Nonlinear effects in thin conductors

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Nonlinear effects in the electric conductivity of thin plates and wires of thickness d much smaller than the conductivity mean free path l are investigated. It is shown that in anisotropic conductors there is a nonlinear effect not due to heating, which can be ascribed to bending of the gliding-electron trajectory by the electric field. This effect should be particularly pronounced in semimetals and result in a logarithmic or square-root dependence of the electric conductivity of the plate on the electric field strength. It is shown that the variation of the current-voltage characteristic of the conductor following reversal of electric current direction can significantly depend on the state of the surface of the anisotropic plate. The influence of an external magnetic field **H** on the nonlinear effects in the electric conductivity of a thin conductor is investigated. The role of the current's magnetic field $\mathbf{H}_j$, which can lead to a static skin effect in metallic plates and conductors even at $\mathbf{H}=0$, is analyzed. The criterion for the observation of the nonlinear effects, the shape of the current-voltage characteristic, and the magnitude of the deviation from Ohm's law are all extremely sensitive to the state of the sample surface. This permits one to use the nonlinear effects for a detailed study of the interaction between the carriers and the conductor surface.

Heating of conduction electrons by electric current is one of the causes of violation of Ohm's law [1]. In semiconductors, even at relatively low current densities j, the electric field can impart to the carriers, over the mean free path l, an energy comparable with their average energy [2]. In metals, owing to the high carrier density n, the average energy of the conduction electrons is high enough and the heating by the current is a weak perturbation of the electron system, and is capable of changing noticeably the current-voltage characteristic of the conductor only in the case when the electron does not have time to transfer to the lattice the energy acquired in the electric field [3,4]. When an electron collides with the lattice, it loses a small fraction of its energy $\Delta\epsilon = s\epsilon_0/v_0$, but nevertheless to observe nonlinear effects connected with electron heating it is necessary to have sufficiently high current densities f, at which the drift velocity of the electrons $v = j/ne$ is comparable with the sound propagation velocity in the crystal s. Here e is the electron charge and v_0 is its velocity on the Fermi surface $\epsilon(\mathbf{p}) = \epsilon_0$.

The heating of the conduction electrons by current is apparently easiest to observe in semimetals, in which the carrier density is low. Borovik [5] observed a deviation from Ohm's law in bismuth at current densities $j \approx 10^5$ A/cm^2. In metals with about one electron per atom, where $nes \approx 10^{11}-10^{12}$ A/cm^2, much larger currents are necessary to observe nonlinear heating effects. Even under the most favorable conditions, when the resistance of the metallic conductor is determined mainly by the interelectron collisions [6], or when the resistance of the conductor increases significantly when a weak magnetic field is turned on [7], the electron heating can lead to violation of Ohm's law in samples of thickness $d \approx l \approx 0.1-1$ cm only at current densities $j \approx 10^7-10^8$ A/cm^2. Such large current densities are attainable only in thin conductors[1)]; the criterion for the observation of the nonlinear heating effect is quite sensitive to the form of the boundary conditions for the conduction-electron distribution function [8,9]. That is to say, an important role is played by the state of the sample surface, which determines the character of the carrier reflection.

No less important in the investigation of nonlinear effects in electric conductivity of metallic conductors is allowance for the self-magnetic field of the current $H_j \approx 4\pi jd/c$. At current densities

$$j \gtrsim nes\frac{c^2}{s\sigma d} \approx \frac{c^2\hbar}{el_{eff}d}n^{1/3} \tag{1}$$

the curvature radius of the electron trajectory in the field H_j is comparable with the effective mean free path l_{eff}, and the current-voltage characteristic of the conductor differs significantly from a straight line. This nonlinear effect is not connected with the electron heating, since the magnetic field does not change the carrier energy. For good conductors such as copper and gold, at low temperatures, $c^2/\sigma ds$ is quite small and Ohm's law in metallic conductors may not hold even in the Knudsen case ($d \ll l$) at current densities at which the electron heating is still negligibly small[2)].

In the electric conductivity of thin conductors with anisotropic carrier dispersion, nonlinear effects are possible and become more significant with decreasing conductor thickness. The reason is that the velocity **v** and the momentum **p** of such electrons are noncollinear and even an electric field along the current direction influences the motion of the charge in the plane of the sample cross section. If the plate surfaces do not respect the conduction electrons specularly, then its electric conductivity is determined mainly by the carriers that glance over the surface of the plate. The effective mean free path of such electrons changes noticeably under the influence of the electric field, if during the free path time the angle through which their trajectory bends becomes comparable with d/l, i.e., Ohm's law does not hold in conductors with anisotropic Fermi surface an electric field in which $\epsilon El/\epsilon_0 \gtrsim d/l$. This nonlinear effect, which takes place at current densities

$$\frac{j}{nes} \gtrsim \frac{v}{s}\frac{dl_{eff}}{l^2} \tag{2}$$

is also quite sensitive to the character of the carrier reflection from the sample boundary. The foregoing estimate (2) is valid for plates whose surfaces reflect the carriers almost diffusely, and the angle between the velocity and quasimomentum vectors of the "glancing" electrons is of the order of unity. If the surfaces of the plate coincide with the symmetry planes of the crystal, then the last condition, generally speaking, does not hold and the criterion for observing such a nonlinear effect turns out to be more stringent than condition (2). On the other hand, the deviation from Ohm's law in wires also sets in at larger current densities than given in condition (2), since the "glancing" electrons whose velocity **v** makes an angle $\vartheta \leq d/l$ with the conductor surface contribute less to the electric conductivity in wires than in plates. The non-heating nonlinear effect connected with the change of the effective free path of the "glancing" electrons under the influence of the electric field should apparently be most strongly pronounced in thin plates of bismuth and antimony, in which the electron luminosity on the Fermi surface does not greatly exceed the sound propagation velocity in the crystal. Only in very thin metallic films $(d/l < (c^2/\sigma v l_{eff})^{1/2}$ can this nonlinear effect compete with the deviation from Ohm's law as a result of the influence of the self-magnetic field of the current on the dynamics of the conduction electrons, and sets in at current densities such that the heating of the carriers is negligibly small.

There is no doubt at present that it is possible to obtain in experiment current densities needed to observe nonlinear effects in the electric conductivity of thin conductors, and a theoretical investigation of these effects, which are extremely sensitive to the shape and surface state of the sample, seems quite timely to us.

1. COMPLETE SYSTEM OF EQUATIONS OF THE PROBLEM

To determine the connection between the density of the constant electric current

$$\mathbf{j}(\mathbf{r}) = \frac{2e}{(2\pi\hbar)^3} \int \mathbf{v} n(\mathbf{r}, \mathbf{p})\, d\mathbf{p} \tag{3}$$

and the electric field intensity **E** it is necessary to solve the Boltzmann kinetic equation

$$\mathbf{v}\frac{\partial n}{\partial \mathbf{r}} + \left(e\mathbf{E} + \frac{e}{c}[\mathbf{v}\times\mathbf{H}]\right)\frac{\partial n}{\partial \mathbf{p}} = \hat{W}\{n_0 - n(\mathbf{r}, \mathbf{p})\}, \tag{4}$$

where $2\pi\hbar$ is Planck's constant and $\hat{W}$ is the collision integral describing the scattering of the carriers inside the volume.

The reflection of the electrons by the sample boundary will be taken into account with the aid of a boundary condition for the conduction-electron distribution function $n(\mathbf{r}, \mathbf{p})$:

$$n(\mathbf{r}_s, \mathbf{p}') = q(\mathbf{p}) n(\mathbf{r}_s, \mathbf{p}) + \chi(\mathbf{r}_s, \varepsilon(\mathbf{p}')). \tag{5}$$

We assume that the scattering of the electrons by the conductor surface $\mathscr{L}(\mathbf{r}_S) = 0$ can be described by introducing a specularity parameter $q(\mathbf{p})$, which is the probability that an electron with momentum **p** incident on the sample boundary will be specularly reflected. The momenta **p** and **p**′ of the incident and specularly-reflected electrons are connected by the relation

$$\varepsilon(\mathbf{p}) = \varepsilon(\mathbf{p}'), \quad [\mathbf{p}\times\mathbf{n}] = [\mathbf{p}'\times\mathbf{n}], \tag{6}$$

where **n** is the inward normal to the conductor surface.

The function χ can be determined with the aid of the macroscopic boundary condition for the electric current—the condition for the continuity of the current on the surface of the conductor. If the conductor takes the form of either a plane-parallel plate or a wire with current contacts located on the ends, then on the entire conductor surface the normal component of the electric current is equal to zero:

$$j_n' = 0. \tag{7}$$

In the nonlinear theory, the kinetic equation should be supplemented by Maxwell's equation

$$\operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\mathbf{j}, \tag{8}$$

which takes into account the change of the magnetic field **H** under the influence of the electric current. Equations (8) and (4) with boundary conditions (5) and (7), together with the electroneutrality condition for the conductor

$$\rho' = 0, \tag{9}$$

where ρ' is the density of the uncompensated charge of the electrons, constitute the complete system of equations with which we can determine the inhomogeneous electric field produced as a result of the presence of sample boundaries, the self-magnetic field of the current, and in final analysis the resistance of the conductor.

It is convenient to seek the solution of (4) in the form

$$n(\mathbf{r}, \mathbf{p}) = n_0(\varepsilon) - \frac{\partial n_0}{\partial \varepsilon} e\psi(\mathbf{r}, \mathbf{p}),$$

where $n_0(\epsilon)$ is the Fermi distribution function of the carriers, but the temperature of the electrons does not equal the lattice temperature if account is taken of the heating of the electrons by the current. In this case it is necessary to specify the boundary condition for the energy flux density, or else the temperature distribution on the conductor boundary, while the temperature of the electron gas can be determined from the heat balance equation, which takes into account the transfer of energy from the electrons to the lattice and to the medium surrounding the conductor [3,4,9].

The collision integral $\hat{W}$, generally speaking, is a nonlinear integral operator. However, the essential nonlinearity is connected mainly with the interelectron collisions. If the interelectron scattering is not accompanied by a reversal of the quasimomentum of the electrons, then this scattering mechanism exerts no influence on the conductor resistance and leads only to a Fermi distribution of the electron gas. The operator for scattering by impurities is linear at any statistics of the conduction electrons. At low temperatures, this mechanism of the momentum flux dissipation is fundamental, and if we disregard the weak nonlinearity of $\hat{W}$, due to the electron-phonon scattering, then the collision integral can be regarded as a linear operator. Then, given the electric and magnetic fields, the kinetic equation is linear with respect to the carrier distribution function

$$\mathbf{v}\frac{\partial\psi}{\partial\mathbf{r}} + \frac{\partial\psi}{\partial t} + \frac{\psi}{t_0(\mathbf{p})} = \mathbf{E}\mathbf{v}. \tag{10}$$

We note that only for the sake of convenience in further calculations do we regard the collision integral as an operator for the multiplication of the function $n(\mathbf{r}, \mathbf{p}) - n_0(\epsilon)$ by the frequency t_0^{-1} of the electron collisions inside the volume. As the variables in momentum space we choose the electron time of motion t along the trajectory in the electric and magnetic fields, as well as two quantities that are conserved along the characteristics of Eq. (10). The choice of the last two quantities does not play a major role.

The solution of Eq. (10)

$$\psi(\mathbf{r}, \mathbf{p}) = \mathscr{E}(\lambda, t) F(\mathbf{r} - \mathbf{r}(t)) + \int_\lambda^t \mathscr{E}(t', t)\mathbf{v}(t')\mathbf{E}(\mathbf{r} + \mathbf{r}(t') - \mathbf{r}(t))\, dt',$$

$$\mathscr{E}(t, t') = \exp\{(t-t')/t_0\}, \quad (11)$$

contains an arbitrary function F, which must be determined from the boundary condition (5) and (7). Here λ is the instant of reflection of the electrons by the sample boundary at the point $\mathbf{r}_S$, i.e., the closest-to-t root of the equation

$$\mathbf{r}-\mathbf{r}_s = \int_\lambda^t \mathbf{v}(t)\,dt; \quad \lambda \leq t, \quad \mathbf{v}(\lambda)\mathbf{n}(\mathbf{r}_s) > 0. \quad (12)$$

Along the electron trajectory in the electric and magnetic fields, the equation of which is a characteristic of the kinetic equation (10), the function F maintains constant value:

$$F(\mathbf{r}-\mathbf{r}(t)) = F(\mathbf{r}_{is}-\mathbf{r}(\lambda_i)) = F_i, \quad (13)$$

where λ_i is the instant of reflection of the electron by the sample surface at the point $\mathbf{r}_{is}$. Equation (5) enables us to establish a recurrence relation with which to determine the change of the function F when an electron collides with the surface of the sample, i.e., to find the connection of F_i with the value of the function F_{i+1} at an area instant λ_{i+1} of the collision of the electron with the sample boundary

$$F_i = q_i \mathscr{E}(\lambda_{i+1}, \lambda_i) F_{i+1} + \frac{\langle (1-q_i)\mathbf{vn}(1-\theta(\mathbf{vn})) F_{i+1} \mathscr{E}(\lambda_{i+1}, \lambda_i)\rangle}{\langle \mathbf{vn}(1-\theta(\mathbf{vn}))\rangle}$$
$$+ q_i \int_{\lambda_{i+1}}^{\lambda_i} \mathscr{E}(t', \lambda_i)\mathbf{v}(t')\mathbf{E}(\mathbf{r}_i + \mathbf{r}(t') - \mathbf{r}(\lambda_i))\,dt'$$
$$+ \left\langle (1-q_i)\mathbf{vn}(1-\theta(\mathbf{vn})) \int_{\lambda_{i+1}}^{\lambda_i} \mathscr{E}(t', \lambda_i)\mathbf{v}(t')\mathbf{E}(\mathbf{r}_i + \mathbf{r}(t') - \mathbf{r}(\lambda_i))\,dt' \right\rangle \cdot$$
$$\cdot [\langle \mathbf{vn}(1-\theta(\mathbf{vn}))\rangle]^{-1}, \quad \lambda_i = \lambda, \quad (14)$$

where the angle brackets denote integration over the Fermi surface with a weight factor $2e/(2\pi\hbar)^3$, and $\theta(x) = \frac{1}{2}(1 + \operatorname{sign} x)$ is the Heaviside function.

Similarly, just as in [11], we can obtain an explicit expression for the function $F(\mathbf{r} - \mathbf{r}(t))$, by applying repeatedly the recurrence relation (14). It will take the same form as expressions (18)–(20) of [11], but the instants λ_i of reflection of the electrons from the sample boundary will be functions of the electric field. Therefore the summation of the resultant series in the expression for $F(\mathbf{r} - \mathbf{r}(t))$ turns out to be more difficult than in the linear theory. The asymptotic expression for $F(\mathbf{r} - \mathbf{r}(t))$ at $d \ll l$ is extremely cumbersome, and we present below the results only for particular cases, when the reflection of the carriers by the sample boundary is either close to specular, $(1 - q) \ll 1$, or close to diffuse, $q \ll 1$. It is to these cases that we must confine oneself in general, since it is hardly correct to use the parameter $q(\mathbf{p})$ to describe the electron reflection that is close to neither specular nor diffuse.

Calculating the electric current at a specified magnetic field **H**, we determine with the aid of (8)

$$\operatorname{rot}\mathbf{H}(\mathbf{r}) = \frac{4\pi}{c}\left\{\left\langle \mathbf{v}\left(F(\mathbf{r}-\mathbf{r}(t))\mathscr{E}(\lambda, t) + \int_\lambda^t \mathscr{E}(t', t)\mathbf{v}(t')\mathbf{E}(\mathbf{r}')\,dt'\right)\right\rangle\right\} \quad (8a)$$

the self-magnetic field of the current and take into account in the determination of the resistance of the conductor.

2. ELECTRIC CONDUCTIVITY OF THIN PLATES AND WIRES

We consider first the case when there is no external magnetic field, and the thickness of the conductor, a plane-parallel plate or else a cylindrical wire with arbitrary cross section, is so small that in the calculation of their electric conductivity we can disregard the magnetic field due to the current. At H = 0, the momentum of the electron during the time between two collisions with the sample boundary remains practically unchanged:

$$|\Delta \mathbf{p}/p| \leq eEl/\varepsilon_0 \ll 1,$$

and the curvature of the Fermi surface can be regarded as constant on the entire section of the electron path, i.e., its acceleration

$$\dot{v}_i = \frac{\partial^2 \varepsilon}{\partial p_i \partial p_k}\dot{p}_k = \alpha_{ik}\dot{p}_k = e\alpha_{ik}E_k \quad (15)$$

is constant, and the connection between the velocity and the momentum of the electron is linear. This makes it possible to solve our problem under the most general assumptions concerning the carrier dispersion.

If the current contacts lie on the ends of the conductor and the distance between them greatly exceeds the mean free path of the electron, then for single-crystal samples, at least in one direction, (along the wire—the μ axis), the electric field is homogeneous and the electrostatic potential φ should be sought in the form

$$\varphi(\xi, \eta, \mu) = \varphi_1(\xi, \eta) - \mu E_\mu. \quad (16)$$

The inhomogeneous electric field $E_\xi = -\partial\varphi_1/\partial\xi$ and $E_\eta = -\partial\varphi_1/\partial\eta$ in the plane of the conductor cross section can be determined from the electroneutrality condition (9), which is the integral equation

$$-\varphi_1(\xi, \eta)\langle 1\rangle + \left\langle \frac{1}{t_0}\int_\lambda^t \mathscr{E}(t', t)\varphi_1(\xi + \xi(t') - \xi(t), \eta(t') + \eta - \eta(t))\,dt' \right\rangle$$
$$+ \langle v_\mu t_0 (1 - \mathscr{E}(\lambda, t))\rangle E_\mu + \langle \mathscr{E}(\lambda, t)\{F(\xi - \xi(t), \eta - \eta(t)) + \varphi_1(\xi_s, \eta_s)\}\rangle = 0. \quad (17)$$

The coordinates ξ and η are chosen such that the coordinate surface $\xi = \xi_S$ is the surface of the conductor, and the ξ axis coincides with the normal to surface at $\mathbf{r} = \mathbf{r}_S$. In single-crystal plates, the function $\varphi_1(\xi, \eta)$ depends linearly on η:

$$\varphi_1(\xi, \eta) = \Phi(\xi) - \eta E_\eta, \quad (16a)$$

since the electric field is homogeneous in the entire $\eta\mu$ plane.

It is easy to show that the asymptotic expression for the electric current in the wires

$$j_i(\xi, \eta) = \left\langle v_i(t)\left\{\mathscr{E}(\lambda, t)F(\xi - \xi(t), \eta - \eta(t)) + \int_\lambda^t \mathscr{E}(t', t)\mathbf{v}(t')E(\xi + \xi(t') - \xi(t), \eta + \eta(t') - \eta(t))\,dt'\right\}\right\rangle,$$

the maximum diameter of which is much less than l, is determined mainly by the homogeneous electric field and by the value of the potential φ_1 on the surface of the conductor. Consequently, to determine the sample resistance it is not so important to know the distribution of the inhomogeneous electric field over the sample cross section. If we assume that this field is homogeneous, then Eq. (12) for λ becomes elementary, since the velocity of the electrons under these assumptions depends linearly on the time:

$$v_i(t) = \alpha_{ik}(\mathbf{p}^0)p_k(t) = \alpha_{ik}(\mathbf{p}^0)[p_k^0 + eE_k t]$$
$$= v_i(\lambda) + eE_k\alpha_{ik}(t - \lambda).$$

In thin plates, the equation for $t - \lambda$ assumes the form:

$$-e\alpha_{\xi k}E_k\frac{(t-\lambda)^2}{2} + v_\xi(t)(t-\lambda) = \xi - \xi_s, \quad (18)$$

where $E_\xi = [\Phi(0) - \Phi(d)]/d$, and $\xi_S = (0, d)$. Substituting the solution of (18) in the expression for the electric field in the plate:

$$j_i(\xi)=\frac{2e^2}{(2\pi\hbar)^3}\int\frac{dv_\xi\,dv_\eta\,dv_\mu}{\det\{\alpha_{ik}\}}\,\delta(\varepsilon(\mathbf{p})-\varepsilon_0)\cdot$$
$$\cdot\left[\mathscr{E}(\lambda,t)F(\xi-\xi(t))+\int_\lambda^t\mathscr{E}(t',t)\mathbf{v}(t')\mathbf{E}(\xi+\xi(t')-\xi(t))\,dt'\right]v_i, \qquad (19)$$

and eliminating with the aid of (17) the field E_ξ, we obtain the connection between the current density and the electric field intensity in the plane of the plate.

In diffuse scattering of the carriers by the plate surfaces, the electric current in the plate is determined mainly by the "glancing" electrons, i.e., an important role is played in the components of the electric conductivity tensor

$$\sigma_{\alpha\beta}=\frac{1}{d}\int_0^d d\xi\int_{v_\xi^{min}}^{v_\xi^{max}}dv_\xi\,(1-\mathscr{E}(\lambda,t))\,f_{\alpha\beta}(v_\xi),$$

$$f_{\alpha\beta}(v_\xi)=\frac{2e^2}{(2\pi\hbar)^3}\int\frac{dv_\eta\,dv_\mu}{\det\{\alpha_{ik}\}}\,v_\alpha v_\beta t_0\delta(\varepsilon(\mathbf{p})-\varepsilon_0),\quad \alpha,\beta=\eta,\mu \qquad (20)$$

by integration over the region of small values of v_ξ. Since $f_{\alpha\beta}(v_\xi)$ is a slowly varying function, then we can replace it with sufficient accuracy by the quantity $f_{\alpha\beta}(0)$. In further calculation of $\sigma_{\alpha\beta}$, it is advantageous to introduce in place of v_ξ a new integration variable $z = (t-\lambda)/t_0$, where $t-\lambda$ is the root of Eq. (18). The symmetric tensor $f_{\alpha\beta}(0)$ can be diagonalized and, if we are not interested in the anisotropy of $\sigma_{\alpha\beta}$ in the plane of the plate, we obtain for the electric conductivity the expression

$$\sigma=\sigma_0\frac{d}{l}\int_{d/l}^{L}\frac{e^{-z}}{z}\,dz+\sigma'(E),$$
$$L\equiv(\varepsilon_0 d/eEl^2)^{1/2}, \qquad (21)$$

where σ_0 is the electric conductivity of a bulky plate. The first term in (21) depends logarithmically on the electric field at $eEl/\epsilon_0 \gg d/l$, and the second term does not exceed $\sigma_0 d/l$ in order of magnitude, and can be expanded in powers of E at $eEl/\epsilon_0 \ll d/l$ or else in powers of $\epsilon_0 d/eEl^2$ at $eEl/\epsilon_0 \gg d/l$.

It is easy to note that the nonlinear effects in the electric conductivity, which are connected with the heating of the electrons and with the change of the effective mean free path of the glancing electrons, are fully separated at $d \ll l$. The last mechanism of Ohm's-law violation is due to the influence of the electric field on the solution of the equation of the characteristics, while the influence of the heating of the electrons corresponds to inclusion in the integrand of (19) of the change of the electron velocity v(t) in the electric field, which leads to a dependence of the smooth function $f_{\alpha\beta}(v_\xi)$ on **E**. In formula (21), this dependence was not taken into account, and the expression presented for the electric conductivity of the plate is valid, strictly speaking, only if $d/l \ll (s/v)^{1/2}$. When account is taken of the heating of the electrons, naturally, the electric conductivity of the plates remains logarithmically dependent on the electric field at $eEl/\epsilon_0 > d/l$, and only the factor in front of the logarithm changes:

$$\sigma=\sigma_0(E)\frac{d}{l}\ln\frac{\varepsilon_0}{eEd}. \qquad (22)$$

Here $\sigma_0(E)$ is the electric conductivity of a bulky plate with allowance for the heating of the electrons by the current, which in the case of weak heating takes the form

$$\sigma_0(E)\approx\sigma_0\left\{1-\left(\frac{eElv}{\varepsilon_0 s}\right)^2\right\}. \qquad (23)$$

In the electric conductivity of wires, the role of the glancing electrons is less significant than in plates (the solid angle of the directions of motion of the carriers gliding along the surface of the conductor is of the order of $(d/l)^2$). Therefore the logarithmic dependence on E arises only in the next higher terms of the expansion of the electric conductivity in powers of d/l. Simple calculations enable us to obtain at q = 0 and $eEl/\epsilon_0 > d/l$ the following expression for the electric conductivity of a wire with an anisotropic carrier dispersion law:

$$\sigma=\sigma_0\frac{d}{l}\left\{1+\frac{d}{l}\ln\frac{\varepsilon_0}{eEd}\right\}. \qquad (24)$$

At an arbitrary character of the carrier reflection by the sample boundary, the expressions for the electric conductivity of the plate and wires are quite cumbersome. For example, even assuming that the specularity parameter q does not depend on the angle of incidence of the carriers, the expression for the electric conductivity of wires takes at $eEl/\epsilon_0 \ll d/l$ the form

$$\sigma=\left(\frac{eEl}{\varepsilon_0 d}\right)^2\int_0^{d/2}r\,dr\int_0\cos^4\theta\sin\theta\,d\theta\int_0 d\varphi\left\{1-\frac{1-q}{1-qe^{-2\alpha}}\left[1+\frac{r^2-d^2/4}{2l^2\sin^2\theta}\right.\right.$$
$$\left.\left.+\left(1+\alpha\frac{1+qe^{-2\alpha}}{1-qe^{-2\alpha}}\right)\left(\frac{r\sin\varphi}{l\sin\theta}+\alpha\frac{1+qe^{-2\alpha}}{1-qe^{-2\alpha}}\right)\right]\exp\left[-\frac{r\sin\varphi}{l\sin\theta}-\alpha\right]\right\}+\sigma(E=0), \qquad (25)$$

where $\alpha = (d^2/4 - r^2\cos^2\varphi)^{1/2}/l\sin\theta$, and d is the wire diameter. We have left out throughout inessential numerical factors of the order of unity.

However, a simple analysis shows that the electric conductivity of plates and of wires is of the same order as the electric conductivity of bulky conductors if the reflection of the carriers by the boundary is close to specular. It is easy to show that at $(1-q) \lesssim d/l$, when after l/d collisions with the boundary of the sample there is still some degree of correlation between the momenta of the initial and final states of the electron, the electric conductivity of such conductors coincides with $\sigma_0(E)$, apart from a numerical factor on the order of unity. At $1-q \gg d/l$, the electric conductivity of the plate and of the wire in the principal approximation in d/l is described by formulas (20)–(24), and only the corrections, particularly $\sigma'(E)$, depend significantly on the specularity parameter of the carrier reflection by the sample boundary. At near-diffuse reflection ($q \ll 1$), the electric conductivity of anisotropic plates is given by

$$\sigma=\sigma_0\left\{\frac{d}{l}\ln\left(\frac{\varepsilon_0}{eFd}\right)^{1/2}+q\frac{\varepsilon_0}{eEd}\right\}. \qquad (26)$$

The dependence of the electric conductivity of plates on the electric field turns out to be quite peculiar if the states of the conductor surfaces are essentially different. Let one of the plate surfaces, say $\xi = 0$, reflect the carriers diffusely, and the other specularly. If $E' = e\alpha_{\alpha k}E_k > 0$, i.e., the electric field that bends the electron trajectory tends to bring it closer to the specular surface $\xi = d$, then the main contribution to the electric conductivity of the plate is made by electrons that do not collide with the surface $\xi = 0$. The mean free path of such electrons is the same as in bulky samples, i.e., equal to l, and their number can be determined from simple considerations. From the equation of motion of the charge it follows that the electron, starting with the surface $\xi = d$ with velocity $v_\xi < v\sqrt{eEd/\epsilon_0}$, will never reach the opposite surface of the plate. From this we easily obtain that the number of electrons glancing along the specular surface of the plate is proportional to $\sqrt{eEd/\epsilon_0}$, and their contribution to the electric conductivity of the plate at $eEd/\epsilon_0 > (d/l)^2$ turns out to be fundamental, i.e.,

$$\sigma\approx\sigma_0\sqrt{eEd/\varepsilon_0}. \tag{27}$$

To calculate the electric-conductivity tensor we can use in this case formulas (18) and (19), replacing in them the time of motion of the charge from the surface $\xi = 0$ to a given point inside the conductor (ξ, η, μ) by the following expression:

$$\begin{gathered} t-\lambda=\frac{v_\xi}{E'}-\left[\left(\frac{v_\xi}{E'}\right)^2-\frac{2\xi}{E'}\right]^{1/2}, \quad v_\xi>\sqrt{2E'\xi}, \\ t-\lambda=\infty, \quad |v_\xi|<\sqrt{2E'\xi}, \\ t-\lambda=\frac{v_\xi}{E'}+2\left[\left(\frac{v_\xi}{E'}\right)^2-\frac{2(d-\xi)}{E'}\right]^{1/2}-\left[\left(\frac{v_\xi}{E'}\right)^2-\frac{2\xi}{E'}\right]^{1/2}, \quad v_\xi<-\sqrt{2E'\xi}. \end{gathered} \tag{28}$$

After elementary calculations it is easy to verify that the components of the matrix $\sigma_{\alpha\beta}$ have a square-root dependence on the electric field. The exact expression for $\sigma_{\alpha\beta}$ differs from (27) only in numerical factors of order unity, which depend on the concrete form of the carrier dispersion law.

When the direction of the electric field is reversed, the roles of the plate surfaces are interchanged, and the electric field now tends to bring the electron closer to the rough surface. The effective mean free path of the carriers colliding with only one surface of the plate is

$$l_{eff}=\min\left\{d\left(\frac{\varepsilon_0}{eEd}\right)^{1/2},\ l\right\},$$

and their contribution to the electric conductivity of the conductor is at any rate not larger than $\sigma_0 d/l$. In this case a more important role is played by the logarithmic dependence of the electric conductivity of the plate on the electric field, and formulas (20)—(24) hold for σ.

Thus, differences between the states of the plate surface can cause a significant difference between the current-voltage characteristics of thin anisotropic conductors when the electric-current direction is reversed.

3. NONLINEAR EFFECTS IN AN EXTERNAL MAGNETIC FIELD AND ROLE OF THE SELF-MAGNETIC FIELD OF THE CURRENT

In a strong magnetic field (electron trajectory curvature radius r smaller than the conductor thickness), the criterion for observing nonlinear effects depends essentially on the orientation of the magnetic field relative to the surface of the sample and the direction of the electric current. If the magnetic field is inclined to the surface of the sample, then there are no glancing electrons and consequently the nonlinear effects described in the preceding section do not appear. In a longitudinal field $\mathbf{H} \parallel \mathbf{j}$, all the electrons with closed orbits in momentum space move along the surface of the sample and in the Ohm's-law approximation their effective mean free path is equal to l. By virtue of the anisotropy of the carrier dispersion, the carriers can drift in the cross section plane of the conductor with velocity cE/H, which can lead with increasing electric field to a decrease of the effective electron mean free path, if their reflection by the sample boundary is nonspecular. For example, in diffuse reflection we have $l_{eff} = \min\{l, vHd/cE\}$ and at $cE/\hbar > vd/l$ the density of the electric current on the current-voltage characteristic of the conductor tends to saturation. However, this nonlinear effect can be observed only in very thin conductors, with thickness $d < ls/v$, for at $cE/H > s$ generation of coherent phonons by electrons takes place. At $s/v < d/l < (s/v)^{1/2}$, the main mechanism that leads to violation of Ohm's law is heating of the electrons.

In metals whose resistance increases without limit with increasing strong magnetic field **H**, the criterion for observing nonlinear effects connected with electron heating becomes less stringent with increasing **H**, and depends essentially on the character of the reflection of the carriers by the sample boundary, although in the Ohm's-law approximation the longitudinal resistance is indifferent to the state of the conductor surface. For example, Ohm's law does not hold if $j/nes \gtrsim n/r$ for specular reflection of the electrons, and if $j/nes \gtrsim d/r$ for diffuse reflection.

In conductors that are not too thin, when $d/l > (s/v)^{1/2}$, one can no longer ignore in the investigation of the nonlinear effects the self-magnetic field of the current, the appearance of which is the main cause of the deviation from Ohm's law. The self-magnetic field, as follows from Maxwell's equations, is always perpendicular to the current propagation direction, and in plates it is directed along the plate surface. Allowance for the self-magnetic field of the current reduces to a solution of the problem of the resistance of the conductor in an inhomogeneous magnetic field. The kinetic equation is in this case an integro-differential equation, even when the collision integral can be regarded as an operator for the multiplication of the nonequilibrium increment to the electron distribution function by the frequency of the elastic collisions. Naturally, the solution of the kinetic equation can be constructed properly only in the limiting cases of weak and strong fields H_j. If the drift velocity v of the electrons satisfies the inequality $v \ll v_0/na_0^2 l^2$, where $a_0 = e^2/mc^2$ is the radius of the electron, then the distribution function can be represented in the form of an expansion in the reciprocal of the electron-trajectory curvature radius (we assume that there is no external magnetic field), and we can solve the kinetic equation by successive approximations. The solution of (8a), which can be rewritten for a plate in the form

$$\operatorname{rot}\mathbf{H}(\mathbf{r})=\frac{4\pi}{c}\left\langle \mathbf{v}\left\{-E_\alpha v_\alpha t_0+[F+E_\alpha v_\alpha t_0-\Phi(\xi s)]\exp\left\{-\frac{\xi-\xi_s}{v_\xi t_0}\right\}+\Phi(\xi)-\frac{1}{v_\xi t_0}\int_{\xi_s}^{\xi}\Phi(\xi')\exp\left\{\frac{\xi'-\xi}{v_\xi t_0}\right\}d\xi'\right\}\right\rangle, \quad \alpha=\eta,\mu, \tag{8b}$$

is then

$$\begin{aligned} H_\eta=-\frac{4\pi}{c}\Big\langle v_\mu\Big\{ & v_\xi t_0(F+F_\alpha v_\alpha t_0-\Phi(\xi_s))\cdot\left(\exp\left\{-\frac{\xi-\xi_s}{v_\xi t_0}\right\}-\exp\left\{-\frac{d-2\xi_s}{2v_\xi t_0}\right\}\right)+E_\alpha v_\alpha t_0\left(\frac{d}{2}-\xi\right) \\ & +\int_{\xi_s}^{\xi}\Phi(\xi')\exp\left\{\frac{\xi'-\xi}{v_\xi t_0}\right\}d\xi'-\int_{\xi_s}^{d/2}\Phi(\xi')\exp\left\{\frac{2\xi'-d}{v_\xi t_0}\right\}d\xi'\Big\}\Big\rangle. \end{aligned} \tag{29}$$

The arbitrary function of the characteristics F and the distribution of the electric potential $\Phi(\xi)$ are determined from (5), (7), and (9).

In the considered region of currents, the principal role in the electric conductivity of the plate is played by glancing electrons, the self-magnetic field bends slightly the trajectory of motion of these electrons, and the effective mean free path increases with increasing j (the scattering by the plate surface is assumed diffuse):

$$l_{eff}\approx d\left\{\ln\frac{l}{d}+(na_0l^2)\left(\frac{eEd}{\varepsilon}\right)^2\right\}. \quad (30)$$

For a sample in the form of a wire, the role of the glancing electrons is not so significant. The successive-approximation method makes it possible to determine the effective mean free path for an arbitrary character of the carrier scattering. At near-specular electron reflection $(1-q) \leq d/l$, it is the collisions inside the volume that are predominant, and the boundary effects are weak. At $(1-q) \gg d/l$, the electron distribution function depends little on the collision integral (for an isotropic quadratic carrier dispersion in a thin wire, the linear-approximation distribution function is entirely independent of the relaxation time), so that the actual expression for $\hat{W}$ in (4) is immaterial. The change of the effective mean free path of the electrons in the wire is proportional to the square of the electric field E, and in the case of near-diffuse scattering, $q \ll d/l$, is given by

$$\Delta l_{eff}=l_{eff}-l_{eff}^{(H_j=0)}=d\left(na_0d^2\frac{eEd}{\varepsilon_0}\right)^2. \quad (31)$$

If the self-magnetic field of the current bends the charge trajectory significantly, so that the curvature radius r_j is much smaller than d, then the distribution function must be expanded in powers of r_j. In metals whose Fermi surface is open or in which the number of electrons n_1 and holes n_2 is compensated, the electric current flows at $r_j \ll d$ near the surface of the conductor, and the character of the electron reflection plays an important role. Simple calculations show that at $v \gg v_0/na_0^2d^2$ the potential difference is proportional to j^2 in specular reflection of the carriers by the sample boundary and is proportional to j^3 in diffuse reflection.

Thus, the criterion for observing nonlinear effects, the form of the current-voltage characteristics, and the magnitude of the deviation from Ohm's law, all depend significantly on the state of the conductor surface, on its thickness, and on the orientation of the external magnetic field. An investigation of the nonlinear effects in the electric conductivity of thin conductors makes possible a detailed analysis of the mechanism whereby the conduction electrons interact with the surface of the sample, and by reversing the direction of the current one can detect even a slight difference in the surface finish of the plate, and determine in final analysis the state of the conductor surface.

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1)Yanson and Bogatina have succeeded in producing a current density $j \approx nes$ in thin lead channels and observed generation of phonons by hot electrons [10].

2)For example, in conductors of thickness $d \approx 1$ mm, the electric conductivity of which is $\sigma \approx 10^{22}$ sec^{-1}, putting $s = 10^5$ cm/sec and a speed of light $c \approx 10^{10}$ cm/sec, we obtain $c^2/s\sigma d \approx 10^{-5}$, i.e., the self-magnetic field of the current can lead to nonlinear effects already at $j \approx 10^6$ A/cm^2. It was precisely at these current densities that Borovik observed a deviation from Ohm's law in copper [5].

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