

# Recursion equations in gauge field theories

A. A. Migdal

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences

(Submitted April 28, 1975)

Zh. Eksp. Teor. Fiz. 69, 810-822 (September 1975)

An approximate recursion equation is formulated, describing the scale transformation of the effective action of a gauge field. In two-dimensional space-time the equation becomes exact. In four-dimensional theories it reproduces asymptotic freedom to an accuracy of 30% in the coefficients of the  $\beta$ -function. In the strong-coupling region the  $\beta$ -function remains negative and this results in an asymptotic prison in the infrared region. Possible generalizations and applications to the quark-gluon gauge theory are discussed.

PACS numbers: 11.10.Np

## INTRODUCTION

Many experiments favor the model of confined quarks. According to this model the quarks are relatively light and are almost free inside the hadrons. The forces between the quarks must increase with the distance, as if they were connected by strings. Recently a hypothesis was put forward<sup>[1]</sup> according to which such strings are formed from the non-Abelian gauge fields called gluons which interact with the color of the quarks.

As was stressed by Wilson,<sup>[2]</sup> it is essential for this that the vacuum be gauge-invariant. Then there arise local selection rules which interdict the decay of gauge-invariant excitations such as

$$\bar{\psi}(r) \hat{T} \exp \left\{ i \int_0^r B, dr, \right\} \psi(0) |0\rangle, \quad (0)$$

into free quarks  $\psi$  and gluons  $B$ , or into other colored objects. The local selection rules follow directly from the transformation laws of the fields

$$\begin{aligned} \Omega \psi_2(r) \Omega^\dagger &= S(r) \psi_2(r), \\ \Omega B, \Omega^\dagger &= S^{-1}(r) B, S(r) - i S^{-1}(r) \partial, S(r) \end{aligned}$$

and the invariance of the vacuum  $\Omega |0\rangle = |0\rangle$ .

The usual vacuum of quantum electrodynamics which is realized in perturbation theory is not gauge-invariant. It is filled with the longitudinal field which participates in the gauge transformations, so that the local selection rules are inapplicable. They are replaced by the infrared catastrophe: charged objects are accompanied by transverse quanta which are Goldstone excitations in that degenerate vacuum.

The existence of an invariant vacuum was proved in two-dimensional models<sup>[3]</sup>, and, what is more interesting, in four-dimensional lattice gauge theories in the strong coupling region<sup>[2,4]</sup>. Strong coupling implies a large bare coupling constant  $g_0$  and a large bare mass  $m_0$  of the quark (measured in units of the cutoff radius  $a^{-1}$ ). The state  $|0\rangle$  becomes in this limit an eigenstate with the energy

$$V(r) = \sigma r/a^2.$$

The energy is confined to the lines of the electric field which join the quarks and this is a prototype of the string.

In Lagrangian language in Euclidean space the electric forces manifest themselves as the contribution  $\sim \exp\{-\text{area of the quark loop}\}$ <sup>[2]</sup>. More precisely, the contribution of a quark loop (e.g., to two-current Green's function) is proportional to  $\exp(-\sigma S/a^2 - \rho P/a)$ , where  $P$  is the length of the quark loop and  $S$  is the area surrounded by it. Summation over all loops and surface areas is understood; the details can be found in

Wilson's paper<sup>[2]</sup>.

The coefficients  $\sigma$  and  $\rho$  depend on the bare parameters  $g_0$  and  $m_0 a$ . In the local limit  $a \rightarrow 0$  these parameters should be varied in such a manner that the zeroes of  $\sigma$  and  $\rho$  are determined. Then, most likely, one will arrive at the dual string models<sup>[5]</sup>.

On the other hand, in the local theory there exists the notorious asymptotic freedom. This means that if we let  $g_0^2$  tend to zero according to the law  $c_1/(c_2 - \ln a)$  with an appropriate constant  $c_1$  we obtain the local limit of the lattice theory. The second bare parameter  $m_0 a$  must also be made to tend to zero if  $\gamma_5$ -invariance, which removes the linear divergences in the self-energy, is respected.

Now there appears a difficulty, since this limit is opposite to the strong-coupling limit where the invariance of the vacuum was proved. A phase transition to the usual vacuum might take place for some  $g_0$  and  $m_0 a$ , such that  $\sigma \equiv 0$ . The phase transition problem is related to the existence of fixed points of the renormalization group. If there are no fixed points except the free ultraviolet one then the effective interaction will increase indefinitely in the infrared region, which apparently characterizes the asymptotic prison, i.e., the absence of a phase transition<sup>[6]</sup>, and also G. 't Hooft, communication at the Marseille Conference, 1972).

In brief, the situation is as follows: in usual perturbation theory the ultraviolet freedom is obvious and the infrared prison causes problems, whereas in the lattice strong coupling expansion it is the other way around. If, as we hope, both the prison and the freedom coexist in the same theory, a third computational method is needed, which is not related to the strength of the interaction.

We intend to develop numerical renormalization group methods analogous to the recursion equations of statistical mechanics.<sup>[7]</sup> The main problem is connected with the requirement of local gauge invariance. In the present paper we propose a first version of a gauge recursion equation which is as yet far from perfect. Nevertheless this equation is sensitive to the existence of phase transitions. We have verified this on the example of the discrete gauge group  $Z_2$  and on the Abelian group  $U(1)$ , where the existence of a phase transition is known. In the non-Abelian theory the equation yields a phase transition for dimension  $D > 4$  and no phase transition for  $D \leq 4$ .

## 1. THE EFFECTIVE ACTION OF THE GAUGE FIELD

The classical action of the gauge field proposed by Yang and Mills

$$A_{cl} = - \int d^D x \sum_{\mu\nu} \frac{1}{4g_{cl}^2} \text{Tr}(\partial_\mu B_\nu - \partial_\nu B_\mu + i[B_\mu, B_\nu])^2 \quad (1)$$

needs to be redefined in quantum theory. In addition to the usual ultraviolet divergences related to the infinite number of degrees of freedom per unit volume, the functional integrals contain divergences related to the invariance of the action under local gauge transformations

$$B_\mu \rightarrow S^{-1}(x) B_\mu S(x) - iS^{-1}(x) \partial_\mu S(x). \quad (2)$$

The most adequate method of regularization for our problem is to consider (1) as the limit of a lattice theory. Gauge transformations on a lattice have been considered in<sup>[2,8]</sup> and independently by Polyakov in 1974 (private communication). One carries out a Wick rotation, after which the D-dimensional Euclidean space is split up into the cells of a hypercubic lattice. The action of the lattice theory is represented as the sum of the actions of each cell

$$A_{lat} = \sum_x A(x). \quad (3)$$

The action  $A(x)$  of one cell depends on the loop products

$$v_\Gamma = \prod_\Gamma \exp\{iB_\mu(y) \Delta y_\mu\}. \quad (4)$$

Here  $\prod_\Gamma$  denotes the product of matrices ordered along the loop  $\Gamma$  in the boundary of the cell,  $\Delta y_\mu$  are the sides of the loop and  $B_\mu(y)$  are the gauge fields at the centers of the sides. Thus, the loop product around the square in Fig. 1 is of the form

$$v_{\square} = \exp\{iaB_\mu(x - 1/2ae_\nu)\} \exp\{iaB_\nu(x + 1/2ae_\nu)\} \\ \times \exp\{-iaB_\mu(x + 1/2ae_\nu)\} \exp\{-iaB_\nu(x - 1/2ae_\nu)\}. \quad (4')$$

The traces of closed loop products are invariant with respect to generalized gauge transformations

$$\exp\{iaB_\mu(y)\} \rightarrow S^{-1}(y - 1/2ae_\mu) \exp\{iaB_\mu(y)\} S(y + 1/2ae_\mu). \quad (5)$$

The traces can be calculated in any representation of the group. The traces in unitary irreducible representations  $p$ , i.e., the characters  $\chi_p(v)$  of the group element  $v$ , will play a distinguished role in the sequel.

A simple example for the cell action has been proposed by Wilson<sup>[2]</sup>:

$$A(x) = \sum_{\Gamma} \frac{1}{4g_0^2} [\chi_{adj}(v_\Gamma) - \chi_{adj}(1)]. \quad (6)$$

where  $\chi_{adj}$  is the character in the adjoint representation.

As  $a \rightarrow 0$  the loop products reduce to the components of the field strength  $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + i[B_\mu, B_\nu]$  so that the classical lattice action (3), (6) reduces to the classical action (1) in the naive local limit ( $g_0^2 a^{4-D}$  fixed). For this it suffices that the loops  $\Gamma$  be situated in all possible  $\mu\nu$  planes leading to the appearance of all terms in the sum

$$\sum_{\mu\nu} \text{Tr} B_{\mu\nu}^2.$$

The existence of the local limit of the classical action does not yet guarantee that the quantum theory will be local. First of all there must exist a sufficient number of connections between the cells, i.e., the same field  $B_\mu$  must enter into sufficiently many actions of neighboring cells. The number of connections depends on the loops  $\Gamma$ . Thus, in the Wilson model the loops consist of the four edges of the cell (Fig. 2 for  $D = 3$ ). Then each field corresponds to an edge and enters into

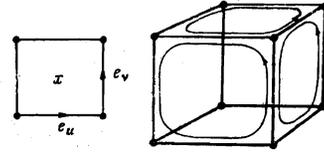


FIG. 1

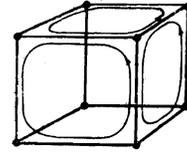


FIG. 2

the actions of  $2(D - 1)$  cells with a common edge. This number of connections is sufficient to correlate the quantum fluctuations over the whole lattice. The correlations of fluctuations over distances much larger than the lattice unit must go over into the Green's functions of the local  $O(D)$ -invariant theory.

The lattice systems are quantized by means of the usual integration of  $e^A$  over the fields. One must only worry about the gauge invariance, and for this one equips the integration with the group invariant measure  $d\Omega(e^{iaB})$ . The gauge invariance of this measure follows from the multiplicativity

$$d\Omega(v) = d\Omega(vu) = d\Omega(wv). \quad (7)$$

In all practically important cases explicit expressions are known for the group measure. Thus, in the case of  $SU(2)$

$$d\Omega(e^{iaB}) = \frac{a}{8\pi^2} \frac{d^3B}{B^2} \sin^2\left(a \frac{|B|}{2}\right). \quad (8)$$

We have used the spin  $1/2$  representation:  $B = \mathbf{B} \cdot \boldsymbol{\tau}/2$ .

For compact groups the integration volume is finite, e.g.,  $|B| < 4\pi/a$ , for  $SU(2)$ , so that there is no need to add to the action terms which determine the gauge and ensure convergence. In the sequel all integrations over the fields reduce to the orthogonality relation

$$\int d\Omega(v) \chi_p(v_1v) \chi_p(v^+v_2) = \frac{\chi_p(v_1v_2) \delta_{p,p_1}}{d_p}, \quad (9)$$

where  $d_p = \chi_p(1)$  is the dimension of the representation  $p$ . A more detailed discussion of the formalism of gauge theories on a lattice can be found in<sup>[2,4,8]</sup>.

We must now define the effective action. We join  $2^D$  neighboring cells into one cell with twice the edge length and integrate the expression

$$\exp\left\{\sum_{i=1}^{2^D} A(x_i)\right\}$$

over all the internal fields, fixing the values of the fields on the boundary. We call this integral the correlation functional  $Z_{2a}$  and its logarithms the effective action  $A_{2a}$  of the doubled cell. When such an integration is carried out over the whole volume we obtain the doubled lattice with the effective action replacing the bare action. Since the effective action is gauge invariant, it depends on the loop products along the boundaries of the doubled cell. The new loops consist of portions of the old ones and in the general case the number of loops increases. Repeating this operation (this is the recursion) we gradually integrate over all the fields.

The recursion equation which relates cells of sizes  $2L$  and  $L$  can symbolically be written in the form:

$$Z_{2L} = \mathcal{R}[Z_L]. \quad (10)$$

It is convenient to separate the averages

$$\langle Z_L \rangle = \int \prod_{\Gamma} d\Omega(v_\Gamma) Z_L, \quad (11)$$

by introducing the normalized correlation functional

$$W_L = Z_L / \langle Z_L \rangle. \quad (12)$$

We are then led to the normalized recursion equation

$$W_{z_L} = \hat{R}[W_L] / \langle \hat{R}[W_L] \rangle \quad (13)$$

and to the equation for the average

$$\langle Z_{z_L} \rangle = \langle Z_L \rangle^{2^D} \langle \hat{R}[W_L] \rangle. \quad (14)$$

The latter equation can be solved in general form (we have set  $\langle Z_a \rangle = 1$ ):

$$\langle Z_L \rangle = \exp \left\{ L^D \sum_{a \leq L_n = 2^n a \leq L} L_n^{-D} \ln \langle \hat{R}[W_{L_{n-1}}] \rangle \right\} \quad (15)$$

This quantity is related to the energy spectrum of a cube of size  $L$  with periodic boundary conditions<sup>[2]</sup>

$$\langle Z_L \rangle = \text{Tr} \exp \{-L\hat{H}(L)\}. \quad (16)$$

For  $L = \infty$  we obtain the energy density of the vacuum in the infinite system:

$$\epsilon = - \sum_{n=0}^{\infty} L_n^{-D} \ln \langle \hat{R}[W_{L_{n-1}}] \rangle. \quad (17)$$

Subtracting  $L^{D-1}$  from the Hamiltonian, we obtain the relation

$$\sum_{L=1}^{\infty} L_n^{-D} \ln \langle \hat{R}[W_{L_{n-1}}] \rangle = L^{-D} \ln \text{Tr} \exp \{\epsilon L^D - L\hat{H}(L)\}. \quad (18)$$

The right-hand side contains all possible surface corrections to the vacuum energy density, and the contribution of the excited states is of the order  $e^{-mLD-D}$ , where  $m$  is the lowest mass. Thus, the asymptotic behavior of the  $W$ -functional is related to the mass spectrum.

In principle the lowest mass can be found directly, introducing the sources of the appropriate quantum numbers. We do not intend to discuss this question now, but hope to return to it after the quarks have been included.

The fixed points of the renormalization group are the solutions of the normalized recursion equation which do not depend on  $L$ . Asymptotic freedom corresponds to the quadratic action (1) with  $g_c^2 \rightarrow 0$ , i.e., to a delta-functional like fixed point  $W \sim \Pi \delta(B_{\mu\nu})$ . This fixed point must be realized in nonabelian theories at small distances. The asymptotic prison for  $L \rightarrow \infty$  corresponds to vanishing action, i.e.,  $W = 1$  is the unit fixed point. This fixed point exists on account of the compactness of the gauge group, otherwise the integrals over the fields would diverge in  $\hat{R}[W]$  for  $W = 1$ .

If there are no other fixed points, the  $W$ -functional will change from the delta-functional to one as  $L$  varies from zero to infinity. The vacuum energy in (17) will diverge for  $L \sim a \rightarrow 0$ , but the excitation energy will be determined by the region  $L^{-1} \sim 1 \text{ GeV}$ . Apparently they will correspond to dual models (after quarks are included into the recursion equations).

In order to realize this program one must approximate the functional recursion equations by means of integral equations, reducing the number of degrees of freedom.

## 2. A TWO-DIMENSIONAL GAUGE THEORY

We start with the trivial example of a two-dimensional theory. Usually the two-dimensional theory is solved in the Coulomb gauge  $B_t = 0$ , when the commutator  $[B_{\mu}, B_{\nu}]$  vanishes. This theory exhibits ultraviolet freedom and an infrared prison simply because

the Coulomb potential is proportional to the distance. We reproduce these results with the help of the recursion approach, in order to develop an apparatus which will be useful in the four-dimensional theory.

The elementary cell of the two-dimensional lattice is the square in Fig. 1. There is only one loop product (4') along the perimeter of the square. We consider two neighboring cells 1 and 2 and determine the correlation functional  $Z_{12}$  of the joint cell 1 + 2 (Fig. 3). We have to calculate the integral

$$Z_{12} = \int d\Omega(u) Z_a(u, u^+) Z_a(u, u_2) \quad (19)$$

over the internal field  $u = \exp[iaB((x_1 + x_2)/2)]$ . We have separated from the loop products  $v_1$  and  $v_2$  the matrices  $u$  corresponding to the common part of the loops. The matrices  $u_1$  and  $u_2$  correspond to the remaining parts of the contours; their product  $v_{12} = u_1 u_2$  corresponds to the perimeter of the joint cell.

The integral is calculated by means of the Fourier expansion on the group

$$Z_a(v) = \sum_p Z_{a,p} d_p \chi_p(v). \quad (20)$$

Making use of the orthogonality relation (9) we find

$$Z_{12} = \sum_p Z_{a,p}^2 d_p \chi_p(v_{12}). \quad (21)$$

It is clear that the form of the  $Z$ -functional reproduces itself when cells are joined together and the Fourier coefficients get multiplied.

It is easy to prove by induction that this rule does not depend on the form and size of the cells. If the boundary between two cells consists of several steps (Fig. 4), the integration with respect to the field of the first edge reduces to a traction, as above, after which the remaining inner fields cancel in the product  $v_1 v_2$ , since the common part of the loop has opposite orientations in the cells 1 and 2. Now the remaining integrations yield the unit volume of the group.

As a result the  $Z$ -functional for a region  $\Gamma$  with area  $S$  has the form

$$Z_S(v_\Gamma) = \sum_p (Z_{a,p})^{S/a^2} d_p \chi_p(v_\Gamma). \quad (22)$$

Here  $v_\Gamma$  is the loop product along  $\Gamma$  and  $S/a^2$  is the number of elementary cells. This is the general solution of the 2-dimensional recursion equations.

In the local limit  $a \rightarrow 0$  one must take the limit  $Z_{a,p} \rightarrow 1 - a^2 \epsilon_p$ , so that

$$Z_S(v_\Gamma) \rightarrow \sum_p \exp\{-S\epsilon_p\} d_p \chi_p(v_\Gamma). \quad (23)$$

The loop product  $v_\Gamma$  goes over into the T-exponential of the circulation

$$v_\Gamma \rightarrow \hat{T} \exp \left\{ i \oint_{\Gamma} B_{\mu} dx^{\mu} \right\}. \quad (24)$$

The vacuum expectation value corresponds to the zero Fourier coefficient

$$\langle Z_S \rangle = \exp\{-S\epsilon_0\}. \quad (25)$$

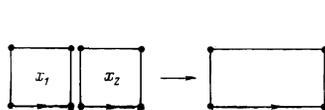


FIG. 3

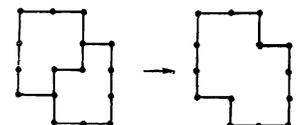


FIG. 4

so that  $\epsilon_0$  is the vacuum energy density.

The normalization corresponds to the subtraction  $\epsilon_p \rightarrow \epsilon_p - \epsilon_0$  in (23). These differences must be positive:

$$\epsilon_p - \epsilon_0 = g_p^2 \geq 0, \quad (26)$$

in order to yield a reasonable W-functional

$$W_s(v_r) = \sum_p \exp\{-g_p^2 S\} d_p \chi_p(v_r). \quad (27)$$

This functional varies from  $W_0 = \delta(v - 1)$  to  $W_\infty$ , i.e., no phase transition that would violate the gauge invariance of the vacuum occurs.

We now introduce the fields  $\psi$  (colored quarks) belonging to the  $q$ -representation of the gauge group. The Green's function of the gauge invariant currents  $J_\mu \sim \bar{\psi} \gamma_\mu \psi$  can be represented as sums over closed quark paths  $\Gamma$  of the quantity  $\text{Tr } v_\Gamma^+ = \chi_q(v_\Gamma^+)$ , averaged over the gauge fields with the weight  $\exp[\text{action}]$  (cf. [2]).

For simplicity we consider only simply connected paths without self-intersections (such paths dominate<sup>[3]</sup> for high-rank gauge groups). Integrating  $\chi_q(v_\Gamma^+)$  over all the fields we can first integrate over the fields inside  $\Gamma$ , which yields  $W_S(v_\Gamma)$  and then outside, which yields  $W_{\bar{S}}(v_\Gamma^+)$ . The external region  $\bar{S}$  has infinite area, so that  $W_{\bar{S}} = 1$ . The remaining integrations over the fields on  $\Gamma$  will reduce, on account of the multiplicativity of the measure, to the simple integral over  $v_\Gamma$ :

$$\langle \chi_q \rangle = \int d\Omega(v_r) \chi_q(v_r^+) W_s(v_r). \quad (28)$$

This integral is easily calculated (only the quark representation contributes), and we find

$$\langle \chi_q \rangle = d_q \exp\{-g_q^2 S\}. \quad (29)$$

This is none other than Wilson's law of areas.<sup>[2]</sup>

In the usual solution of the two-dimensional theory this law follows from the one-dimensional Coulomb law  $u(x) = g^2(T^2)_q |x|$ ; in place of  $g_q^2$  in (29) there appears  $g^2(T^2)_q$ , where  $T^2$  is the Casimir operator. The more general law (29) appeared because we have not required that the bare action go over into the Yang-Mills action in the local limit. This requirement is not necessary in the two-dimensional theory, where there is no renormalizability problem.

### 3. APPROXIMATE RECURSION EQUATIONS

In this section we propose approximate recursion equations for the  $D$ -dimensional gauge theory. We first consider the three-dimensional case. The  $Z$ -functional of a cube of size  $L \gg a$  depends, generally speaking, on all possible loop products along its surface.

We approximate this functional by a function of the average loop products. The average loops are square contours with angles in the middles of the edges of the cube. There are a total of three such loops:  $\gamma_{xy}$ ,  $\gamma_{yz}$  and  $\gamma_{xz}$ . They are situated in distinct planes (Fig. 5). We shall consider the average loop products as the independent variables and shall search for the  $Z$ -functional in the factorized form:

$$Z_L = F(L, v_{xy}) F(L, v_{yz}) F(L, v_{xz}). \quad (30)$$

When eight  $L$ -cubes are joined into one  $2L$ -cube, the average loops are independently joined into larger contours in each plane. Thus, in the  $xy$ -plane four loops are joined into a square with transverse coordinate  $z = -L/2$  and the other four into a square with

$z = L/2$  (Fig. 6).

The integration over the fields reduces to a contraction as in the two-dimensional case

$$\square = F(L, v) = \sum_p F_p(L) d_p \chi_p(v), \quad (31)$$

$$\square\square = \sum_p F_p^4(L) d_p \chi_p(v) = \Phi(v). \quad (32)$$

After this the  $Z$ -functional of the  $2L$ -cube turns out to be factorized as in (30), but with

$$\Phi(v_{xy}(-L/2)) \Phi(v_{xy}(L/2)) \quad (33)$$

in place of  $F(2L, v_{xy}(0))$ , and similarly for  $yz$  and  $xz$ . Here  $v_{xy}(z)$  is the loop product over the  $xy$ -square with side  $2L$  and coordinate  $z$ .

As can be seen, our assumption (30) is not reproduced by the recursion equation—the average loop splits into two.

The splitting  $\Delta z$  is smaller than the size of the  $2L$ -loop, so that in a rough approximation we may neglect it. We then obtain a recursion equation for  $F$ :

$$F(2L, v) = \left[ \sum_p F_p^4(L) d_p \chi_p(v) \right]^2 \quad (34)$$

or, in the Fourier representation

$$F_p(2L) = \sum_{p_1 p_2} F_{p_1}^4(L) F_{p_2}^4(L) C_{p_1 p_2}^p \quad (35)$$

where

$$C_{p_1 p_2}^p = \frac{d_{p_1} d_{p_2}}{d_p} \int d\Omega(v) \chi_{p_1}(v) \chi_{p_2}(v) \chi_p(v) \quad (36)$$

is the appropriate Wigner coefficient.

We expect this equation to have a quantitative meaning when  $L$  is smaller than a typical Compton wavelength of the bound states: in this region there are no dimensional parameters, so that the splitting  $z$  should be compared only to  $2L$ . Then, since the terms linear in  $z$  cancel, the error may be small. Below we shall modify this equation somewhat to decrease the error.

First we wish to generalize the equation to the case of arbitrary dimension  $D$  of the space. We introduce  $D(D-1)/2$  average loops  $\gamma_{\mu\nu}$  on the  $d$ -dimensional cubic lattice. The angles of the squares  $\gamma_{\mu\nu}$  are at the points  $(x_\mu, x_\nu = \pm L/2, x_\perp = 0)$ , if the vertices of the cube are at the points  $(\pm L/2, \pm L/2, \dots, \pm L/2)$ .

In our approximation the  $Z$ -functional of the cube will be a product of functions of the average loop products:

$$Z_L = \prod_{\mu < \nu} F(L, v_{\mu\nu}(0)). \quad (37)$$

As above, the integration is carried out independently in each plane, and joining  $2^D$   $L$ -cubes into one  $2L$ -cube, we obtain

$$Z_{2L} = \prod_{\mu < \nu} \prod_{x_\perp} \sum_p F_p^4(L) d_p \chi_p(v_{\mu\nu}(x_\perp)), \quad (38)$$

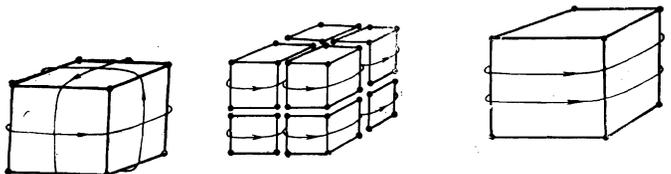


FIG. 5

FIG. 6

where  $x_{\perp,i} = \pm L/2 (i = 1, 2, \dots, D-2)$  are the transverse coordinates of the resulting  $2L$ -squares. We now neglect the dependence on  $x_{\perp}$  of the  $2L$  loop products  $v_{\mu\nu}(x_{\perp})$ , so that the form (37) is reproduced, and obtain the recursion equation

$$F(2L, v) = \left[ \sum_p F_p^s(L) d_p \chi_p(v) \right]^{D-2}, \quad (39)$$

which generalizes (34).

The second generalization is the transition to an arbitrary change of scale  $L \rightarrow \lambda L$  in place of the doubling. If our approach would be exact the results would not depend on  $\lambda$ , but the approximate equation can be improved by varying  $\lambda$  as an independent free parameter.

A scale transformation with nonintegral  $\lambda$  can be defined by means of the following trick. We first consider integral values of  $\lambda$  and find the corresponding recursion equation, joining  $\lambda^D$  cubes into one. Integration with respect to the internal fields yields, in place of (38)

$$Z_{\lambda L} = \prod_{\mu < \nu} \prod_{x_{\perp,i}} \sum_p F_p^s(L) d_p \chi_p(v_{\mu\nu}(x_{\perp})), \quad (40)$$

where

$$x_{\perp,i} = \pm 1/2 L, \pm 3/2 L, \dots, \pm 1/2 (\lambda-1) L; \quad i=1, 2, \dots, D-2.$$

Neglecting the splittings  $x_{\perp,i}$  we obtain

$$F(\lambda L, v) = \left[ \sum_p F_p^s(L) d_p \chi_p(v) \right]^{\lambda^{D-2}} = \hat{R}_{\lambda}[F(L)]. \quad (41)$$

For noninteger  $\lambda$  we define a recursion operation  $\hat{R}_{\lambda}$  by means of the same equation (41).

The relations (10)–(19) are also generalized to the case of an arbitrary scale parameter  $\lambda \neq 2$ . In particular, in place (15) and (16) we have

$$\langle Z_{\lambda} \rangle = \text{Tr} \exp\{-L \hat{H}(L)\} = \exp\left\{L^D \frac{D(D-1)}{2} \sum_a L_n^{-D} \ln \langle \hat{R}_{\lambda}[f(L_{n-1})] \rangle\right\}, \quad (42)$$

where  $L_n = \lambda^n a$  and  $f(L, v)$  satisfies the normalized equation

$$f(\lambda, L) = \hat{R}_{\lambda}[f(L)] / \langle \hat{R}_{\lambda}[f(L)] \rangle. \quad (43)$$

The averaging  $\langle \dots \rangle$  denotes here an integration over the normalized group measure; the factor  $D(D-1)/2$  in (42) appears on account of factorization.

#### 4. ASYMPTOTIC FREEDOM AND THE INVARIANT VACUUM

The equation (41) is the main result of this paper. In a future paper we shall discuss in detail this equation for different theories. Now we demonstrate the renormalizability and the asymptotic freedom on the example of the gauge group  $SU(2)$  and list some results obtained by numerical methods.

A representation of the group  $SU(2)$  is characterized by the spin  $T = 0, 1/2, 1, 3/2, \dots$ :

$$\chi_r(v) = \frac{\sin(T+1/2)\theta}{\sin(\theta/2)}, \quad (44)$$

$$d_r = \chi_r(1) = 2T+1. \quad (45)$$

In our case the angle  $\theta$  corresponds to the Yang-Mills field strength.

We introduce the "vertices"  $g_n(L)$ , such that

$$F(L, v) = \exp\left\{-q_0(L) - q_2(L) \frac{\theta^2}{2!} - q_4(L) \frac{\theta^4}{4!} - \dots\right\}. \quad (46)$$

The Fourier coefficients

$$F_T(L) = \int_0^{4\pi} \frac{d\theta}{4\pi} \frac{\sin(\theta/2) \sin(T+1/2)\theta}{T+1/2} F(L, v) \quad (47)$$

can be characterized by another set of variables  $p_i$ :

$$F_T(L) = \exp\left\{-p_0 - p_2 \left(T + \frac{1}{2}\right)^2 - \frac{1}{2!} p_4 \left(T + \frac{1}{2}\right)^4 - \frac{1}{4!} p_6 - \dots\right\}. \quad (48)$$

The inversion formula (47) allows one to express the  $p_i$  in terms of the  $q_j$ . In terms of the functions  $p_i(q_2, q_4, \dots)$  the recursion equation (44) can be rewritten as

$$\lambda^2 p_i[q_n(L)] = p_i[\lambda^{2-D} q_n(\lambda L)], \quad i=2, 4, 6, \dots \quad (49)$$

These equations for  $q_i(L)$  can be integrated with respect to the effective coupling constant

$$g^2(L) = 1/q_2(L). \quad (50)$$

As it should be in renormalized perturbation theory, the vertices  $q_4, q_6, \dots$  depend on  $L$  only through  $g^2(L)$ :

$$q_n(L) = A_n + B_n g^2(L) + C_n g^4(L) + \dots \quad (51)$$

The coefficients  $A_n, B_n, C_n, \dots$  are determined successively from (49) together with the expansion coefficients of  $g^2(\lambda L)$  in powers of  $g^2(L)$ :

$$g^2(\lambda L) = \lambda^{-D} g^2(L) + 1/2 (\lambda^2 - \lambda^{-D}) g^4(L) + \dots \quad (52)$$

It is convenient to introduce the Gell-Mann–Low function

$$\beta(g^2) = -\frac{\partial g^2}{\partial \ln L} = (D-4)g^2 - \beta_4 g^4 - \beta_6 g^6 + \dots \quad (53)$$

Its coefficients are easily determined by differentiating (52) with respect to  $\ln L$  and expressing in the left-hand side  $g^2(\lambda L)$  in terms of  $g^2(L)$  from (52).

In four-dimensional space we obtain

$$\beta_4 = (\lambda^2 - 1)/12 \ln \lambda, \quad (54)$$

$$\beta_6 = \lambda^2 \beta_4 / 12 (\lambda^2 + 1). \quad (55)$$

These coefficients are positive for all  $\lambda$ , so that the effective  $c$  coupling constant vanishes for  $L \rightarrow 0$  according to the law

$$g^2(L) \rightarrow (\beta_4 t + \beta_4^{-1} \beta_6 \ln t)^{-1}, \quad (56)$$

$$t = \ln(L/L_0) \gg 1. \quad (57)$$

Here  $L_0$  is some arbitrary normalization length. When  $L$  increases and approaches  $L_0$  perturbation theory becomes inapplicable. In (47) there appear terms of the order

$$g^3(L) \exp\left(-\frac{2\pi^2}{g^2(L)}\right) \sim \exp\left\{-2\pi^2 \beta_4 \ln \frac{L_0}{L} - \left(3 + \frac{2\pi^2}{\beta_4} \beta_6\right) \ln \ln \frac{L_0}{L}\right\} \quad (58)$$

determined by the upper integration limit. These terms lead to a rapid growth of the effective coupling constant. It is impossible to follow its behavior further by means of analytic methods and a numerical solution on a computer is required. Such a solution was obtained for  $\lambda = \sqrt{2}$ , when the equation (41) simplifies in the Fourier representation. The result for  $g^2(L)$  is exhibited in Fig. 7.

It can be seen that the solution (56) corresponding to the first corrections of perturbation theory is valid almost up to  $L_0$ . After that  $g^2(L)$  increases like

$$g^2(L) \propto \exp\{\text{const} \cdot (L/L_0)^2\}. \quad (59)$$

The solution for  $f_T(L)$  tends to the unit fixed point  $f_T = \delta T_0$  according to the law of areas:

$$f_T(L) \rightarrow \exp\{-\text{const} \cdot T(L/L_0)^2\}. \quad (60)$$

(We recall that  $f_T(L)$  has the meaning of the average

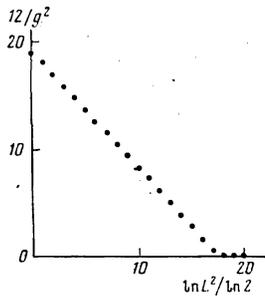


FIG. 7

loop product along the sides of the square of area  $L^2$ , averaged over the fields.) Such a behavior corresponds to the absence of a phase transition.

It is interesting that  $D = 4$  is the critical value of the dimension of space. For  $D = 4 + \epsilon$  there will be a phase transition for  $g_c^2 = \epsilon/\beta_4 + O(\epsilon^2)$ , so that  $f_T(\infty) = 1$  for  $g_0^2 < g_c^2$  and  $f_T(\infty) = \delta T_0$  for  $g_0^2 > g_c^2$ . For  $D \leq 4$  there is never a phase transition.

In the general case the absence of a phase transition in a pure gauge theory in four dimensions corresponds to the following behavior of the function  $\sigma(g_0^2)$  introduced by Wilson (cf. the Introduction)

$$\sigma(g_0^2) \rightarrow \exp\left(\frac{2\beta_4}{\beta_4^2} \ln \frac{1}{g_0^2}\right) \exp\left(-\frac{2}{\beta_4 g_0^2}\right) (c_1 + c_2 g_0^2 \ln g_0^2 + \dots). \quad (61)$$

The coefficient  $\sigma/a^2$  in front of the area of the loop (the tension in the string) remains finite in the local limit of the lattice theory. For large  $g_0$  the function  $\sigma(g_0^2)$  behaves like  $\ln g_0^2 + O(g_0^{-2})^{[2]}$ .

The asymptotic behavior of (61) as  $g_0^2 \rightarrow 0$ , which appears naturally in the recursion approach, seems complicated from the point of view the strong-coupling expansions. It is possible that such a behavior can be justified on the basis of a modified weak-coupling expansion, taking into account the compactness of the gauge group.

## 5. DISCUSSION

In our model the vacuum remains invariant in asymptotically free theories. The invariance of the vacuum leads, in particular, to the unobservability of colored objects. To what extent can our model be trusted?

In the weak coupling region it yields the values (54), (55) for the coefficients of the  $\beta$ -function. The exact values of  $\beta_4$  and  $\beta_6$  are respectively equal to  $11/12\pi^{2[6]}$  and  $17/48\pi^{4[9]}$ . For  $\lambda = 2^{-1/2}$  the error is smaller or approximately equal to 30% in both cases. For  $\lambda = 2$

the error is already large, but the signs remain valid for all  $\lambda$ . One can also check that for the group  $SU(N)$  the coefficients  $\beta_4, \beta_6/\beta_4, \dots$  are proportional to  $C_2(G) = N$ , as it should be. It is probable that this is true for an arbitrary nonabelian group.

The Abelian group  $U(1)$  and the discrete group  $Z_2$  are discussed in a following paper. In both cases the equation yields a correct qualitative picture of the phase transition and correct values for the critical dimensions of the space  $D_c = 4$  and  $D_c = 2$ . The numerical errors in the critical indices and coupling constants are also of the order of 30% for  $D = 4$ . Thus, our equation is capable of detecting a phase transition and of describing its main properties.

The applicability of the equation for  $L > L_0$  is more dubious. For such distances the splitting of the average loop is comparable to  $L_0$  and not to  $L$ , so that our approximation can become inapplicable. Nevertheless, we hope that the vacuum energy and the masses of the excitations will be determined by the region  $L \lesssim L_0$  and can be estimated by means of recursion equations. For this one must make some more steps, and in the first order of business, include the quarks.

I am grateful to A. Polyakov and K. Wilson for useful discussions and also to E. Bogomol'nyi and V. Lysin for numerically solving the equations.

<sup>1</sup>H. Fritsch, M. Gell-Mann, and G. Leutwyler, CalTech Preprint 68-409-1973. S. Weinberg, Phys. Rev. Lett. 31, 494 (1973). G. 'tHooft, Nucl. Phys. B72, 461 (1974). P. Olsen, Preprint, Niels-Bohr-Institute, NBI-HE-74-1, Copenhagen, 1974.

<sup>2</sup>K. Wilson, Phys. Rev. D10, 2445 (1974).

<sup>3</sup>A. Casher, J. Kogut, and L. Susskind, Preprint, Tel-Aviv University, TAUP 373-73, Ramat Aviv, June 1973.

<sup>4</sup>J. Kogut and L. Susskind, Preprint CLNS-276, August 1974 (Cornell?).

<sup>5</sup>S. Mandelstam, Physics Reports C 13, 261 (1974).

<sup>6</sup>D. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1434 (1973). H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).

<sup>7</sup>K. Wilson and J. Kogut, Phys. Reports C 12, 75 (1974).

<sup>8</sup>R. Ballian, J. Ddrouffe, and C. Izykson, Saclay Preprints; Phys. Rev. D 10, 3376 (1974).

<sup>9</sup>A. A. Belyavin and A. A. Migdal, ZhETF Pis. Red. 19, 317 (1974) [JETP Lett. 19, 181 (1974)]. W. Caswell, Phys. Rev. Lett. 33, 244 (1974).

Translated by M. E. Mayer

86