

Expression for energy density in the electrodynamics of a dispersive and absorbing medium

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The method recently proposed (Pekar, 1975) for determining the expression for energy density and evolved heat in the electrodynamics of a dispersive and absorbing medium is discussed. It is demonstrated that the method is erroneous.

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The problem of the energy relations in the electrodynamics of continuous media in the presence of dispersion and absorption as a nontrivial one, and has already been considered repeatedly (see^[1], Sec. 22 and^[2], Sec. 3, where there are also references to a number of books and papers). It was made clear, in particular, that the use of Poynting's theorem is generally completely insufficient for us to obtain an expression for the energy density U . Furthermore, when account is taken of absorption it is generally impossible to express U directly in terms of the dielectric constant $\epsilon(\omega)$ in any sort of general case. This latter is clear both from general considerations and from examples of specific media or electric circuits.

However, it was asserted in a recently published work by Pekar^[3] that U and the loss density Q can be expressed separately and in general form in terms of $\epsilon(\omega)$. Specifically, according to Pekar,

$$\left(\frac{dU}{dt}\right)_P = \frac{1}{4\pi} \frac{\partial D_{\text{Even}}}{\partial t} \mathbf{E} + \frac{1}{4\pi} \frac{\partial \mathbf{B}}{\partial t} \mathbf{B}, \quad Q_P = \frac{1}{4\pi} \frac{\partial D_{\text{Odd}}}{\partial t} \mathbf{E}; \quad (1)$$

$$D_{\text{Even}} = \sum_{p=0}^{\infty} \epsilon_{2p} \frac{d^{2p} \mathbf{E}}{dt^{2p}}, \quad D_{\text{Odd}} = \sum_{p=0}^{\infty} \epsilon_{2p+1} \frac{d^{2p+1} \mathbf{E}}{dt^{2p+1}}, \quad (2)$$

$$\epsilon(\omega) = \sum_{n=0}^{\infty} \epsilon_n (-i\omega)^n.$$

Here $\mathbf{D} = \mathbf{D}_{\text{Even}} + \mathbf{D}_{\text{Odd}}$ is the total displacement vector; for simplicity we limit ourselves to an isotropic non-magnetic medium without spatial dispersion, in which case, for a field of the form $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$, the following are valid: $\mathbf{D} = \epsilon(\omega) \mathbf{E}$, $\mathbf{B} = \mathbf{H}$, $\epsilon(\omega) = \epsilon' + i\epsilon''$, $\epsilon' = \text{Re} \epsilon$, $\epsilon'' = \text{Im} \epsilon$.

The generalization to the case of a magnetic medium described by the permeability $\mu(\omega)$, and also allowance for the anisotropy, do not contribute anything new. On the contrary, for the case of spatial dissipation, the relation

$$\frac{1}{4\pi} \left(\frac{\partial \mathbf{D}}{\partial t} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \mathbf{B} \right)$$

contains, in addition to $dU/dt + Q$, a term which is connected with the energy flux $S^{(1)}$, which is added to the Poynting vector $\mathbf{S} = c(\mathbf{E} \times \mathbf{B})/4\pi$. There is however, no need of discussing such a more general case immediately, inasmuch as the recipe (1) suggested by Pekar is not valid even when applied to a medium characterized by a complex dielectric constant $\epsilon(\omega)$, as will be shown below.

The subdivision of the general expression

$$\frac{1}{4\pi} \left(\frac{\partial \mathbf{D}}{\partial t} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \mathbf{B} \right)$$

into parts, in accord with (1), was referred to above as a recipe, inasmuch as in our opinion it has no basis. It is indicated in^[1] only that $\mathbf{D}_{\text{Odd}} = 0$ in a nonabsorbing medium. This is actually obvious, since

$$i \text{Im} \epsilon(\omega) = \sum_{p=0}^{\infty} \epsilon_{2p+1} (-i\omega)^{2p+1}.$$

But it still does not follow that in (1) U_P is the energy density and Q_P the evolved heat. That the subdivision of (1) is incorrect can be verified by indicating an example, for which this subdivision leads to erroneous results.

By way of such an example or, more precisely, a whole class of examples, we consider a medium consisting of a set of independent oscillators described by the equations

$$\ddot{\mathbf{r}}_k + \nu_k \dot{\mathbf{r}}_k + \omega_k^2 \mathbf{r}_k = \frac{e_k}{m_k} \mathbf{E}, \quad k=1, 2, \dots \quad (3)$$

It is well known that in linear electrodynamics of continuous media, such a model has a very broad range of applicability; in particular, for $\omega_k = 0$, it corresponds to a plasma and leads to results similar to those obtained in the case when the kinetic equation is used (see^[1], Sec. 22.).

We now consider the action on the medium of the field $\mathbf{E} = \text{Re}\{\mathbf{E}_0 e^{-i\omega t}\}$. For this case, we can, as we also do, assume the field $\mathbf{B} = \mathbf{H}$ to be equal to zero, for which, to be sure, certain external currents \mathbf{j}_{ext} must exist. Therefore, we must introduce the term $\mathbf{j}_{\text{ext}} \mathbf{E}$ in the Poynting relation but we shall not dwell on this question, inasmuch as it is not touched on in^[3] and is not important for what follows.

As follows from a simple calculation, for the model considered (see also^[1], Sec. 22; N_k is the number of oscillators of type k) we have

$$\epsilon(\omega) = 1 - \sum_k \frac{\Omega_k^2}{\omega^2 - \omega_k^2 + i\omega\nu_k}, \quad \Omega_k^2 = \frac{4\pi e_k^2 N_k}{m_k}, \quad (4)$$

$$\begin{aligned} \bar{U} = \bar{U}_E &= \frac{\bar{E}^2}{8\pi} + \sum_k N_k \left(\frac{m_k \bar{\dot{\mathbf{r}}}_k^2}{2} + \frac{m_k \omega_k^2 \bar{\mathbf{r}}_k^2}{2} \right) \\ &= \left\{ 1 + \sum_k \frac{\Omega_k^2 (\omega_k^2 + \omega^2)}{(\omega^2 - \omega_k^2)^2 + \omega^2 \nu_k^2} \right\} \frac{|E_0|^2}{16\pi}, \quad Q = \sum_k N_k \nu_k m_k \bar{\dot{\mathbf{r}}}_k^2, \end{aligned} \quad (5)$$

the bar denotes averaging over time. As is clear from (4) and (5), the energy \bar{U} , not to speak of U and Q , when account is taken of absorption, is not expressed in terms of ϵ . If there is no absorption, i.e., $\nu_k = 0$ and $\epsilon = \text{Re} \epsilon$, then

$$\bar{U} = \frac{d(\omega \epsilon)}{d\omega} \frac{|E_0|^2}{16\pi}.$$

For $\omega_k = 0$, we get the plasma model for which, in the case of only a single component with $\Omega_k = \Omega$ and $\nu_k = \nu$, we have

$$\varepsilon = 1 - \frac{\Omega^2}{\omega(\omega + i\nu)}, \quad \bar{U}_E = \left\{ 1 + \frac{\Omega^2}{\omega^2 + \nu^2} \right\} \frac{|E_0|^2}{16\pi},$$

$$\bar{Q} = \frac{\nu\Omega^2}{\omega^2 + \nu^2} \frac{|E_0|^2}{8\pi}. \quad (6)$$

In this case, $\bar{U}_E = \{2 - \text{Re } \epsilon\} |E_0|^2 / 16\pi$, i.e., it is expressed directly in terms of ϵ also in the presence of absorption, but this is an exception.

We now apply the formula (1) to the same case. According to the formula (36) of [3], for a quasimonochromatic electric field,

$$\bar{U}_P = \frac{d(\omega \epsilon_{\text{even}}) |E_0|^2}{d\omega} \frac{1}{16\pi} = \frac{d(\omega \text{Re } \epsilon) |E_0|^2}{d\omega} \frac{1}{16\pi}, \quad (7)$$

where the equality $\epsilon_{\text{even}} = \text{Re } \epsilon$ follows from the fact that

$$\epsilon_{\text{even}}(\omega) = \sum_{p=0}^{\infty} \epsilon_{2p} (-i\omega)^{2p},$$

while

$$\epsilon(\omega) = \sum_{n=0}^{\infty} \epsilon_n (-i\omega)^n,$$

where ϵ_n are real coefficients (see [3], formula (4), and the definition of ϵ_{even} preceding formula (13)). Substituting the expression (4) in (7), we obtain

$$\bar{U}_P = \left\{ 1 + \sum_k \frac{\Omega_k^2 (\omega_k^2 + \omega^2) [(\omega^2 - \omega_k^2)^2 - \omega^2 \nu_k^2]}{[(\omega^2 - \omega_k^2)^2 + \omega^2 \nu_k^2]^2} \right\} \frac{|E_0|^2}{16\pi} \quad (8)$$

or, for a single-component plasma,

$$\bar{U}_P = \left\{ 1 + \frac{\Omega^2 (\omega^2 - \nu^2)}{(\omega^2 + \nu^2)^2} \right\} \frac{|E_0|^2}{16\pi}. \quad (9)$$

Obviously, the expressions (8) and (9) differ, in the presence of absorption, from the elementary and clearly valid expressions (5) and (6). Incidentally, the fact expression (7), which can be negative, i.e.,

$$\frac{d(\omega \epsilon')}{d\omega} \frac{\bar{E}^2}{8\pi} = \frac{d(\omega \text{Re } \epsilon)}{d\omega} \frac{\bar{E}^2}{8\pi},$$

does not represent energy density in the presence of absorption was specially noted in [1] (see p. 366; we note that in [1], $\text{Re } \epsilon$ is denoted by ϵ and U_E by W_E). It is not difficult to establish the fact that the expression (1) for Q_P , which takes the form of (15) from [3] for a quasimonochromatic field, is also incorrect and only the average value \bar{Q}_P is equal to the correct expression

$$\bar{Q} = \frac{\omega \text{Im } \epsilon}{8\pi} |E_0|^2.$$

In the plasma case, especially for $\omega^2 \ll \nu^2$, it is very easy to trace how the expression

$$\frac{1}{4\pi} \frac{\partial D}{\partial t} \mathbf{E} = \frac{dU_E}{dt} + Q$$

breaks up into the parts dU_E/dt and Q , and therefore, $U_E \neq (U_E)_P$ and $Q \neq Q_P$.

¹V. L. Ginzburg, *Rasprostanenie élektromagnitnykh voln v plazme* (Propagation of Electromagnetic Waves in Plasma), Nauka, 1967.

²V. M. Agranovich and V. L. Ginzburg, *Kristaloptika s uchétom prostranstvennoi dispersii i teoriya eksitonov* (Crystal Optics with Account of Spatial Dispersion and Exciton Theory), Nauka, 1965.

³S. M. Pekar, *Zh. Eksp. Teor. Fiz.* **68**, 866 (1975) [*Sov. Phys.-JETP* **41**, 430 (1975)].

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129