical.

In addition to the continuous spectrum, we shall be interested in solutions $\psi_a(x)$ that correspond to values from the forbidden bands. It can be established that for these bands $\operatorname{Im} p > 0$ correspond to two segments: $(0, \omega)$ and $(\omega + i\omega', i\omega')$ in the complex plane of a.

We now determine the C(t) dependence. Solving Eqs. (6) and using the formulas for the addition of elliptic functions (see[6]), we find that

$$C_a(t)=C_a(0)\exp[-4\wp'(a)t]. \tag{11}$$

Now, knowing the particular solution of Eqs. (5), (6), we construct the general solution:

$$F(x,y,t)=\int_c\rho(a,t)\psi_a(x)\psi_a(y)\,da+\sum_n M_n^2(t)\psi_n(x)\psi_n^*(y),$$

where the integral is taken over the entire continuous spectrum $(-\infty < p < \infty)$, and the discrete summation over the values of a_n for which $\operatorname{Im} p(a_n) > 0$.

The given solution is a general one in the class of functions which vanish at $+\infty$. The solutions $\rho(a, t)$ and $M_n^2(t)$ vary with time according to (11). The values of $\rho(a, t)$ and $M_n^2(t)$, as will be shown below, are connected with the data on scattering from the total potential $V = u_0 + u$.

3. THE INVERSE SCATTERING PROBLEM

By the inverse scattering problem, we mean the problem of the determination of K(x, y, t) from a given F(x, y, t). Here, by virtue of the Marchenko equation and Eq. (7), there arises a connection between $\rho(a, t)$ and $M_n^2(t)$ and the data on scattering from the potential V.

Before moving to this problem, we make two observations. First, it can be shown (see, for example,[9]) that the wave functions $\psi_a(x)$, which correspond to a continuous spectrum, form a complete set of functions. Moreover, we can establish the fact that these functions are orthogonal:

$$\int_{-\infty}^{\infty}\psi_a(x)\psi_b(x)\,dx=2\pi\left|\frac{\zeta(\omega)}{\omega}+\wp(a)\right|\delta(p(a)+p(b)). \tag{12}$$

Second, the functions ψ_a and ψ_n, which correspond to the continuous and discrete spectra, are linearly independent. For this reason, the general solution for K(x, y) can be put in the form

$$K(x,y)=-\int_C \rho(a,t)\varphi_a(x)\psi_a(y)\,da-\sum_n M_n^2(t)\varphi_n(x)\psi_n^*(y).$$

Substituting this solution in the Marchenko equation, we find the triangular representation for the functions φ_a, φ_n:

$$\varphi_a(x)=\psi_a(x)+\int_x^\infty K(x,y)\psi_a(y)\,dy,$$
$$\varphi_n(x)=\psi_n(x)+\int_x^\infty K(x,y)\psi_n(y)\,dy, \tag{13}$$

where the functions φ_a and φ_n satisfy the Schrödinger equation

$$\left(\frac{d^2}{dx^2}+V(x)\right)\varphi_a=-\wp(a)\varphi_a, \tag{14}$$

i.e., φ_a is a wave function of the continuous spectrum and φ_n corresponds to the bound state, inasmuch as φ_n decays as $x\to\infty$, and, as will be shown below, falls off also if $x\to-\infty$.

As is well known,[2,4] the solutions of the KdV equation for decaying potentials, which correspond to the continuous and discrete spectra, can be regarded separately asymptotically as $t\to\infty$. A similar proposition is also valid for the given problem. By virtue of this, we set all $M_n^2 = 0$ and consider the asymptote $\varphi_a(x)$ as $x\to-\infty$.

Since, $\varphi_a(x)$ tends to $\psi_a(x)$ as $x\to\infty$, as follows from the triangular representation, it can be decomposed at the other infinity into the functions $\psi_a(x)$ and $\psi_a^*(x)$:

$$\varphi_a(x)=\alpha(a)\psi_a(x)+\beta(a)\psi_a^*(x). \tag{15}$$

Here the scattering data $\alpha(a)$ and $\beta(b)$ carry all the information on the potential u(x). From Eq. (8), one can easily determine the behavior of $\alpha(a, t)$ and $\beta(a, t)$. It can be established directly that $\alpha(a, t)$ does not depend on time, while $\beta(a, t)$ varies according to the law

$$\beta(a,t)=\beta(a,0)\exp(-4\wp'(a)t).$$

We now establish the connection of $\rho(a)$ with the scattering data; it follows from (15) that, as $x\to-\infty$,

$$K(x,y)=K_1(x,y)+K_2(x,y);$$
$$K_1(x,y)=-\int_C \rho(a,t)\alpha(a)\psi_a(x)\psi_a(y)\,da,$$
$$K_2(x,y)=-\int_C \rho(a,t)\beta(a,t)\psi_a^*(x)\psi_a(y)\,da.$$

Since the functions $\psi_a(x)$ are rapidly oscillating, it follows that $K_1(x, y)$, $F(x, y)$ differ from zero in the region $x\sim -y$, and $K_2(x, y)$ in the region $x\sim y$. It then follows from the Marchenko equations (4) that, as $x\to-\infty$ (cf. with[2])

$$K_2(x,y)+\int_{-\infty}^\infty K_1(x,s)F(s,y)\,ds=0.$$

Solving this equation by the Fourier method and using the orthogonality of the functions $\psi_a(x)$ (12), we obtain

$$\rho(a,t)=-\frac{1}{2\pi i}\frac{\beta^*(a,t)}{\alpha(a)}$$

Thus the quantity $\rho(a)$ is connected with the scattering data for the continuous spectrum. As will be shown in the following section, the quantities $M_n^2(t)$ are also connected with the scattering data for the bound states. To each discrete level there corresponds a solitary wave—a soliton.

4. N-SOLITON SOLUTIONS

We now consider solutions u(x, t) that correspond to discrete degrees of freedom. We shall first make clear to what solution of the equations a soliton or solitary wave corresponds. For this purpose, we keep only a single term in the discrete sum (13):

$$K(x,y)=-M_n^2(t)\varphi_n(x)\psi_n^*(y);$$

it then follows from the triangular representation that

$$\varphi_n(x)=\psi_n(x)\Big/\left\{1+M_n^2(t)\int_x^\infty|\psi_n(y)|^2\,dy\right\}.$$

Thus

$$V(x,t)=u_0(x)+2\frac{d}{dx}K(x,x,t)=u_0(x)+2\frac{d^2}{dx^2}\ln\left(1+M_n^2(t)\int_x^\infty|\psi_n|^2\,dy\right). \tag{16}$$

We shall call this solution a soliton. We now investigate it.

The integral in (16) is rather simple to calculate if Eq. (14) is used:

$$\int_x^\infty|\psi_n|^2\,dx=|\psi_n|^2\frac{\sigma(x'+2\,\mathrm{Re}\,a)\sigma(x')|\sigma(a)|^2}{\sigma(2\,\mathrm{Re}\,a)\sigma(x'+a)\sigma(x'+a^*)},\qquad x'=x+i\omega'.$$

It is then obvious that $u(x, t)\to -2M_n^2\,d|\psi_n|^2/dx$ as $x\to\infty$, i.e., u decays exponentially as $x\to\infty$. At the other infinity $(x\to-\infty)$

$$\varphi_n(x)\to\frac{\sigma(2\,\mathrm{Re}\,a)\sigma(a^*)}{M_n^2(t)\sigma(a)}\psi_{-a^*}(x+2\,\mathrm{Re}\,a)\exp(-2\zeta^*(a)\,\mathrm{Re}\,a),$$
$$V(x,t)\to u_0(x+2\,\mathrm{Re}\,a) \tag{17}$$

and the contribution to the potential $u_0(x + 2\mathrm{Re}\,a)$ is exponentially small as $x\to-\infty$.

The solution (16) naturally does not have the form of a stationary wave. However, this wave moves in the mean with a constant velocity in the medium. We determine it from the following: we find the time τ in which the soliton passes through one period of the soliton lattice 2ω. Inasmuch as

$$\sigma(x+2\omega)=-\sigma(x)\exp(2\zeta(\omega)(x+\omega)),$$

this time $\tau = -\mathrm{Im}\,p(a)\omega/\wp'(a)$, i.e., the mean velocity is

$$v=2\omega/\tau=-2\wp'(a)/\mathrm{Im}\,p(a).$$

Thus the soliton in the given case represents a nonstationary solitary dislocation which propagates with some mean velocity v along the cnoidal wave. By investigating the sign of v, we can establish that all solitons having a from the lowest forbidden band move to the right and those having a from the upper band to the left. A graph of these solutions, calculated on an electronic computer, is shown in Fig. 2. The completely natural question arises as to the interaction of two dislocations having two different velocities. It is obvious that there exists instant of time t when the interaction of the solitons becomes significant. It is clear beforehand that in a collision of solitons, the parameters a remain as before. A similar proposition is, of course, satisfied for N-soliton interactions. As will be shown below, the collisions are pair collisions; therefore in what follows we can limit ourselves to study of collision of only two solitons.

Thus, as $t\to-\infty$, let there be two solitons with velocities v_a, v_b ($v_b > v_a$). We determine the matrix of scattering from the two soliton potentials. As $x\to\infty$, let the asymptotic form of $\varphi_a(x)\to\psi_a(x)$. We

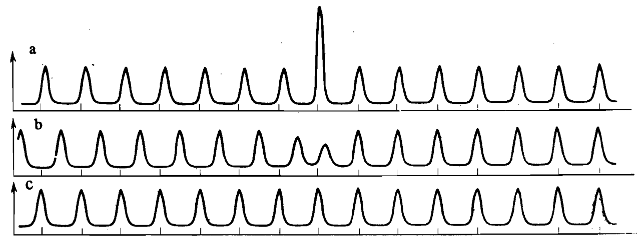

FIG. 2. Plot of solutions V(x) for dislocations which are propagated a) to the right (a = 0.3), b) to the left (a = 0.5 + iω′); c) for the soliton lattice (ω = 1, ω′ = 1/π).

determine $\varphi_a(x)$ as $x \to -\infty$. After the first soliton (a), we have (17):

$$\varphi_a(x)=\frac{\sigma(2\operatorname{Re}a)\sigma(a^*)}{M_n^2(t)\sigma(a)}\psi_{-a^*}(x+2\operatorname{Re}a)\exp(-2\zeta^*(a)\operatorname{Re}a).$$

This expression represents an intermediate asymptotic form, as $t \to -\infty$, between two solitons which have moved a great distance apart. Our problem now is to calculate the scattering matrix due to the second soliton (b).

The kernel $K_b(x, y)$ is expressed only in terms of the functions $\varphi_b(x + 2\operatorname{Re} a)$:

$$K_b(x,y)=-M_b^2\varphi_b(x+2\operatorname{Re}a)\psi_b^*(y+2\operatorname{Re}a)$$
$$=-\frac{M_b^2\psi_b(x+2\operatorname{Re}a)\psi_b^*(y+2\operatorname{Re}a)}{1+M_b^2\int_x |\psi_b(s+2\operatorname{Re}a)|^2 ds}.$$

Here the eigenfunction φ_a is determined from the triangular representation (13)

$$\varphi_a(x)=S(a)\Big(\psi_{-a^*}(x+2\operatorname{Re}a)$$
$$+\int_x^\infty K_b(x+2\operatorname{Re}a,y+2\operatorname{Re}a)\psi_{-a^*}(y+2\operatorname{Re}a)\,dy\Big).$$

As is shown in the Appendix, as $x \to -\infty$,

$$\varphi_a(x)=S(a,\ b,\ -\infty)\psi_{-a^*}(x+2\operatorname{Re}a+2b^*),$$
$$S(a,b,-\infty)=-S(a)\frac{\sigma^*(a+b)}{\sigma^*(a-b)}\exp[-\zeta(a^*)b^*].$$

In connection with the fact that $\varphi_a(x)$ is proportional to ψ_{-a}^* as $x \to -\infty$, it follows that the scattering matrix is

$$S(a,\ b,\ t)=S(a,\ b,\ -\infty)\exp(4\wp'(a)t). \tag{18}$$

On the other hand, as $t \to \infty$, the solitons a and b change places. Carrying out similar calculations, we establish the fact that the scattering matrix in this case has the form

$$S(a,b,\infty)=-\frac{\sigma(2\operatorname{Re}a)\sigma(a^*)\sigma(a-b)}{M_a^2(\infty)\sigma(a)\sigma(a+b)}\exp[-2\zeta^*(a)\operatorname{Re}a+2\zeta(a)b].$$

The time behavior of the scattering matrix can be found from this expression by continuing it in time to small t. Setting the result equal to (18) at t = 0, we find that as a result of collision of the slow soliton (a) with the fast soliton (b), M_a^2 has been changed:

$$\left|\frac{M_a^+}{M_a^-}\right|^2=\left|\frac{\sigma(a-b)}{\sigma(a+b)}\right|^2\exp[4\operatorname{Re}(\zeta(a)b)].$$

Repeating these discussions for soliton (b), we obtain

$$\left|\frac{M_b^+}{M_b^-}\right|^2=\left|\frac{\sigma(a+b)}{\sigma(a-b)}\right|^2\exp[-4\operatorname{Re}(\zeta(b)a)].$$

In particular, the conclusion then follows that the collisions are pair collisions. Only the phase of the soliton

$$\Delta\varphi=\ln|M_a^+/M_a^-|^2$$

is changed as a result of the collisions. Here, in contrast with ordinary solitons, an additional phase shift appears in $\Delta\varphi$ that is equal to $-4\operatorname{Re}(\zeta(b)a)$ for the fast soliton and to $4\operatorname{Re}(\zeta(a)b)$ for the slow one. These are connected with the fact that the soliton is a dislocation of the soliton lattice. In the case in which the perturbation u is a localized one, there is a limitation on the number of solitons:

$$\sum a_n=p\omega+iq\omega',\quad (p,q-\text{are integers}), \tag{19}$$

which is connected with the fact that as $x \to -\infty$ the lattice is unperturbed. From this it follows, in particular, that the minimum number of possible dislocations is equal to 2. Expression (19) can be regarded in a sense as a "parametric" perturbation of the dislocations. In concluding this section, we write out the explicit expression for the N-soliton solution

$$V=u_0(x)+2\frac{d^2}{dx^2}\ln\Delta,$$

$$\Delta=\det\left\|\delta_{nm}+M_n^2(t)\int_x^\infty\psi_n(s)\psi_m^*(s)\,ds\right\|.$$

One can also establish directly from investigation of this solution that the interaction of solitons is of paired nature, i.e., the change of phase of each soliton represents the sum of phases $\Delta\varphi$ in the scattering by each individual soliton.

5. ASYMPTOTIC STATES

In this section, we shall prove that the asymptotic state of any initial condition is a set of solitons. The latter is equivalent to the fact that the soliton lattice turns out to be stable. As follows from the results of the previous section, the lattice can only change its phase by an amount equal to twice the sum of phases of all the dislocations which leave on the right.

Thus, we must prove that the effect of the nonsoliton part can be neglected asymptotically as $t \to \infty$. For this purpose, as was shown in[4], it suffices to establish that as $t \to \infty$ the kernel F(x, y, t), which corresponds to the continuous spectrum, tends to zero. First we shall prove the condition of unitarity for the scattering matrices α, β. For this purpose, we introduce two functions (Jost functions): φ_a, Φ_a, which are solutions of Eq. (14):

$$\varphi_a(x)\to\psi_a(x)\ \text{as}\ x\to\infty,\quad \Phi_a(x)\to\psi_a(x)\ \text{as}\ x\to-\infty.$$

Here, as it is not difficult to see, Φ_a and Φ_a^* are linearly independent, whence it follows from comparison with Sec. 3 that

$$\varphi_a(x)=\alpha(a)\Phi_a(x)+\beta(a)\Phi_a^*(x).$$

Calculating the Wronskian $\{\varphi_a, \varphi_a^*\}$, which does not depend on x, we obtain the unitarity condition

$$|\alpha(a)|^2-|\beta(a)|^2=1.$$

It then follows that $|\rho(a, 0)| \le 1/2\pi$. We now consider the expression

$$F(x,y,t)=\frac{1}{2\pi i}\int_c\frac{\beta^*(a,t)}{\alpha(a)}\psi_a(x)\psi_a(y)\,da.$$

We note that $\beta^*(a, t) \sim \exp(-4y'(a)t)$, i.e., as $t \to \infty$ the given integral is the integral of a rapidly oscillating function. It then also follows that $F(x, y, t) \to 0$ as $t \to \infty$. It is obvious that this is also valid as $t \to -\infty$.

Thus the effect of the nonsoliton part can be neglected as $t \to \infty$ and the asymptotic state of any initial condition is a set of solitons. This also proves the fact of stability of the soliton lattice.

In conclusion, the authors thank V. E. Zakharov for his interest in the work and S. M. Manakov for useful discussions.

APPENDIX

We consider the expression

$$\varphi_a(x)=\psi_a(x)+\int_x^\infty K_b(x,y)\psi_a(y)\,dy,$$
$$K_b(x,y)=-M_b^2\psi_b(x)\psi_b^*(y)\Big/\left\{1+M_b^2\int_x^\infty |\psi_b(y)|^2\,dy\right\}, \qquad \text{(A.1)}$$

where the quantities a and b correspond to values of the energy from the forbidden bands with $\operatorname{Im} p(a) > 0$ and $\operatorname{Im} p(b) > 0$. We find the asymptotic form of $\varphi_a(x)$ as $x \to \infty$. We first note that the functions $\varphi_a(x)$ and $\psi_b(x)$ increase exponentially as $x \to -\infty$. Further, using Eq. (14) and the formula for the addition of ζ functions,[6] we find that the integrals in (A.1) are expressed in terms of the $\wp$ function:

$$\int_x^\infty \psi_a(y)\psi_b(y)\,dy$$
$$=\frac{1}{2}\psi_a\psi_b\frac{1}{\wp(a)-\wp(b)}\left[\frac{\wp'(x')-\wp'(b)}{\wp(x')-\wp(b)}-\frac{\wp'(x')-\wp'(a)}{\wp(x')-\wp(a)}\right]$$

It is convenient in what follows to express this answer in terms of σ functions. Using the representation of the $\wp$ functions in terms of σ functions (see[6], p. 323):

$$\wp(u)-\wp(v)=-\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)}$$
$$\frac{1}{2}\begin{vmatrix}1 & \wp(x) & \wp'(x)\\ 1 & \wp(a) & \wp'(a)\\ 1 & \wp(b) & \wp'(b)\end{vmatrix}=-\frac{\sigma(x+a+b)\sigma(x-a)\sigma(x-b)\sigma(a-b)}{\sigma^3(x)\sigma^3(a)\sigma^3(b)}$$

we get

$$\int_x^\infty \psi_a(y)\psi_b(y)\,dy=\psi_a(x)\psi_b(x)\frac{\sigma(x'+a+b)\sigma(a)\sigma(b)\sigma(x')}{\sigma(a+b)\sigma(x'+a)\sigma(x'+b)}$$

It then follows immediately that, as $x \to -\infty$,

$$\varphi(x)=\psi_a(x)\Phi(x',a,b),$$
$$\Phi(x,a,b)=1-\frac{\sigma(x+a+b)\sigma(a)\sigma(2b)\sigma(x+b)}{\sigma(a+b)\sigma(b)\sigma(x+a)\sigma(x+2b)} \qquad \text{(A.2)}$$

We now consider the function $\Phi(x, a, b)$. This function is elliptic in the arguments x, a with poles at the points $a = -b$, $x = -a$, $x = -2b$ and points comparable with them. Since that the given function is elliptic, $\Phi(x, a, b)$ has only three zeros in the fundamental square. The zeros of the function $\Phi(x, a, b)$ are rather easily calculated: $a = b$, $x = 0$, $x = -a - 2b$. It then follows that

$$\Phi(x,a,b)=C\frac{\sigma(a-b)\sigma(x)\sigma(x+a+2b)}{\sigma(x+a)\sigma(x+2b)\sigma(a+b)}$$

Calculating the residue to $\Phi(x, a, b)$, for example, at the point $x = -a$, we find that $C = -1$. Substituting this expression in (A.2), we obtain

$$\varphi_a(x)=\psi_a(x+2b)S(a,b), \qquad S(a,b)=-\frac{\sigma(a-b)}{\sigma(a+b)}\exp[2\zeta(a)b].$$

1. V. I. Karpman, Nelineĭnye volny v dispergiruyushchikh sredakh (Nonlinear Waves in Dispersive Media), Nauka, 1973.
2. C. Gardner, G. Green, M. Kruskal, and R. Miura, Phys. Rev. Lett. 19, 1095 (1967).
3. P. D. Lax, Comm. on Pure and Appl. Math. 21, 467 (1968).
4. V. E. Zakharov, Zh. Eksp. Teor. Fiz. 60, 993 (1971) [Sov. Phys.-JETP 33, 538 (1971)].
5. A. B. Shabat, Dokl. Akad. Nauk SSSR 211, 1311 (1973).
6. E. T. Whittaker and G. N. Watson, Modern Analysis, Part 2 Cambridge University Press, 4th ed., 1962.
7. V. E. Zakharov, and A. B. Shabat, Funktsional'nyĭ analiz (Functional Analysis) 8, No. 3 (1974) (in press).
8. J. Kay and M. E. Moses, Nuovo Cim. 3, 277 (1956).
9. B. M. Levitan and I. S. Sargsyan, Vvedenie v spektral'nuyu teoriyu (Introduction to Spectral Theory), Nauka, 1970.

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