Particle creation from the vacuum near a homogeneous isotropic singularity

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A study is made of the creation of boson pairs in Friedmann models of open, closed, and quasi-Euclidean types. A method for regularizing the energy–momentum tensor based on a time-dependent operation of normal ordering is proposed. The "cosmological" {Ya. B. Zel'dovich and A. A. Starobinskiĭ, Zh. Eksp. Teor. Fiz. **61**, 2161 (1971) [Sov. Phys. JETP **34**, 1159 (1972)]} problem of pair creation when initial conditions are specified at the singularity is solved. The vacuum corrections to the energy density and pressure of a scalar field in the closed model are calculated. Expressions are found for the number density of created pairs and also their energy density and pressure for both $t \ll m^{-1}$ and $t \gg m^{-1}$. It is concluded that the creation takes place most intensively when $t \sim m^{-1}$, after which it virtually ceases.

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1. INTRODUCTION

In this paper, we consider the creation of boson pairs in Friedmann models of open, closed, and quasi-Euclidean types. This problem has already been studied on a number of occasions (see, for example, [1–6]), but some of the problems that arise in connection with the description of the creation process have remained unresolved. For example, in[7–10] different methods were proposed for eliminating divergences from the vacuum expectation values of the energy–momentum tensor; these different methods lead to different results, and none of them are adequately justified. According to Hawking,[11] the infinities in the expressions for the energy and pressure cannot be eliminated by means of the operation of normal ordering since its direct application conflicts with the condition that the energy–momentum tensor must be conservative.

Zel'dovich and one of the authors[3] have proposed a method for eliminating the divergences from the expressions for ε and P that keeps the renormalized quantities conservative and does not use the operation of normal ordering. In[3], in particular, they obtained general expressions for ε and P in the quasi-Euclidean case. In this paper (see Secs. 2 and 3) these results, and also analogous results for the open and closed models are obtained by application of an appropriately defined operation of normal ordering. The method developed in[3] thus acquires a more rigorous field-theoretical justification.

In the literature[2,3,12,13] the choice of the vacuum state at an arbitrary time t for the case of a nonstationary external gravitational field has also been discussed. It is shown below that the only physically acceptable vacuum at any time t is the state which is annihilated by the annihilation operators of quasiparticles in terms of which the instantaneous Hamiltonian $H(t)$ constructed from the metric energy–momentum is diagonal. In Sec. 2 it is shown that the use of this "metric" Hamiltonian is equivalent to the use of the canonical Hamiltonian obtained from a Lagrangian that differs from the original Lagrangian by a four-divergence. Since the vacuum state depends on the time, the operation of normal ordering is different for different times.

It should be noted that the operation of normal ordering gives ε and P only for the created particles. In the case of the closed Friedmann model, the vacuum energy density and pressure of a scalar field (i.e., ε and P for the state which is the vacuum at the given time) are not equal to zero. This fact was noted for the first time in[6,14,15] in the case of massless scalar particles. In Sec. 3, the corresponding terms in ε and P are calculated for a massive scalar field. These corrections appear in the vacuum expectation values because the topology of the spatial section of the closed Friedmann model (S^3) is different from that of the spatial section of the flat and the open model (R^3), in which there are no "topological" corrections.

In[3], the problem was solved of pair creation during isotropic collapse, and an expression was found for the time dependence of the particle number, the energy, and the pressure as the singularity is approached. In[3], the possibility was also pointed out of posing, in the isotropic case, the opposite, "cosmological," problem in which the initial conditions are specified in the singularity itself (which in this case is a regular point of the corresponding equation). In Sec. 4, the cosmological problem of pair creation is solved under the condition $t \ll m^{-1}$. It is shown that the created matter satisfies the "vacuum-like" equation of state $P \approx -\varepsilon$. Further, the cosmological problem of pair creation is solved under the condition $t \gg m^{-1}$. In this case, the estimate $|P| \ll \varepsilon$ is obtained for the pressure. Comparison with the preceding case shows that the creation takes place most intensively for $t \sim m^{-1}$. This fact justifies the use of the approximation of a classical gravitational field in the cosmological problem of particle creation since the effects associated with quantization of gravitation can appear only when $t \sim t_{Pl} \ll m^{-1}$, where $t_{Pl} = \sqrt{G} \sim 10^{-43}$ sec. Here and below, we use units in which $\hbar = c = 1$.

2. SCALAR FIELD IN HOMOGENEOUS ISOTROPIC SPACE

We take the metric of the homogeneous isotropic models in the conformally static form

$$ds^2=a^2(\eta)[d\eta^2-dl^2]=a^2(\eta)[d\eta^2-d\chi^2-f^2(\chi)(d\theta^2+\sin^2\theta d\varphi^2)], \quad (1)$$

where $f(\chi)=\sinh\chi$, χ, $\sin\chi$, respectively, for the cases when the three-space with metric dl^2 has curvature $\varkappa=-1, 0, 1$; $a(\eta)d\eta=dt$.

The Lagrangian density of a charged scalar field with conformal coupling is

$$\mathscr{L}=\sqrt{-g}L=\sqrt{-g}\left[g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}^{*}-\left(m^2-\frac{R}{6}\right)\Phi\Phi^{*}\right], \quad (2)$$

where R is the scalar curvature of spacetime. The Lagrangian (2) corresponds to the field equation

$$(\nabla_\mu\nabla^\mu+m^2-R/6)\Phi(x)=0, \quad (3)$$

where ∇_μ is the covariant derivative. The metric energy-momentum tensor of the field Φ is[12]

$$T_{\mu\nu}=2\Phi_{,(\mu}\Phi_{,\nu)}^{*}-g_{\mu\nu}L+\frac{1}{3}(R_{\mu\nu}-\nabla_\mu\nabla_\nu+g_{\mu\nu}\nabla_\lambda\nabla^\lambda)\Phi\Phi^{*}. \quad (4)$$

The solutions of Eq. (3) can be obtained by separation of variables:

$$\Phi(x)=a^{-1}\varphi(x)=a^{-1}(\eta)u_\lambda(\eta)\psi_J(\mathbf{x}), \quad (5)$$

where $\psi_J(\mathbf{x})$ are eigenfunctions of the Laplacian $\Delta_{(3)}$; $J\equiv\{\lambda, l, m\}$, where $0<\lambda<\infty$, $l=0, 1, 2, \ldots$ in the open case, $\lambda=1, 2, 3, \ldots$, $l=0, 1, 2, \ldots, \lambda-1$ in the closed case, and $-l\leq m\leq l$ is integral. After the separation of the variables in (5) we obtain for $u_\lambda(\eta)$ an equation of oscillator type with variable frequency:

$$d^2u_\lambda(\eta)/d\eta^2+\omega^2(\eta)u_\lambda(\eta)=0,$$
$$\omega^2(\eta)=\lambda^2+m^2a^2(\eta). \quad (6)$$

The explicit expressions for the functions $\psi_J(\mathbf{x})$ and also a number of their properties are given in[16,17].

In (5), the scale factor $a(\eta)$ is separated out explicitly. The value of this derives from the fact that the metric (1) is conformally static, and Eq. 3 for $m=0$ is invariant under a conformal transformation of the metric, $g_{\mu\nu}\to\tilde{g}_{\mu\nu}=e^{-2\sigma}g_{\mu\nu}$, and a simultaneous transformation of the field $\Phi\to\varphi=e^{\sigma}\Phi$.[12,18] At the same time, the Lagrangian (2) can be expressed in terms of $\varphi(x)$ in the form

$$L=L'-\frac{1}{\sqrt{-\tilde{g}}}\partial^\mu(\sqrt{-\tilde{g}}\,\sigma_{,\mu}\varphi\varphi^{*}),$$
$$L'=\tilde{g}^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}^{*}-(m^2e^{2\sigma}-\tilde{R}/6)\varphi\varphi^{*} \quad (7)$$

($\tilde{R}$ is the scalar curvature in the metric $\tilde{g}_{\mu\nu}$; in our case (1), we have $\tilde{R}=-6\varkappa$).

As can be seen from (7), the Lagrangian L' differs from L by a four-divergence, and therefore leads to the same equations of motion as L in (2). One can show that the canonical Hamiltonian constructed from L' is equal to the metric Hamiltonian $H(\eta)$ constructed from the energy-momentum tensor (4). At the same time, the canonical Hamiltonian $H^{\text{can}}(\eta)$ constructed from the original Hamiltonian L (7) differs from $H(\eta)$ by a quantity that in quantum theory is an infinite C-number. Below, we use the metric Hamiltonian $H(\eta)$, which leads to finite results for the number density of the created pairs.

Obviously, the field $\varphi(x)$ can be interpreted as a field with variable mass $\mu^2(\eta)=m^2a^2(\eta)$ in the static spacetime with metric $\tilde{g}_{\mu\nu}$. This is a consequence of the fact that the metric (1) is conformally static. Note that the metric (1) is also conformally flat for $\varkappa=-1, 0, 1$.

The operator of the charged boson field can be represented as an expansion with respect to the complete orthonormal system (5):

$$\Phi(x)=\frac{1}{\sqrt{2}a}\int dJ\{\psi_J(\mathbf{x})u_\lambda^{*}(\eta)a_J^{(-)}+\psi_J^{*}(\mathbf{x})u_\lambda(\eta)a_J^{(+)}\}, \quad (8)$$

where $\int dJ$ denotes integration with respect to the continuous quantum numbers and summation over the discrete ones; $a_J^{*(+)}$ and $a_J^{(-)}$ are operators of creation and annihilation of particles and $a_J^{(+)}$ and $a_J^{*(-)}$ are those for antiparticles; these satisfy the usual commutation relations. The initial conditions on the functions $u_\lambda(\eta)$ are fixed below.

The vacuum state corresponding to the operators $a_J^{(\pm)}$ and $a_J^{*(\pm)}$ is determined by the equations

$$a_J^{(-)}|0\rangle=a_J^{*(-)}|0\rangle=0. \quad (9)$$

Using (4), (5), (8), and also the properties of the functions $\psi_J(\mathbf{x})$,[16,17] we obtain an expression for the Hamiltonian of the field $\Phi(x)$ in terms of the creation and annihilation operators:

$$H(\eta)=\int d\sigma^\mu T_{0\mu}(x)=\int dJ\,\omega(\eta)\{E_\lambda(\eta)(a_J^{*(+)}a_J^{(-)}+a_J^{*(-)}a_J^{(+)})$$
$$+F_\lambda(\eta)a_J^{*(+)}a_{\bar{J}}^{(+)}+F_\lambda^{*}(\eta)a_{\bar{J}}^{*(-)}a_J^{(-)}\}, \quad (10)$$

where

$$E_\lambda(\eta)=(2\omega)^{-1}[|u_\lambda'|^2+\omega^2|u_\lambda|^2],$$
$$F_\lambda(\eta)=(2\omega)^{-1}[u_\lambda'^2+\omega^2u_\lambda^2],$$

and $\bar{J}=\{\lambda, l, -m\}$ (the prime denotes differentiation with respect to η). As can be seen from (10), the Hamiltonian is nondiagonal in the creation and annihilation operators, which, in complete analogy with the case of the electrodynamics of nonstationary external fields,[19-22] leads to the creation of particle-antiparticle pairs from the vacuum.

We assume that at a certain time η_0 the boson field under consideration is in the vacuum state. This means that $E(\eta_0)=1$, $F(\eta_0)=0$; the last condition is ensured, in accordance with (10), by the following initial conditions for Eq. (6):

$$u_\lambda(\eta_0)=\omega^{-1/2}(\eta_0), \quad u_\lambda'(\eta_0)=i\omega^{1/2}(\eta_0). \quad (11)$$

Since the point $\eta=0$, a singular point for the Friedmann models, is a regular point of Eq. (6), we can set $\eta_0=0$ in (11) ("cosmological" formulation of the problem[3]). It then becomes possible to calculate the energy, pressure, and number density of the particles created during the whole of the evolution of the universe from the singular state.

3. GENERAL EXPRESSIONS FOR THE PARTICLE NUMBER DENSITY, ENERGY DENSITY, AND PRESSURE

An exact solution of the problem of determining the number of particles, the energy, and the pressure can be obtained by diagonalizing the Hamiltonian (10) by means of a Bogolyubov transformation of the creation and annihilation operators:

$$\begin{gathered} a_J^{(-)}=\alpha_\lambda^*(\eta)b_J^{(-)}(\eta)-(-1)^m\beta_\lambda^*(\eta)b_J^{(+)}(\eta), \\ a_J^{(+)}=\alpha_\lambda(\eta)b_J^{(+)}(\eta)-(-1)^m\beta_\lambda(\eta)b_J^{(-)}(\eta), \\ |\alpha_\lambda|^2-|\beta_\lambda|^2=1. \end{gathered} \tag{12}$$

Substituting (12) into (10) and requiring that there be no nondiagonal terms $b_J^{*(+)}b_J^{(+)}$, $b_J^{*(-)}b_J^{(-)}$, we obtain

$$\frac{\beta_\lambda^*}{\alpha_\lambda}=\frac{E_\lambda-1}{F_\lambda^*},\quad |\beta_\lambda|^2=\frac{1}{4\omega}(|u_\lambda'|^2+\omega^2|u_\lambda|^2-2\omega). \tag{13}$$

From (6) and (13) we can obtain a system of two linear equations directly for α_λ and β_λ:

$$\begin{gathered} \alpha_\lambda'=\frac{\omega'}{2\omega}\beta_\lambda\exp\left(2i\int_{\eta_0}^{\eta}\omega\,d\eta\right), \\ \beta_\lambda'=\frac{\omega'}{2\omega}\alpha_\lambda\exp\left(-2i\int_{\eta_0}^{\eta}\omega\,d\eta\right), \end{gathered} \tag{14}$$

with initial conditions $\alpha_\lambda(\eta_0)=1$, $\beta_\lambda(\eta_0)=0$. The values of α_λ and β_λ are equal to the values introduced in[3] on the basis of other arguments. The functions $u_\lambda(\eta)$ and $u_\lambda'(\eta)$ can be expressed in terms of $\alpha_\lambda(\eta)$ and $\beta_\lambda(\eta)$ as follows:

$$\begin{gathered} u_\lambda=\frac{1}{\sqrt{\omega(\eta)}}\left[\alpha_\lambda^*\exp\left(i\int_{\eta_0}^{\eta}\omega\,d\eta\right)+\beta_\lambda^*\exp\left(-i\int_{\eta_0}^{\eta}\omega\,d\eta\right)\right], \\ u_\lambda'=i\sqrt{\omega(\eta)}\left[\alpha_\lambda^*\exp\left(i\int_{\eta_0}^{\eta}\omega\,d\eta\right)-\beta_\lambda^*\exp\left(-i\int_{\eta_0}^{\eta}\omega\,d\eta\right)\right]. \end{gathered} \tag{15}$$

The Hamiltonian (10) takes the form

$$H(\eta)=\int dJ\,\omega(\eta)[b_J^{(-)}b_J^{*(+)}+b_J^{(+)}b_J^{*(-)}]. \tag{16}$$

The "quasiparticle" operators $b_J^{(\pm)}(\eta)$ and $b_J^{*(\pm)}(\eta)$ give one the possibility of representing the boson field in the external gravitational field at any time as a collection of free particles with energies $\omega_\lambda(\eta)$, the interaction with the external field appearing as a redefinition of the concept of a particle at each subsequent time.[2,22,23] Accordingly, the physical vacuum state defined by the equations

$$b_J^{(-)}(\eta)|0_\eta\rangle=b_J^{*(-)}(\eta)|0_\eta\rangle=0, \tag{17}$$

depends on the time. For $\eta=\eta_0$, the state $|0_{\eta_0}\rangle$ agrees with $|0\rangle$ in (9) since $\beta_\lambda(\eta_0)=0$, $\alpha_\lambda(\eta_0)=1$.

To obtain finite results from the calculation of the matrix elements, it is necessary to express the operators of the physical quantities in normal form and subtract the energy of the vacuum fluctuations. And, since the physical vacuum defined in (17) is different at different times, the energy of the vacuum fluctuations subtracted during the reduction of the operators to normal form depends on the time. Thus, the normally ordered energy–momentum tensor has the form

$$N_\eta(T_{\mu\nu}(x))=T_{\mu\nu}(x)-\langle 0_\eta|T_{\mu\nu}(x)|0_\eta\rangle. \tag{18}$$

Using the rule we have formulated, we now calculate the number density of the created particles, and also their energy and pressure.

The operators of the number density of particles and antiparticles in state J in normal form are, respectively

$$n_J=b_J^{*(+)}(\eta)b_J^{(-)}(\eta),\quad \bar n=b_J^{(+)}(\eta)b_J^{*(-)}(\eta). \tag{19}$$

Using (12), we obtain an expression for the number of particles and antiparticles created from the vacuum during the time that elapses from η_0 to η in unit volume of physical space:

$$n(\eta)=\bar n(\eta)=\frac{1}{2\pi^2a^3}\int d\lambda\,\lambda^2\langle 0|n_J|0\rangle=\frac{1}{2\pi^2a^3}\int d\lambda\,\lambda^2|\beta_\lambda|^2. \tag{20}$$

Similarly, we can calculate the energy and pressure of the created pairs:

$$\varepsilon(\eta)=\langle 0|N_\eta(T_0^0(x))|0\rangle,\quad P(\eta)=-\langle 0|N_\eta(T_i^i(x))|0\rangle \tag{21}$$

($i=1,\ 2,\ 3$; no summation over i). For this, we use the composition formulas for the functions $\psi_J(x)$ obtained in[16,17]. As a result, for the functions (21) we obtain

$$\begin{gathered} \varepsilon(\eta)=\frac{1}{\pi^2a^4}\int d\lambda\,\lambda^2\omega(\eta)|\beta_\lambda(\eta)|^2, \\ P(\eta)=\frac{1}{3\pi^2a^4}\int d\lambda\,\lambda^2\omega(\eta)[|\beta_\lambda(\eta)|^2-\gamma_\lambda(\eta)], \end{gathered} \tag{22}$$

where

$$\gamma_\lambda(\eta)=\frac{m^2a^2}{2\omega}\left(|u_\lambda|^2-\frac{1}{\omega}\right).$$

In the closed model, the integration in (20) and (22) is replaced by the summation over $\lambda=1,\ 2,\ldots$.

It is easy to show that $\langle 0|N_\eta(T^\nu_\mu(x))|0\rangle=0$ for the nondiagonal components of the energy-momentum tensor, as must be in the homogeneous isotropic case. One can show that the vacuum expectation values of the normally ordered energy–momentum tensor have the property of being conservative:

$$\nabla_\nu\langle 0|N_\eta(T_\mu^\nu(x))|0\rangle=0.$$

Equation (20) for the special case of the quasi-Euclidean Friedmann model ($\varkappa=0$) was obtained in[1,2]. Equations (22) in the case $\varkappa=0$ were obtained by a different method in[3] without the use of the normal ordering operation.

By its very definition, the operation of normal ordering (18) gives ε and P only for the created particles. In quantum electrodynamics it is well known that the vacuum (i.e., the state without particles) can also have nonzero ε and P. An example of this is the Casimir effect, in which the presence of vacuum energy leads to a force of attraction between two parallel uncharged metallic or dielectric plates.[24,25] This effect predicted by theory has been experimentally confirmed.

By analogy, we must expect that in the problem considered here the vacuum energy may, under certain conditions, be nonzero. Let us therefore consider in more detail the divergent terms that were ignored in the energy-momentum tensor of the scalar field when we applied the normal ordering operation. In the case $\varkappa = 0, -1$, these terms have the form

$$\varepsilon_0 = \frac{1}{2\pi^2 a^4}\int_0^\infty d\lambda\, \lambda^2 \omega(\eta) = \frac{1}{2\pi^2}\int_0^\infty dp\ p^2(p^2+m^2)^{1/2},$$
$$P_0 = \frac{1}{6\pi^2 a^4}\int_0^\infty d\lambda \frac{\lambda^4}{\omega(\eta)} = \frac{1}{6\pi^2}\int_0^\infty dp \frac{p^4}{(p^2+m^2)^{1/2}}, \quad (23)$$

where $p = \lambda/a$ is the physical momentum.

As usual, we shall assume that the energy density and the pressure of the vacuum in the flat pseudo-Euclidean space time are equal to zero. It can be seen that for $\varkappa = 0, -1$ no additional corrections to the flat spacetime values of ε_0 and P_0 arise. Therefore, in the flat and open Friedmann models, the renormalized energy density ε_v and the pressure P_v of the vacuum $|0_\eta\rangle$ are equal to zero.

For $\varkappa = 1$, the integration with respect to λ in Eqs. (23) is replaced by summation over $\lambda = 1, 2, \ldots$. We show that the resulting vacuum corrections ε_v and P_v to the expressions for flat spacetime,

$$\varepsilon_v = \frac{1}{2\pi^2 a^4}\left(\sum_{\lambda=1}^\infty \lambda^2(\lambda^2+m^2a^2)^{1/2} - \int_0^\infty d\lambda\, \lambda^2(\lambda^2+m^2a^2)^{1/2}\right),$$
$$P_v = \frac{1}{6\pi^2 a^4}\left(\sum_{\lambda=1}^\infty \frac{\lambda^4}{(\lambda^2+m^2a^2)^{1/2}} - \int_0^\infty d\lambda \frac{\lambda^4}{(\lambda^2+m^2a^2)^{1/2}}\right), \quad (24)$$

are finite and do not depend on the renormalization method. We introduce into the sum and integral in (24) a cutoff function $F(\lambda)$, which we shall assume is analytic for $\mathrm{Re}\lambda \geq 0$ and decreases sufficiently rapidly as $\mathrm{Re}\lambda \to \infty$ for the integral and the sum to converge. One can then apply the summation formula (see, for example, [26], p. 38 of the Russian translation)

$$\sum_{n=0}^\infty \varphi(n) = \frac{\varphi(0)}{2} + \int_0^\infty \varphi(\tau)\,d\tau + i\int_0^\infty \frac{\varphi(i\tau)-\varphi(-i\tau)}{e^{2\pi\tau}-1}\,d\tau. \quad (25)$$

In view of the very good convergence of the last integral in (25), we can set $F(\lambda) \equiv 1$ in it. It is then obvious that the expression (24) do not depend on the form of the cutoff function $F(\lambda)$. Bearing in mind that the function $\varphi(\lambda) = \lambda^2(\lambda^2 + m^2a^2)^{1/2}$ has branch points for $\lambda = \pm ima$, so that

$$\varphi(i\lambda) = \varphi(-i\lambda) = -\lambda^2(m^2a^2-\lambda^2)^{1/2} \quad \text{for} \quad \lambda < ma,$$
$$\varphi(i\lambda) = -\varphi(-i\lambda) = -i\lambda^2(\lambda^2-m^2a^2)^{1/2} \quad \text{for} \quad \lambda > ma$$

(the branch points of $\varphi(\lambda)$ are passed in the half-plane $\mathrm{Re}\lambda > 0$), we obtain

$$\varepsilon_v = \frac{1}{\pi^2 a^4}\int_{ma}^\infty \frac{\lambda^2(\lambda^2-m^2a^2)^{1/2}}{e^{2\pi\lambda}-1}\,d\lambda,$$
$$P_v = \frac{1}{3\pi^2 a^4}\int_{ma}^\infty \frac{\lambda^4\,d\lambda}{(\lambda^2-m^2a^2)^{1/2}(e^{2\pi\lambda}-1)}. \quad (26)$$

Thus, in the case of the closed Friedmann model, the total expectation value of the operator of the energy-momentum tensor of the scalar field is made up of the energy density and pressure of the created particles (22) and the vacuum corrections (26):

$$\varepsilon_{tot} = \varepsilon + \varepsilon_v, \quad P_{tot} = P + P_v.$$

ε_v and P_v satisfy the condition of being conservative:

$$\frac{1}{a^3}(\varepsilon_v a^3)' = -3P_v\frac{a'}{a},$$

so that the total expectation value of $T_{\mu\nu}$ also satisfies this condition. Note that direct application to the unrenormalized expectation values $\varepsilon + \varepsilon_0$ and $P + P_0$ of the renormalization method proposed in[3] leads to the same expressions for ε_{tot} and P_{tot}.

One can say that ε_v and P_v arise from the non-Euclidean topology of the spatial section of the closed Friedmann model (S^3) since it is the change in the topology which causes the continuous spectrum of energies and momenta of the scalar particles to become discrete and the integral in (23) to be replaced by a sum. In the case $m = 0$, the topological effect appears in its cleanest form: For $m = 0$ it follows from Eqs. (14) that $\beta_\lambda \equiv 0$ so that $\varepsilon = P = 0$, and therefore

$$\varepsilon_{tot} = \varepsilon_v = \frac{1}{240\pi^2 a^4(\eta)}, \quad P_{tot} = P_v = \frac{1}{3}\varepsilon_v \quad (27)$$

(in the case of a neutral massless scalar field the result is half this; see[14,15]). It is interesting that, as can be seen from (26), the distribution of the virtual particles over the energies for $m = 0$ is exactly the same as a Planck distribution with temperature (in ordinary units) $kT = \hbar c/2\pi a(\eta)$. An analogous result in the problem of the creation of massless particles by black holes has been obtained by Hawking.[27] If $m \neq 0$, then ε in (26) gives a distribution of the virtual particles that is the same as the distribution of a Bose gas of relativistic particles with chemical potential equal to zero.

If $ma(\eta) \gg 1$, then ε_v and P_v are exponentially small:

$$\varepsilon_v = e^{-2\pi ma}\frac{(ma)^{5/2}}{4\pi^3 a^4},$$
$$P_v = e^{-2\pi ma}\frac{(ma)^{7/2}}{6\pi^2 a^4} \gg \varepsilon_v. \quad (28)$$

We now turn to the calculation of n, ε, and P near the singularity.

4. CALCULATION OF THE PARTICLE NUMBER DENSITY, ENERGY, AND PRESSURE OF THE CREATED MATTER IN THE EARLY EVOLUTION OF THE UNIVERSE

Near the singularity ($t = 0$) for $\varkappa = 0, \pm 1$ one can assume $a(t) = a_0 t^q$, where $0 < q < 1$. This behavior of $a(t)$ is characteristic of the solutions of Einstein's equations near the singularity; the background matter, which determines the metric, has the equation of state $P_b = (2/3q - 1)\varepsilon_b$. Then

$$a(\eta) = a_1\eta^{q/(1-q)}, \quad a_1 = a_0^{1/(1-q)}(1-q)^{q/(1-q)}.$$

For real models of the Universe, the influence of the spatial curvature on the expansion at $t \sim m^{-1}$ is still very small, i.e., $a(t \sim m^{-1}) \gg m^{-1}$, and therefore $ma_1 \gg 1$ and $\eta(t \sim m^{-1}) \ll 1$.

We specify the initial conditions (11) at the singularity; then for $t=\eta=0$ we must have $\beta_\lambda=0$, $\alpha_\lambda=1$. We consider first the region $mt\ll 1$. For $\lambda\eta\ll 1$, the exponentials in the system of equations (14) can be omitted. We obtain

$$\alpha_\lambda=\frac{1}{2}\left(\sqrt{\frac{\omega}{\lambda}}+\sqrt{\frac{\lambda}{\omega}}\right),\quad \beta_\lambda=\frac{1}{2}\left(\sqrt{\frac{\omega}{\lambda}}-\sqrt{\frac{\lambda}{\omega}}\right)$$
$$|\beta_\lambda|^2=\frac{(\omega-\lambda)^2}{4\lambda\omega},\quad \gamma_\lambda=\frac{m^2a^2(\omega-\lambda)}{2\omega^2\lambda}. \tag{29}$$

Then the density (20) of created particles for $mt\ll 1$ is

$$n=\bar{n}=m^3/24\pi^2. \tag{30}$$

In the first approximation, n does not depend on the expansion law $a(t)$ (however, it is necessary that the integral $\int dt/a=\eta$ converge at the point $t=0$). Thus, for $t\ll m^{-1}$ the particle creation proceeds at just such a rate as to keep the particle density constant, despite the expansion of the Universe. The following term in the expansion of $n(t)$ for $mt\ll 1$ is of order $m^3\cdot mt$ and depends on $a(t)$. Note that the main contribution to the integral with respect to λ in Eq. (20) is given by the region $\lambda\sim ma$.

To calculate ε and P, it is insufficient to use the approximation (29), since in this case the region of large momenta λ makes an important contribution. Let us consider the region $\lambda\gg ma$. In it, $|\beta_\lambda|\ll 1$, $\alpha_\lambda\approx 1$, and therefore the system of equations (14) can be solved by the method of successive approximations. In the first approximation,

$$\beta_\lambda=\frac{m^2}{4\lambda^2}\int_0^\eta\frac{da^2(\eta_1)}{d\eta_1}e^{-2i\lambda\eta_1}d\eta_1,\quad \alpha_\lambda=1,$$
$$|\beta_\lambda|^2=\frac{m^4}{16\lambda^4}\int_0^\eta d\eta_1\frac{da^2(\eta_1)}{d\eta_1}\int_0^\eta d\eta_2\frac{da^2(\eta_2)}{d\eta_2}\cos 2\lambda(\eta_1-\eta_2), \tag{31}$$
$$\gamma_\lambda=\frac{m^2a^2}{\lambda}\left[|\beta_\lambda|^2+\frac{m^2}{4\lambda^2}\int_0^\eta\frac{da^2(\eta_1)}{d\eta_1}\cos 2\lambda(\eta-\eta_1)d\eta_1\right].$$

The regions of applicability of the approximations (29) and (31) for $mt\ll 1$ overlap. Substituting (29) and (31) into (22), we find that when $mt\ll 1$ (see Appendix I for the details of the calculation)

$$\varepsilon=\frac{m^4}{16\pi^2}\left(\ln\frac{1}{mt}+D\right),\quad P=-\frac{m^4}{16\pi^2}\left(\ln\frac{1}{mt}+D-\frac{1}{3q}\right), \tag{32}$$

where

$$D=\Psi\left(\frac{1+q}{1-q}\right)+\ln(1-q)+\frac{1}{4q}-\frac{1}{2}.$$

Here, $\Psi(z)=\Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function. If, in particular, $q=\frac{1}{2}$ (radiation-dominated background matter), then $D=-C-\ln 2+\frac{3}{2}$, where $C=0.577\ldots$ is Euler's constant. Note that for all power laws of expansion the created matter satisfies the "vacuum-like" equation of state $P\approx-\varepsilon$ [28,29] for $t\ll m^{-1}$. At the same time $\varepsilon+P\approx m^4/48\pi^2q>0$ but $\varepsilon+3P<0$, i.e., for the created matter the dominant-energy condition is violated (see[11]). It should however be borne in mind that the energy density of the background appreciably exceeds the value given by (32).

Equations (30) and (32) refer to the case of the open and quasi-Euclidean Friedmann models ($\varkappa=0,\ -1$). Because of the condition $ma_1\gg 1$, they are also true for the closed model ($\varkappa=1$) for $t\ll m^{-1}$, except for a very short interval of time at the start of expansion ($t\lesssim t_1=m^{-1}(ma_1)^{(q-1)/q}$) when $ma(\eta)\lesssim 1$. The values of n, ε, and P for $ma(\eta)\ll 1$ are given in Appendix I. Estimates show that for real models of the Universe ma_1 is so large (for example, $ma_1\gtrsim 10^{40}$ for $q=\frac{1}{2}$ and $m=10^{-24}\ g$) that the time t_1 is less than t_{Pl}, and therefore the region $0<t\lesssim t_1$ is not of physical interest. For $ma(\eta)\gg 1$, as can be seen from (25), the corrections that arise from the replacement of the integrals in (20) and (22) by sums are exponentially small; in addition, $\varepsilon_v\ll\varepsilon$. Thus, Eqs. (30) and (32) are true for $\varkappa=0,\ \pm 1$ and $t\ll m^{-1}$.

We now consider the region $t\gg m^{-1}$. The solution of the system of equations (14) in the first approximation in $(mt)^{-1}$ has the form

$$\alpha_\lambda=\alpha_0-\beta_0\frac{i\omega'}{4\omega^2}\exp\left(2i\int_0^\eta\omega\,d\eta\right),$$
$$\beta_\lambda=\beta_0+\alpha_0\frac{i\omega'}{4\omega^2}\exp\left(-2i\int_0^\eta\omega\,d\eta\right), \tag{33}$$
$$|\beta_\lambda|^2=n(\lambda)+\frac{\omega'^2}{16\omega^4}+i\left[\alpha_0\beta_0^*\exp\left(-2i\int_0^\eta\omega\,d\eta\right)\right.$$
$$\left.-\alpha_0^*\beta_0\exp\left(2i\int_0^\eta\omega\,d\eta\right)\right]\frac{\omega'}{4\omega^2},$$

where α_0 and β_0 are functions of λ, $n(\lambda)=|\beta_0|^2=\lim|\beta_\lambda|^2$, $t\to\infty$. As $t\to\infty$, we have α_λ, $\beta_\lambda\to$ const, so that the particle creation ceases. Forming the Wronskians from the different solutions of Eq. (6), we can readily verify that $n(\lambda)$ is equal to the function $n(k)$ introduced in Sec. 3 of[3] in the problem of particle creation in the case of isotropic collapse. Therefore, the number of particles created in the case of expansion is the same as in the case of contraction with the same law of variation $a(|\eta|)$.

The contribution to ε and n from the third, oscillating term in the expression (33) for $|\beta_\lambda|^2$ is always much less than the contribution of the first term. The second term in $|\beta_\lambda(\eta)|^2$ gives the following contribution to n and ε:

$$n^{(2)}=\frac{m}{512\pi}\frac{1}{a^2}\left(\frac{da}{dt}\right)^2,\quad \varepsilon^{(2)}=\frac{m^2}{48\pi^2}\frac{1}{a^2}\left(\frac{da}{dt}\right)^2. \tag{34}$$

For $0<q<\frac{2}{3}$, the main contribution to n and ε is made by the first term in $|\beta_\lambda|^2$, i.e., $n(\lambda)$, and for $t\gg m^{-1}$ we then obtain

$$n=\bar{n}=\frac{1}{2\pi^2a^3}\int_0^\infty d\lambda\,\lambda^2n(\lambda)\propto m^3(mt)^{-3q},$$
$$\varepsilon=m(n+\bar{n})=2mn. \tag{35}$$

The main contribution to the integrals over λ in (35) comes from the region of small momenta $\lambda/a\sim m(mt)^{-q}\ll m$, so that the energy of a created pair is nearly equal to twice the rest mass.

Let us investigate the properties of the limit distribution function $n(\lambda)$ for small and large momenta. For $a(t)=a_0t^q$, the function $n(\lambda)$ depends only on the dimensionless parameter

$$\delta=\frac{\lambda}{a_0m^{1-q}}=\frac{\lambda}{(ma_1)^{1-q}}(1-q)^q. \qquad (36)$$

Here, δ is the ratio of the physical momentum λ/a at time $t=m^{-1}$ (at this time, the creation process is taking place most intensively) to the mass. In the case of the quasi-Euclidean Friedmann model, δ is invariant under a scale transformation of the spatial coordinates.

When $\delta\ll 1$ ($\lambda\ll(ma_1)^{1-q}$; we recall that $ma_1\gg 1$) we can ignore λ^2 in Eq. (6) compared with $m^2a^2(\eta)$, and then the solution of this equation satisfying the initial conditions (11) for $\eta_0=0$ is

$$u_\lambda\approx\frac{1}{\sqrt{\lambda}}\Gamma\left(\frac{1+q}{2}\right)\left(\frac{mt}{2}\right)^{(1-q)/2}J_{-(1-q)/2}(mt). \qquad (37)$$

Therefore, for $\delta\ll 1$

$$n(\delta)=\frac{1}{\pi}2^{q-2}\Gamma^2\left(\frac{1+q}{2}\right)\delta^{-1}. \qquad (38)$$

For $\delta\gg 1$, the value of $n(\delta)$ is largely determined by the law of variation of the metric near the singularity, and one can therefore formally apply perturbation theory (cf. the determination of the coefficient of superbarrier reflection in the case when the potential has an inflection[30]). In the present case, $U(\eta)=-m^2a^2(\eta)\theta(\eta)$ is the perturbation. The calculations give

$$n(\lambda)=\frac{m^4}{4\lambda^2}\left|\int_0^\infty e^{-2i\lambda\eta}a^2(\eta)\,d\eta\right|^2$$
$$=\frac{1}{16}\left(\frac{1-q}{2}\right)^{4q/(1-q)}\Gamma^2\left(\frac{1+q}{1-q}\right)\delta^{-4/(1-q)} \qquad (39)$$

for $\delta\gg 1$. The integration with respect to η in (39) is performed by introducing the factor $e^{-\mu\eta}$ into the integrand and then setting $\mu=0$ in the result. Thus, $n(\delta)\propto\delta^{-1}$ for $\delta\ll 1$ and $n(\delta)\propto\delta^{-4/(1-q)}$ for $\delta\gg 1$, so that the integral in (35) converges.

The solution of Eq. (6) can be expressed in terms of known transcendental functions for any λ if $q=\frac{1}{2}$ (and then $P_b=(\frac{1}{3})\varepsilon_b$) or $q=\frac{1}{3}$ ($P_b=\varepsilon_b$). These solutions and the exact expressions obtained from them for $n(\lambda)$ are given in Appendix II. In the case of greatest interest for astrophysical applications, $q=\frac{1}{2}$, numerical integration of the exact solution gives the following result: for $t\gg m^{-1}$

$$n\approx 5.3\cdot 10^{-4}\,m^3(mt)^{-3/2},\qquad \varepsilon=2mn. \qquad (40)$$

For $0<q<\frac{1}{2}$ and $t\gg m^{-1}$, the pressure satisfies $|P|\ll\varepsilon$. For $q=\frac{1}{2}$, we have the estimate

$$|P|<10^{-3}\,m^{5/2}t^{-3/2}/\ln mt,$$

so that under the additional condition $\ln mt\gg 1$ we again have $|P|\ll\varepsilon$. In the interval $\frac{1}{2}<q<\frac{2}{3}$, $P\sim\varepsilon\sin mt$, and the nonoscillating part of the pressure is of order $t^{-2}\ll\varepsilon$. When $\frac{2}{3}<q<1$, we obtain $P=(2/3q-1)\varepsilon\sim t^{-2}$, where ε is given by the expression (34).

Introducing the Lagrangian volume $V(t)\sim a^3\sim t^{3q}$, for the number of particles in this volume for $0<q\leqslant\frac{2}{3}$ we obtain from (30) and (35)

$$N(t)\sim nt^{3q}\sim\begin{cases} m^3(mt)^{3q} & \text{for } mt\ll 1\\ \text{const} & \text{for } mt\gg 1\end{cases}.$$

This dependence $N(t)$ enables us to conclude that the creation of particles takes place most intensively when $t\sim m^{-1}$, after which it virtually ceases. For $t\gg m^{-1}$ and $0<q\leqslant\frac{2}{3}$ we have basically particles created during the earlier stages in the evolution of the Universe.

If it is assumed that the matter density in the Universe is critical, then $q=\frac{1}{2}$ up to the time t_c at which the equation state changes ($t_c\sim 10^{13}$ sec), and then $q=\frac{2}{3}$. Then the ratio of the energy density ε of the created particles to the energy density ε_b of the background matter for $t>t_c$ (including at the present epoch) is constant and very small for all known elementary particles:

$$\varepsilon/\varepsilon_b\sim Gm^2(mt_c)^{1/2}\sim 10^{-20}, \qquad (41)$$

if $m=10^{-24}$ g.

APPENDIX I

To derive Eqs. (32), we introduce an auxiliary quantity L_0 such that $ma\ll L_0\ll\eta^{-1}$ and split the domain of integration with respect to λ in Eq. (22) into two: $(0, L_0)$ and (L_0,∞). In the first domain, we can use the approximation (29), and in the second the approximation (31). Assuming that for $ma\ll L_0\ll\eta^{-1}$

$$\int_0^{L_0}\lambda(\omega-\lambda)^2d\lambda\approx\frac{1}{4}m^4a^4\left(\ln\frac{2L_0}{ma}-\frac{1}{4}\right),$$
$$\int_{L_0}^\infty\frac{d\lambda}{\lambda}\cos 2\lambda(\eta_1-\eta_2)\approx-\ln(2L_0|\eta_1-\eta_2|)-C, \qquad (I.1)$$

we obtain

$$\varepsilon=\frac{m^4}{16\pi^2}\left(\ln\frac{1}{ma}-C-\frac{1}{4}\right.$$
$$\left.-\frac{1}{a^4}\int_0^\eta d\eta_1\frac{da^2(\eta_1)}{d\eta_1}\int_0^\eta d\eta_2\frac{da^2(\eta_2)}{d\eta_2}\ln|\eta_1-\eta_2|\right),$$
$$P=-\frac{m^4}{16\pi^2}\left(\ln\frac{1}{ma}-C-\frac{7}{12}\right. \qquad (I.2)$$
$$+\frac{1}{3a^4}\int_0^\eta d\eta_1\frac{da^2(\eta_1)}{d\eta_1}\int_0^\eta d\eta_2\frac{da^2(\eta_2)}{d\eta_2}\ln|\eta_1-\eta_2|$$
$$\left.-\frac{4}{3a^2}\int_0^\eta\frac{da^2(\eta_1)}{d\eta_1}\ln(\eta-\eta_1)\,d\eta_1\right).$$

The auxiliary L_0 has disappeared from the result. Setting in (I.2)

$$a(\eta)=a_1\eta^{q/(1-q)},\qquad (1-q)a(\eta)\eta=t$$

and using the values of the following integrals:

$$\int_0^1 z^a\ln z\,dz=-\frac{1}{(a+1)^2},$$
$$\int_0^1 z^a\ln(1-z)\,dz=-\frac{\Psi(a+2)+C}{a+1}, \qquad (I.3)$$
$$\int_0^1 dx\,x^a\cdot\int_0^1 dy\,y^b\ln|x-y|$$
$$=-\frac{1}{a+b+2}\left[\frac{\Psi(a+2)+C}{a+1}+\frac{\Psi(b+2)+C}{b+1}+\frac{1}{(a+1)(b+1)}\right]$$

we arrive at the results (32).

For $ma(\eta) \ll 1$, the approximation (31) is already satisfied for the first mode $\lambda = 1$ in the closed model. Direct summation in (20) and (22) with the use of (31) and the integrals (I. 3) in this case gives

$$
\begin{aligned}
n &= m^4 a(t)/192, \\
\varepsilon &= \frac{m^4}{16\pi^2}\left[(1-q)\ln\frac{1}{mt}+D_1\right], \\
P &= -\frac{m^4}{16\pi^2}\left[(1-q)\ln\frac{1}{mt}+D_1-\frac{1-q}{3q}\right]
\end{aligned}
\qquad \text{(I. 4)}
$$

where

$$
D_1=(1-q)\ln[(1-q)ma_1]-\ln 2+\Psi\left(\frac{1+q}{1-q}\right)+C+\frac{1-q}{4q}.
$$

Note however that for $ma(\eta) \ll 1$ we have $\varepsilon \ll \varepsilon_v$, $P \ll P_v$, where ε_v and P_v are given approximately by the expressions (27), i.e., the energy density of the created particles is much less than the vacuum density.

APPENDIX II

Suppose $q = \frac{1}{2}$, $a = a_0\sqrt{t}$. Then Eq. (6) reduces to Weber's equation.[31] Its exact solution with the initial conditions (11) is

$$
\begin{aligned}
u &= \frac{1}{\sqrt{\lambda}}\frac{i}{2}\{[E^{*\prime}(-\delta^2,0)-i\delta E^*(-\delta^2,0)]E(-\delta^2,x) \\
&\quad +[i\delta E(-\delta^2,0)-E'(-\delta^2,0)]E^*(-\delta^2,x)\},
\end{aligned}
\qquad \text{(II. 1)}
$$

where $x = 2\sqrt{mt}$, and δ is determined in (36),

$$
E(A,x)=\sqrt{2}\exp\left[\frac{\pi A}{4}+i\left(\frac{\alpha}{2}+\frac{\pi}{8}\right)\right]D_{-iA-1/2}(xe^{-i\pi/4}),
$$

$\alpha = \arg\Gamma(\frac{1}{2}+iA)$, $D_\nu(z)$ is a parabolic cylinder function, and

$$
E'(A,0)=\frac{d}{dx}E(A,x)|_{x=0}.
$$

The function u_λ can also be expressed in terms of confluent hypergeometric functions. The exact expression for $n(\delta)$ when $q = \frac{1}{2}$ is

$$
n(\delta)=\frac{\pi}{\delta\sqrt{2}}\exp\left(-\frac{\pi\delta^2}{2}\right)\left[\frac{1}{|\Gamma(1/4+i\delta^2/2)|^2}+\frac{\delta^2}{2}\frac{1}{|\Gamma(3/4+i\delta^2/2)|^2}\right]-\frac{1}{2}.
\qquad \text{(II. 2)}
$$

The asymptotic behaviors of $n(\delta)$ for $\delta \ll 1$ and $\delta \gg 1$ are determined by (38) and (39).

For $x \gg 1$, WKB type asymptotic behaviors[31] can be used for the functions $E(-\delta^2, x)$. As regards the values of these functions for $x = 0$, which occur in the coefficients of the exact solution (II. 1), when $\delta > \delta_m \gg 1$ the asymptotic behavior (31) is valid, while numerical calculation must be used in the region $0 < \delta < \delta_m$. At the same time, it can be shown that it is sufficient to take $\delta_m = \sqrt{2}$. Carrying out numerical integration in Eqs. (20) and (22) in the region $0 < \delta < \delta_m$ and estimating the integral for $\delta > \delta_m$ by the method of stationary phase, we obtain the result (40).

Suppose $q = \frac{1}{3}$, $a = a_0 t^{1/3}$. The exact solution of Eq. (6) with the initial conditions (11) is

$$
u_\lambda=\frac{1}{\sqrt{\lambda}}\left(\frac{(mt)^{2/3}}{\delta^2}+1\right)^{1/2}[C_1 J_{1/3}(y)+C_2 J_{-1/3}(y)],
\qquad \text{(II. 3)}
$$

where

$$
\begin{aligned}
y &= [(mt)^{2/3}+\delta^2]^{3/2}, \quad \delta=\lambda/a_0 m^{2/3}, \\
C_1 &= -\frac{\pi\delta^3}{\sqrt{3}}\left[J'_{-1/3}(\delta^3)+\left(-i+\frac{1}{3\delta^3}\right)J_{-1/3}(\delta^3)\right], \\
C_2 &= \frac{\pi\delta^3}{\sqrt{3}}\left[J_{1/3}'(\delta^3)+\left(-i+\frac{1}{3\delta^3}\right)J_{1/3}(\delta^3)\right];
\end{aligned}
$$

the prime denotes differentiation with respect to the argument. Then the function $n(\delta)$ has the form

$$
\begin{aligned}
n(\delta) &= \frac{\pi\delta^3}{6}\Bigg\{\Bigg[\frac{1}{2}\left(J_{1/3}'(\delta^3)+\frac{1}{3\delta^3}J_{1/3}(\delta^3)\right)-\frac{\sqrt{3}}{2}J_{1/3}(\delta^3) \\
&\quad -\left(J'_{-1/3}(\delta^3)+\frac{1}{3\delta^3}J_{-1/3}(\delta^3)\right)\Bigg]^2+\Bigg[\frac{\sqrt{3}}{2}\left(J_{1/3}'(\delta^3)+\frac{1}{3\delta^3}J_{1/3}(\delta^3)\right) \\
&\quad +\frac{1}{2}J_{1/3}(\delta^3)-J_{-1/3}(\delta^3)\Bigg]^2\Bigg\}.
\end{aligned}
\qquad \text{(II. 4)}
$$

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Pion interaction in nuclear matter and π condensation

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A number of topics is considered related to the problem of π condensation in nuclear matter ($N=Z$) and in neutron stars ($N \gg Z$). A consistent multiparticle approach is developed to describe exact excitations of a medium with quantum numbers characterizing π mesons—pion quasiparticles. A method is given for calculating the effective Lagrangian of a pion field, the nonlinear terms of which are interpreted as an interaction between pion quasiparticles. An exactly soluble model for the $\pi^+\pi^-$ condensation in a neutron medium is studied, which enables us to calculate by numerical methods the energy of the system in the presence of a π condensate of arbitrary amplitude. In order to illustrate the computation methods the high frequency approximation $\omega \gg k v_F$ is considered within the framework of which we succeed in calculating analytically the critical parameters and the energy of the $\pi^+\pi^-$ condensate. It is shown that the instability discovered by Sawyer and Scalapino [R. F. Sawyer, Phys. Rev. Lett. 29, 382 (1972); D. I. Scalapino, Phys. Rev. Lett. 29, 386 (1972)] is of the same nature as the $\pi^+\pi^-$-instability. The problem of the spatial and isotopic structure of the π condensate in the system with $N=Z$ is investigated. A broad class of solutions is investigated by the Thomas-Fermi method and it turns out that the one-dimensional isotopically asymmetric configurations of the condensate field have the lowest energy. The amplitude of the modulations of the particle density and of the spin density of nucleons in the condensate field is calculated.

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I. FORMULATION OF THE PROBLEM

1. The physical picture

The possibility of a phase transition with the formation of a π-meson condensate was investigated for the first time in[1]. The Klein–Gordon–Fock (KGF) equation was solved in external fields of different type: scalar, electric, in a field produced by nucleons (nuclear matter). It was found that in sufficiently strong external fields two types of instability of the pion field arise which correspond in a scalar field to $\omega^2_{\pi^+,\pi^-,\pi^0} < 0$ and in an electrical field to $(\omega_{\pi^+} + \omega_{\pi^-})^2 < 0$, where ω_π is the energy of the pions in the field. In these cases the single particle treatment becomes inapplicable and in order to obtain the condensate field and the energy of the system it is necessary to solve the nonlinear field problem. In[1] this problem was solved for the pion interaction of the form $\lambda\varphi^4$. The appearance of a Bose-condensate makes the system stable (the energy of all possible excitations becomes greater than zero). In that paper the possibility of a phase transition in nuclei and in neutron stars was demonstrated. But the influence of the nucleon medium was taken into account in the gaseous approximation in terms of the external field. Possible excitations of the nucleon medium were taken into account independently in references[2,3] by essentially different methods. In accordance with this the further development of the theory proceeded along two paths.

Sawyer and Scalapino[2] have put forward the idea of the instability of the matter of a neutron star with respect to the reaction $n \to p + \pi^-$. In order to verify this assertion a model was considered in which the nucleons interact with π^- mesons which are present in the only state with the propagation vector **k**. Since only one type of pions was taken into consideration the approach of[2] based on the method of the average field corresponded to the description of the pion field by a Schrödinger equation and not by the KGF equation. An instability was discovered in this model the meaning of which has become entirely clear only recently. This instability does not correspond to the initial idea of[2]. The instability with respect to the reaction $n \to p + \pi^-$ could have arisen only if the obvious condition $\mu^{(n)} \geq \mu^{(p)} + (\omega_{\pi^-})_{\min}$ is satisfied, which in the absence of an interaction between pions and nucleons goes over into the condition $\mu^{(n)} \geq \mu^{(p)} + m_\pi c^2$ (m_π is the pion mass). At the same time the instability observed in[2] disappears in the case of a weak interaction. Below we shall return to the question of the nature of the instability observed in[2]. We shall show that it represents a manifestation of the instability observed in a realistic model.[3,4]

The stability of the model under consideration with respect to the reaction $n \to p + \pi^-$ was demonstrated in[5]