when $\beta \gg 1$, $\Phi \to 1/(u+1)$ (see (94)). In view of that $\varphi(x) = x^{-1/2}$ and we finally get

$$\Phi_0 = e^{i\beta t/2}/(u+1).$$

Moreover, assuming the term in (77) without β to be small and using the method of successive approximations we can also find the next-order term which has the form

$$\Phi_1 = e^{i\beta t/2}\left(\frac{1}{i\beta(1+u)^2} - \frac{t}{1+u}\right).$$

As a whole up to terms of order $1/\beta$ or t we can write the result in the form

$$\Phi(t,u) \approx e^{-(1-i\beta/2)t}\left[\frac{1}{1+u} + \frac{1}{i\beta(1+u)^2}\right] \approx e^{-(1-i\beta/2)t}\Phi(0,u). \quad (95)$$

We emphasize that this result is already invalid for terms of order β^{-2} or t^2. The transfer of t to the argument of the exponential is justified by the fact that it is clear from what preceded that Φ must be exponentially damped at very large t. This means that when we take integrals over t we must put $i\beta \to i\beta - \gamma$ with $\gamma \ll \beta$. Equation (95) automatically guarantees that.

Further we have[3)]

$$\int_0^\infty \Phi(t,u)\,dt \approx \frac{\Phi(0,u)}{1-i\beta/2} \approx -\frac{2}{i\beta(1+u)} - \frac{4}{(i\beta)^2(1+u)} - \frac{2}{(i\beta)^2(1+u)^2}.$$

Substituting this expression into (81) we find

$$Q = -e^2 v/\pi c + 4ie^2 v/\pi c\beta. \quad (96)$$

Hence, connecting this with what preceded, we find the asymptotic behavior of the dielectric permittivity and of the conductivity:

$$\varepsilon = -4e^2 v/\omega_0^2 = -4\pi n_e e^2/m\omega_0^2,$$
$$\sigma = 2e^2 v/\pi\omega_0^2\tau_2 = 2n_e e^2\tau_2/m(\omega_0\tau_2)^2,$$

where $n_e = p_0/\pi = mv/\pi$ is the number of electrons. As should be the case, when $\omega\tau_2 \gg 1$ the permittivity is given by the formula for free electrons.

1) The introduction of the mutually uncorrelated fields η and ζ as well as Eq. (11) for G which is of first order in $\partial/\partial z$ is valid provided $1/\tau \ll \varepsilon_F$.

2) One should note that equations very close to (57) and (73) were obtained also by Berezinskiĭ.[1] However, in view of the fact that the numerical value of $Q(\omega_0)$ found in[1] is incorrect and that our method for solving the equations is somewhat different we thought it useful to give here the complete calculation right to the end.

3) We note that the same result is obtained if we take the expression for Φ without transferring t to the exponent and put $i\beta \to i\beta - \gamma$, where $\gamma \to +0$.

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Roton spectrum in superfluid He^3–He^4 solutions

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The roton spectrum in superfluid He^3–He^4 solutions is considered by taking into account interactions between impurity excitations and rotons. An equation for the self-energy function of the rotons is obtained within the framework of a model in which this interaction is assumed to be a point interaction. The equation is solved by numerical integration with a computer. The solutions are used to determine the thermodynamic characteristics of the rotons and the energy dependence of the cross sections of various scattering processes in which rotons take part.

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INTRODUCTION

It is known that in superfluid He^3–He^4 solutions there are two excitation branches—Fermi (impurity) and Bose. We are interested in temperatures at which the role of the phonons is negligible, i.e., the only Bose excitations considered are rotons. Information on the spectrum of these excitations in He^3–He^4 solutions can be obtained from measurements of the density of the normal component by the method of the oscillating stack of disks,[1–3] the velocity of fourth sound,[4] or mobility of the positive ions.[5] In the interpretation of the experimental data, the authors of the cited papers have concluded that the roton gap decreases strongly with increasing impurity concentration.

However, the results of experiments on the scattering of photons[6,7] and neutrons[8] by superfluid He^3–He^4

solutions, which have yielded direct information on the parameters of the roton spectrum, have revealed no noticeable change in the roton gap with changing concentration. It was therefore necessary to reconsider the interpretation of the experimental results of[1–5]. Sobolev[9] has noted that in the reduction of the experimental data it is necessary to take into account the deviation of the disperson of the impurity excitation from quadratic. Esel'son *et al.*[10] attempted to reconstruct the dispersion of the impurity excitations from data on the density of the normal component. The contribution of the rotons to the normal density was calculated in this case by using the parameters of the roton spectrum in pure He^4, but the roton damping was taken phenomenologically into account in accordance with[11].

It could thus be concluded that in the interpretation of the experimental data it is necessary to take correct account of the form of the excitation spectrum, particularly the damping of the excitations. This raises the question of the theoretical treatment of the interaction between the rotons and the impurity excitations. Within the framework of the point-interaction model, the self-energy roton function was considered in[12] in second-order perturbation theory. This approach, however, does not yield the correct behavior of the roton spectrum near the roton gap, a behavior that is essential for the consideration of the properties of super fluid He^3–He^4 solutions. The model of the point interaction is used also in the present paper to derive an equation for the self-energy function. This expression makes it possible to find the function by numerical computer integration for different temperatures and impurity concentrations.

The obtained functions can be used to determine the temperature dependences of the number of rotons, of the roton contribution to the normal component, and of other thermodynamic characteristics, and also to determine the energy dependence of the cross sections for the scattering of various particles by superfluid He^3–He^4 solutions with participation of the rotons. The obtained functions can be used to express directly the energy dependence of the cross section for the scattering with production of one roton; numerical integration with a computer yielded the energy dependence of the cross sections of two-roton scattering processes. Likewise, numerical integration within the framework of the considered model yielded the damping of the impurity excitations.

ROTON SPECTRUM

We are dealing with phenomena at temperatures $T \sim 1\,^\circ\mathrm{K}$, so that to describe the interaction of rotons and impurity excitations (in other processes) it is necessary to use the Green's function method at finite temperatures.[13] Actually, however, we are interested not in Matsubara Green's function, but in the behavior, on the real axis, of their analytic continuations, namely the retarded and advanced Green's functions.

We present relations that can be easily obtained by analytic continuation.[14] If $H(i\omega_n)$ is given by the diagram of Fig. 1a, where the solid line is set in correspondence to G and the wavy line to D, then we have

$$\operatorname{Im} H^R(\omega, k)\,n(\omega) = -\int \frac{d^3p}{(2\pi)^3}\int \frac{d\varepsilon}{\pi} \times \{\operatorname{Im} G^R(\varepsilon, p)\,n(\varepsilon)\}\{\operatorname{Im} D^R(\omega-\varepsilon, \mathbf{k}-\mathbf{p})\,n(\omega-\varepsilon)\}, \tag{1}$$

and for the diagram of Fig. 2a

$$\operatorname{Im} H^R(\omega, k)\,n(\omega) = -\int \frac{d^3p}{(2\pi)^3}\int \frac{d\varepsilon}{\pi} \times \exp\left(\frac{\varepsilon}{T}\right)\{\operatorname{Im} G^R(\varepsilon, p)\,n(\varepsilon)\}\{\operatorname{Im} D^R(\omega+\varepsilon, \mathbf{k}+\mathbf{p})\,n(\omega+\varepsilon)\}. \tag{2}$$

where n is respectively a Bose or Fermi distribution function, depending on the statistics of the particles to which G, D, and H correspond. The formulas are expressed in a system of units in which $\hbar = 1$, i.e., the momentum is measured in $\mathrm{\AA}^{-1}$, the energy is measured in °K, and the mass is measured respectively in $\mathrm{\AA}^{-2}\mathrm{K}^{-1}$.

The solid lines will henceforth correspond to the impurity Green's function G, and the wavy lines to the roton function D. In the considered temperature region, the rotons can be regarded as Boltzmann excitations. We shall also regard the impurity excitations as Boltzmann excitations, thus restricting the applicability of the considered model to solutions with He^3 molear concentrations approximately up to 0.2. Thus, in expressions of the type $n(\varepsilon)\operatorname{Im}G^R(\varepsilon)$ we can replace $n(\varepsilon)$ by the Boltzmann value $\exp(-\varepsilon/T)$, as will henceforth be implied in all the formulas that follow.

We shall examine in greater detail the expression given by the diagram of Fig. 1c, which will henceforth be designated F. An explicit expression for $\operatorname{Im}F^R$ in terms of G^R can be obtained from formula (2). If we substitute for the impurity Green's functions the expression for non-interacting quasiparticles $G^R_{\alpha\beta}(\varepsilon, p) = \delta_{\alpha\beta}/(\varepsilon + \lambda - \varepsilon(p) + i0)$, where λ is the chemical potential and $\varepsilon(p)$ is the law governing the dispersion of the impurity excitations, and integrate with respect to energy, then we obtain

$$n_B(\omega)\operatorname{Im} F^R(\omega, k) = -\int \frac{d^3p}{(2\pi)^2}\exp\left(\frac{\lambda-\varepsilon(p)}{T}\right)\delta[\omega-\varepsilon(p)+\varepsilon(\mathbf{p}-\mathbf{k})]. \tag{3}$$

In the effective mass approximation we have $\varepsilon(p) = p^2/2m$, and then (3) can be integrated fully and we obtain

$$n_B(\omega)\operatorname{Im} F^R(\omega, k) = -\frac{\nu}{k}\left(\frac{\pi m}{2T}\right)^{1/2}\exp\left[-\frac{1}{2mT}\left(\frac{m\omega}{k}+\frac{k}{2}\right)^2\right], \tag{4}$$

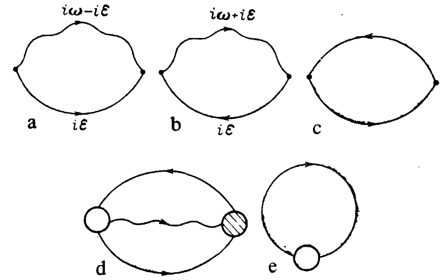

FIG. 1.

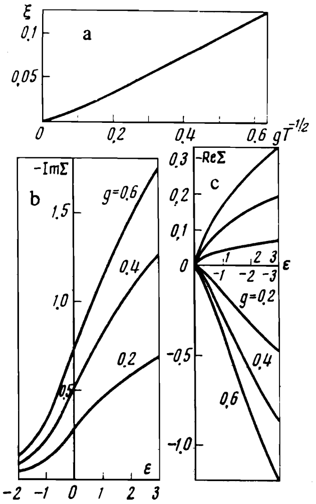

FIG. 2.

where we have substituted the number $\nu=2(mT/2\pi)^{3/2}\times\exp(\lambda/T)$ of impurity quasiparticles per unit volume (in the same effective-mass approximation).

When considering the interactions of the rotons with the impurity excitations, we shall take into account only two-particle elastic scattering, since the process of scattering of an impurity quasiparticle with production of a roton is impossible by virtue of the conservation laws. The corresponding expression for the self-energy of the rotons, which follows from the Dyson equation, consists of two parts given by the diagrams of Fig. 1d and Fig. 1e, where the unshaded vertex corresponds to the bare vertex function and the shaded one to the total vertex function.

The number of rotons in the considered temperature region is small, so that we can neglect the interaction between the rotons (which makes its own contribution to the self-energy function of the rotons) as well as the distortion of the spectrum of the impurity excitations as a result of the interaction between the rotons and impurity excitations (this will be confirmed by subsequent estimates). Neglecting the interaction between the impurity excitations, we can thus substitute for the impurity Green's function the expression for the noninteracting quasiparticles.

An important role in the vertices in the expression for the self-energy function Σ of the rotons is played by the energy transfer $\sim T$ and the momentum transfer $\sim(mT)^{1/2}$, so that it is reasonable to replace the vertex functions by constants. But then the expression given by the diagram of Fig. 1d contains a divergence, the sign of which corresponds to a decrease of the roton gap Δ. This indicates that within the framework of the considered model it is impossible to calculate the shift. However, the sign of the divergence makes it apparently possible to conclude that the sign of the interaction of the rotons and the impurity excitations corresponds to repulsion, because, as already mentioned, no noticeable shift Δ is observed in the experiment, so that the net shift Δ determined by the diagrams of Figs. 1d and 1e should be equal to zero.

We shall assume henceforth that the parameters of the real part of the roton spectrum are renormalized by an integral over those parts of the spectrum which are far from the minimum. Therefore when writing down the equation for Σ it is necessary to exclude from consideration the diagram of Fig. 1e, which yields a pure renormalization of Δ, and regularize the diagram of Fig. 1d. The corresponding expression for Σ^R is obtained by analytic continuation, [14] by replacing each of the vertex functions by a constant and denoting its product by Γ^2:

$$\Sigma^R(\omega,k)=-\Gamma^2\int\frac{d^3p}{(2\pi)^3}\int\frac{d\varepsilon}{\pi}[\mathrm{Im}\,F^R(\varepsilon,p)D^R(\varepsilon+\omega,\mathbf{p}+\mathbf{k})+\mathrm{Im}\,D^R(\varepsilon,p)F^A(\varepsilon-\omega,\mathbf{p}-\mathbf{k})]n_B(\varepsilon). \quad (5)$$

In this formula the integration region significant for the second term is the one near $\varepsilon=\Delta$, and accordingly $n_B(\varepsilon)=\exp(-\varepsilon/T)$, so that the integral of the second term is exponentially small and can be neglected. The equation for Σ^R is obtained by substituting (5) D^R expressed in its terms near the roton minimum

$$D^R(\omega,s)=-\frac{c}{\omega-\Delta-s^2-\Sigma^R(\omega,s)}, \quad (6)$$

where $s=(k-k_0)/\sqrt{2\mu}$, μ is the roton mass, k_0 is the momentum of a roton with zero velocity, and the constant c in (5) and all the succeeding formulas is included in the redefinition of the vertex parts. As indicated above, to calculate $\mathrm{Im}F^R$ it is necessary to use the Green's functions of the noninteracting quasiparticles, and as a result we obtain from (5) in the effective-mass approximation, using (4),

$$\Sigma^R(\omega,k)=\frac{\nu\Gamma^2}{4\pi^2}\left(\frac{m}{2\pi T}\right)^{1/2}\int_0^\infty dp\,p\int_{-1}^{1}d(\cos\theta)\times\int_{-\infty}^{\infty}d\varepsilon\exp\left(-\frac{1}{2mT}\left(\frac{m\varepsilon}{p}+\frac{p}{2}\right)^2\right)\frac{1}{\omega+\varepsilon-\Delta-s^2-\Sigma^R(\omega+\varepsilon,s)},$$

where $s=(|\mathbf{k}+\mathbf{p}|-k_0)/(2\mu)^{1/2}$. The integration with respect to the momentum p is carried out over a region in which $p\ll k_0$, and therefore $|\mathbf{k}+\mathbf{p}|\approx k+p\cos\theta$ and $d(\cos\theta)\approx p^{-1}(2\mu)^{1/2}ds$. The exponential factor indicates that the significant momenta are $p\sim(mT)^{1/2}+(m|\varepsilon|)^{1/2}$, and therefore, by virtue of the smallness of μ/m, we can assume that as $\cos\theta$ varies from -1 to 1 the value of s changes from $-\infty$ to $+\infty$. Integrating with respect to ds, we find that Σ^R does not depend on k, and the equation takes the form

$$\Sigma^R(\omega)=-i\frac{\nu\Gamma^2}{4\pi}\left(\frac{m\mu}{\pi T}\right)^{1/2}\int_0^\infty dp\int_{-\infty}^{+\infty}d\varepsilon\times\exp\left[-\frac{1}{2mT}\left(\frac{m\varepsilon}{p}+\frac{p}{2}\right)^2\right]\frac{1}{[\omega+\varepsilon-\Delta-\Sigma^R(\omega+\varepsilon)]^{1/2}}.$$

Of the two roots, we choose the one with the smaller argument. The integral with respect to dp is now calculated by using the formula

$$\int_0^\infty d\zeta\exp(-\zeta^2-x^2/4\zeta^2)=(\pi^{1/2}/2)\exp(-|x|).$$

We ultimately obtain for Σ the equation (we drop the index R)

$$\Sigma(\varepsilon)=-ig\int_{-\infty}^{+\infty}d\alpha\exp\left[-\frac{(\alpha-\varepsilon)\theta(\alpha-\varepsilon)}{T}\right]\frac{1}{[\alpha-\Sigma(\alpha)]^{1/2}},\qquad(7)$$

where $g=\nu\Gamma^2m\sqrt{2\mu}/4\pi$, and the energy is reckoned from the roton gap, corresponding to the condition $\mathrm{Re}\Sigma(0)=0$. Formally, the integral in the right-hand side of (7) diverges at negative α with large absolute values, since the exponential is equal to unity in this case, and $1/[\alpha-\Sigma(\alpha)]^{1/2}\approx 1/i(|\alpha|)^{1/2}$, but this divergence is connected simply with the already mentioned impossibility of calculating the shift Δ within the framework of the considered model.

We change over from (7) to a differential equation in which there are no divergences at all. To this end we calculate

$$\frac{d}{d\varepsilon}\left[\exp\left(-\frac{\varepsilon}{T}\right)\frac{d}{d\varepsilon}\Sigma(\varepsilon)\right],$$

substituting the right-hand side of (7) for Σ. Double differentiation under the integral sign gives rise to $\delta(\alpha-\varepsilon)$, after which integration with respect to $d\alpha$ yields the equation

$$\Sigma'-T\Sigma''=-ig/(\varepsilon-\Sigma)^{1/2}.\qquad(8)$$

It is seen from this equation that $\Sigma=Tf(g/T^{1/2},\varepsilon/T)$, i.e., a change takes place in only one essential dimensionless parameter $g/T^{1/2}$, which determines the form of the energy dependence of Σ.

From (8) we can obtain an asymptotic expression for Σ as $|\varepsilon|\to\infty$. It takes the form

$$\Sigma\approx-2ig\varepsilon^{1/2}-g^2\ln\varepsilon.\qquad(9)$$

We are interested in the parameter values $g/T^{1/2}\lesssim 1$, so that the criterion for the applicability of (9) is $|\varepsilon|\gg T$. At large negative ε, Eq. (9) yields $\Sigma\approx 2g(|\varepsilon|)^{1/2}-g^2\ln|\varepsilon|$, and in general all the terms of the asymptotic expansion of the expansion of Σ at negative ε are real, since $\mathrm{Im}\Sigma$ decreases exponentially with energy. We shall write down the corresponding asymptotic expression in the form

$$\mathrm{Im}\,\Sigma\approx-g(\pi T)^{1/2}\exp(\xi+\varepsilon/T),\qquad(10)$$

where $\xi=\xi[g/(T)^{1/2}]$. Expression (10) is valid for $\varepsilon\ll-(gT)^{3/2}$ (generally speaking, this criterion is valid if $g/T^{1/2}\lesssim 1$).

Starting from (7), we can obtain an expression for ξ, since the imaginary part of the integral in the right-hand side of (7) contains no divergence. Replacing $\theta(\alpha-\varepsilon)$ by unity in the argument of the exponential as $\varepsilon\to-\infty$, we get

$$\exp(\xi)=\frac{1}{(\pi T)^{1/2}}\int_{-\infty}^{+\infty}d\alpha\exp(-\alpha/T)\,\mathrm{Re}\frac{1}{[\alpha-\Sigma(\alpha)]^{1/2}}.\qquad(11)$$

In the limit of zero concentration (i.e., at $g=0$), we have $\Sigma=0$, and taking the integral in the right-hand side of (11), we obtain $\xi(0)=0$.

We write down also the asymptotic expression at $\varepsilon\ll-T$

$$\mathrm{Re}\frac{1}{(\varepsilon-\Sigma)^{1/2}}\approx\frac{g(\pi T)^{1/2}}{2|\varepsilon|^{3/2}}\exp(\xi+\varepsilon/T).\qquad(12)$$

It is seen from (12) that the integral (11) indeed converges at negative energies.

Equation (8) was integrated with a computer, and the result was the plot of $\xi(g/T^{1/2})$ shown in Fig. 2a. Figs. 2b and 2c show plots of $\mathrm{Im}\Sigma$ and $\mathrm{Re}\Sigma$ against the energies at $T=1$ for various values of g, also obtained by numerical integration.

THERMODYNAMICS OF ROTONS

To determine the thermodynamic characteristics of the rotons within the framework of the considered model it suffices to know the form of the function $\xi(g/T^{1/2})$. We express in its terms the number of rotons per unit volume, N. From the general formula (for Boltzmann excitations) we have

$$N=-\int\frac{d^3k}{(2\pi)^3}\times\int\frac{d\omega}{\pi}\exp\left(-\frac{\omega}{T}\right)\mathrm{Im}\,D^R(\omega,k).$$

Substituting in this formula expression (6) for $D^R(\omega,k)$, changing over to the energy reckoned from Δ, and integrating over the momenta, we obtain

$$N=\frac{k_0(2\mu)^{1/2}}{2\pi^2}\exp\left(-\frac{\Delta}{T}\right)\times\int_{-\infty}^{+\infty}d\varepsilon\,\mathrm{Re}\frac{\exp(-\varepsilon/T)}{[\varepsilon-\Sigma(\varepsilon)]^{1/2}}.\qquad(13)$$

The integral in this expression is expressed by means of formula (11) in terms of ξ. Substituting, we obtain

$$N=\exp(\xi)N_0,\qquad(14)$$

where N_0 is the Landau expression for the number of rotons per unit volume (with renormalized parameters of the roton spectrum):

$$N_0=c\frac{k_0^2}{\pi}\left(\frac{\mu T}{2\pi}\right)^{1/2}\exp\left(-\frac{\Delta}{T}\right).$$

Knowing the $N(T)$ dependence, we can find the various thermodynamic quantities, for example the density $\Omega=-TN$ of the thermodynamic potential, and the contribution $\rho_r=(k_0^2/3T)N$ of the rotons to the normal component of the density. The formulas for these quantities are similar to (14), e.g.,

$$\rho_r=\exp(\xi)\rho_{r0},\qquad(15)$$

where ρ_{r0} is the contribution of the rotons to the normal density without allowance for the interaction.

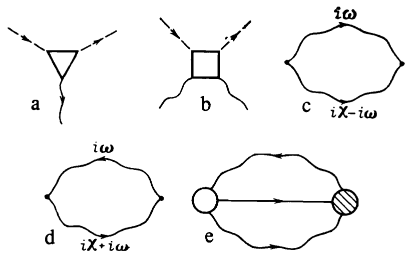

FIG. 3.

SCATTERING PROCESSES WITH PARTICIPATION OF ROTONS

We consider first the simplest process of scattering of a particle by the superfluid solution He^3–He^4 with production of one roton. The amplitude of the corresponding scattering process is represented by the diagram of Fig. 3a, where the dashed lines correspond to the amplitudes of the scattered particle in the initial and final state, and the vertex function is designated f. Using the analytic properties of the Green's functions (see[13], Sec. 17), we can obtain for the cross section corresponding to the diagram of Fig. 3a the expression

$$\frac{d\sigma}{V}=-\frac{1}{v_1}|f(\mathbf{p}_1;\omega,\mathbf{k})|^2\frac{2}{1-\exp(-\omega/T)}\operatorname{Im}D^R(\omega,k)\frac{d^3p_2}{(2\pi)^3}. \quad (16)$$

In this formula V is the volume, v_1 is the velocity of the incident particle, while ω and k are the energy of the momentum of the roton and are connected by the conservation laws with the initial and final momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ of the scattered particle.

The significant energy transfers are at $\omega=\Delta$ and the significant momentum transfer are near $k=k_0$, so that the dependence of the cross section on the angle and on the energy transfer is determined mainly by the factor $\operatorname{Im}D^R(\omega,k)$ and the exponential (16) can be neglected. Substituting expression (6) for $\operatorname{Im}D^R(\omega,k)$ and changing over to the differentials of the scattering angle θ and of the roton energy ω, we obtain

$$\frac{d\sigma}{V}=\frac{p_2^2}{2\pi^2 v_1 v_2}|f(\mathbf{p}_1;\omega,\mathbf{k})|^2\operatorname{Im}\frac{1}{\omega-\Delta-s^2-\Sigma(\omega)}\,d\omega\, d(\cos\theta). \quad (17)$$

To determine the energy dependence of this cross section we must use the $\operatorname{Re}\Sigma$ and $\operatorname{Im}\Sigma$ obtained as a result of the numerical integration and shown in Figs. 2b and 2c. It is seen from (17) that at a given scattering angle the cross section has a Lorentz peak at $\omega-\Delta-s^2-\operatorname{Re}\Sigma=0$, and the width of the peak is determined by $|\operatorname{Im}\Sigma|$. Far from the peak we have $\operatorname{Im}D^R=\operatorname{Im}\Sigma(\omega)/(\omega-\Delta-s^2)^2$; the asymptotic forms of this expression can be obtained in accord with formulas (9) and (10).

In full analogy with (16), we can obtain an expression for the scattering cross section of a two-roton process, the amplitude of which is represented by the diagram of Fig. 3b. Assuming that the corresponding vertex function γ depends only on the summary energies and the momentum transferred by the scattered particle, and introducing the two-roton Green's function K, we get

$$\frac{d\sigma}{V}=-\frac{p_2^2}{2\pi^2 v_1 v_2}|\gamma(\mathbf{p}_1;\chi,\mathbf{q})|^2\frac{\operatorname{Im}K^R(\chi,q)}{1-\exp(-\chi/T)}\,d\chi\, d(\cos\theta). \quad (18)$$

In this formula χ and $\mathbf{q}$ are the energy and momentum transfers, while v and p are the velocities and momenta of the scattered particle in the initial and final states.

We shall take the two-roton processes to be scattering with production of two rotons and elastic scattering by a roton. The diagrams that determine K, neglecting the interaction between the rotons, are shown for these two cases on Figs. 3c and 3d. In accord with formulas (1) and (2) we have

$$n_B(\chi)\operatorname{Im}K_1^R(\chi,q)=-\exp(-\chi/T)\int\frac{d^3k}{(2\pi)^3}\times\int\frac{d\omega}{\pi}\operatorname{Im}D^R(\omega,k)\operatorname{Im}D^R(\chi-\omega,\mathbf{q}-\mathbf{k}) \quad (19)$$

for scattering with production of two rotons and

$$n_B(\chi)\operatorname{Im}K_2(\chi,q)=-\exp(-\chi/T)\int\frac{d^3k}{(2\pi)^3}\times\int\frac{d\omega}{\pi}\exp\left(-\frac{\omega}{T}\right)\operatorname{Im}D^R(\omega,k)\operatorname{Im}D^R(\chi+\omega,\mathbf{q}+\mathbf{k}) \quad (20)$$

for elastic scattering by a roton.

The momentum dependence of $\operatorname{Im}D^R$ in (19) and (20) is determined by $s_1=(k-k_0)/(2\mu)^{1/2}$ and $s_2=(|\mathbf{q}\pm\mathbf{k}|-k_0)/(2\mu)^{1/2}$, so that the regions of importance in the integration with respect to d^3k are those in which k and $|\mathbf{q}\pm\mathbf{k}|$ lie near k_0.

Consider the case when the momentum transfer q is not small. In the half-plane passing through the vectors $\mathbf{k}$ and $\mathbf{q}$, we choose unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ such that $\mathbf{k}_2=\mathbf{k}_1+\mathbf{q}$, where $\mathbf{k}_1=k_0\mathbf{n}_1$ and $\mathbf{k}_2=k_0\mathbf{n}_2$. In the case, say, of formula (20) we assume $\mathbf{k}=\mathbf{k}_1+\beta$, and then $\mathbf{k}+\mathbf{q}=\mathbf{k}_2+\beta$, i.e., the values of k and $|\mathbf{k}+\mathbf{q}|$ close to k_0 correspond to $\beta\ll k_0$. We can therefore put $s_1\approx(\mathbf{n}_1\beta)/(2\mu)^{1/2}$ and $s_2\approx(\mathbf{n}_2\beta)/(2\mu)^{1/2}$. We can thus transform the integration differential $d^2\beta\to 2\mu\sin z\, ds_1 ds_2$, where z is the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$, and accordingly $d^3k\approx 2\pi k_0 d^2\beta\to 4\pi k_0\mu\sin z\, ds_1 ds_2$. The criterion for the applicability of this approximation is $z\gg(2\mu\alpha)^{1/2}/k_0$, where $\alpha=\max(T,|\chi|)$ is the characteristic energy. Recognizing that the angle z is connected with q, namely $2k_0\sin z/2=q$, we can rewrite the criterion in the form $2k_0\gtrsim q\gg(2\mu\alpha)^{1/2}$.

Each of the factors $\operatorname{Im}D^R$ in (19) and (20) depends only on s_1 or s_2, so that the integration with respect to them can be separated. Substituting (6) and integrating with respect to ds_1 and ds_2, we get

$$n_B(\chi)\operatorname{Im}K_2^R(\chi,q)=-\exp(-\chi/T)\exp(-\Delta/T)\frac{\mu k_0}{2\pi}\sin z\times\int d\varepsilon\exp(-\varepsilon/T)\operatorname{Re}\frac{1}{[\varepsilon-\Sigma(\varepsilon)]^{1/2}}\operatorname{Re}\frac{1}{[\varepsilon+\chi-\Sigma(\varepsilon+\chi)]^{1/2}}. \quad (21)$$

In this formula we have changed from integration with respect to the energy reckoned from the roton gap. An analogous formula holds also for $\operatorname{Im}K_1^2$.

Substituting the obtained expressions in (18), we get

$$\frac{d\sigma_1}{V}=\frac{p_2^2\mu k_0\sin z}{4\pi^3 v_1 v_2}|\gamma_1(\mathbf{p}_1;\chi,\mathbf{q})|^2 B_1(\chi-2\Delta)\,d\chi\,d(\cos\theta), \qquad (22)$$

$$\frac{d\sigma_2}{V}=\frac{p_2^2\mu k_0\sin z}{4\pi^3 v_1 v_2}\exp\left(-\frac{\Delta}{T}\right)|\gamma_2(\mathbf{p}_1;\chi,\mathbf{q})|^2 B_2(\chi)\,d\chi\,d(\cos\theta), \qquad (23)$$

where

$$B_1(\alpha)=\int_{-\infty}^{+\infty}d\varepsilon\,\mathrm{Re}\frac{1}{[\varepsilon-\Sigma(\varepsilon)]^{1/2}}\mathrm{Re}\frac{1}{[\alpha-\varepsilon-\Sigma(\alpha-\varepsilon)]^{1/2}}, \qquad (24)$$

$$B_2(\alpha)=\int_{-\infty}^{+\infty}d\varepsilon\,\mathrm{Re}\frac{\exp(-\varepsilon/T)}{[\varepsilon-\Sigma(\varepsilon)]^{1/2}}\mathrm{Re}\frac{1}{[\alpha+\varepsilon-\Sigma(\alpha+\varepsilon)]^{1/2}}. \qquad (25)$$

Let us find the asymptotic forms of (24) and (25). As $\alpha\to\infty$ the main contribution to the integral B_1 builds up at $\varepsilon\sim\alpha$, and we can therefore neglect Σ in comparison with ε, and as a result we get

$$B_1(\alpha)\approx\int_0^\alpha d\varepsilon\frac{1}{\varepsilon^{1/2}}\frac{1}{(\alpha-\varepsilon)^{1/2}}=\pi.$$

This asymptotic expression is valid for $\alpha\gg g^2$. As $\alpha\to-\infty$, the principal part of the integral is built up near $\varepsilon=0$ and $\varepsilon=\alpha$. Accordingly we obtain, substituting (12) and using (11),

$$B_1(\alpha)\approx 2\int d\varepsilon\,\mathrm{Re}\frac{1}{[\varepsilon-\Sigma(\varepsilon)]^{1/2}}\frac{g(\pi T)^{1/2}}{2|\alpha|^{1/2}}\exp\left(\xi+\frac{\alpha}{T}-\frac{\varepsilon}{T}\right)=\frac{\pi gT}{|\alpha|^{1/2}}\exp\left(2\xi+\frac{\alpha}{T}\right).$$

This asymptotic expression is valid at $|\alpha|\gg T$. As to B_2, as $\alpha\to+\infty$ the principal part of the integral builds up near $\varepsilon=0$, therefore

$$B_2(\alpha)\approx\int d\varepsilon\exp\left(-\frac{\varepsilon}{T}\right)\mathrm{Re}\frac{1}{[\varepsilon-\Sigma(\varepsilon)]^{1/2}}\frac{1}{\alpha^{1/2}}=\exp(\xi)\left(\frac{\pi T}{\alpha}\right)^{1/2}$$

where we have again used (11). The criterion for the applicability of this asymptotic expression is $\alpha\gg T$. The values $B_2(\alpha)$ at negative α are expressed in terms of the values at positive α via the formula $B_2(-\alpha)=\exp(-\alpha/T)B_2(\alpha)$, which holds for any integral of the form

$$\int d\varepsilon\exp(-\varepsilon/T)f(\varepsilon)f(\varepsilon+\alpha).$$

Plots of $B_1(\alpha)$ and $B_2(\alpha)$ at $T=1$ for different values of g, obtained by numerical computer integration, are shown in Figs. 4a and 4b.

We consider now the production of two rotons at a low momentum transfer ($q\ll(2\mu T)^{1/2}$). In this case we have in (19) $|\mathbf{q}-\mathbf{k}|\approx k$, and we can make the substitution $d^3k\to4\pi k_0^2(2\mu)^{1/2}ds$, where $s=(k-k_0)/(2\mu)^{1/2}$. We substitute (6), integrate (19) with respect to ds, and substitute in (18). As a result we obtain (we have changed over to the energy reckoned from Δ):

$$\frac{d\sigma}{V}=\frac{(2\mu)^{1/2}k_0^2p_2^2}{8\pi^4v_1v_2}|\gamma(\mathbf{p}_1,\chi,\mathbf{q})|^2B_3(\chi-2\Delta)\,d\chi\,d(\cos\theta), \qquad (26)$$

where

$$B_3(\alpha)=\mathrm{Re}\int_{-\infty}^{+\infty}d\varepsilon\left\{\frac{1}{(a_1+ib_1)^{1/2}}\frac{b_2}{(a_1-a_2+ib_1)^2+b_2^2}+\frac{1}{(a_2+ib_2)^{1/2}}\frac{b_1}{(a_2-a_1+ib_2)^2+b_1^2}\right\} \qquad (27)$$

$$\varepsilon_1=(\alpha+\varepsilon)/2,\quad \varepsilon_2=(\alpha-\varepsilon)/2,$$
$$a_1=\varepsilon_1-\mathrm{Re}\,\Sigma(\varepsilon_1),\quad a_2=\varepsilon_2-\mathrm{Re}\,\Sigma(\varepsilon_2),$$
$$b_1=-\mathrm{Im}\,\Sigma(\varepsilon_1),\quad b_2=-\mathrm{Im}\,\Sigma(\varepsilon_2).$$

Let us find the asymptotic forms of $B_3(\alpha)$ as $\alpha\to\pm\infty$. In this case $b'(\alpha/2)\ll1$, and we can therefore assume that the poles of the integrand in (27) are determined by the equations $(\pm\varepsilon+ib)^2+b^2=0$, where $b=b(\alpha/2)$, i.e., they occur at $\varepsilon=0$ and $\varepsilon=\pm2ib$. The two pole terms at $\varepsilon=0$ cancel each other (so that there is no singularity in the integrand at $\varepsilon=0$), the pole contribution to the integral from the two other pole terms is determined by half the difference between the residues at the corresponding poles (this contribution can be correctly taken into account since $1/(a+ib)^{1/2}$ varies little over the length of b). As a result of the calculation we find that the pole contribution is equal to $\mathrm{Re}\,\pi(2\alpha)^{1/2}$. Thus, at $\alpha\gg g^2$ the asymptotic form of B_3 is $B_3(\alpha)\approx\pi(2/\alpha)^{1/2}$, and that at negative α the pole term is equal to zero. Therefore at $\alpha\ll-T$ the integral builds up near $a_1=0$ and $a_2=0$, i.e.,

$$B_3(\alpha)\approx2\,\mathrm{Re}\int d\varepsilon\frac{1}{(a_1+ib_1)^{1/2}}\frac{b_2}{\alpha^2}.$$

Substituting the asymptotic expression for b_2 from (10) and using (11), we get

$$B_3(\alpha)\approx\frac{4\pi gT}{\alpha^2}\exp\left(2\xi+\frac{\alpha}{T}\right).$$

Plots of $B_3(\alpha)$ at $T=1$ for different values of g, obtained by computer integration, are shown in Fig. 4c.

We consider now the damping η of the impurity excitations as a result of their interaction with the rotons. It is determined by the imaginary part of the advanced self-energy function of the impurity excitations, which is shown in Fig. 3d. The vertex parts are again considered to be constant and their product is replaced by the quantity Γ^2, while G is replaced by the expression for the non-interacting particles. If we now use (1) and then integrate with respect to energy, we get

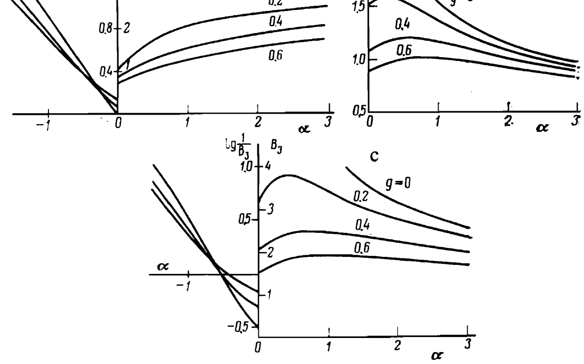

FIG. 4.

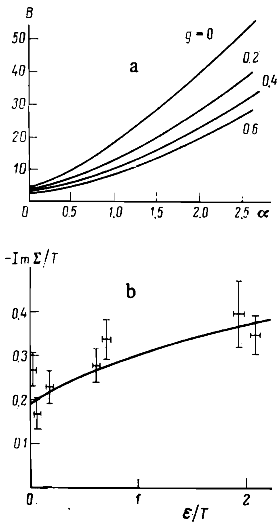

FIG. 5.

$$\eta(\alpha,\beta)=-\Gamma^2\int\frac{d^3k}{(2\pi)^3}\exp\left(\frac{\alpha-\varepsilon}{T}\right)n_B(\alpha-\varepsilon)\,\mathrm{Im}\,K_2^R(\alpha-\varepsilon,\beta-k), \qquad (28)$$

where $\varepsilon=\varepsilon(k)$.

In the integral (28), and the $\beta^2\sim mT$ of interest to us, the significant values are $k^2\sim mT\gg\mu T$, so that we can use the expression for $\mathrm{Im}K_2^R$ in terms of B_2, as determined by formulas (21) and (25). Recognizing that $k^2\sim mT\ll k_0^2$, we substitute $\sin z\approx q/k_0$ and obtain, integrating over the angles,

$$\eta(\alpha,\beta)=\frac{\mu\Gamma^2}{3(2\pi)^2\beta}\exp\left(-\frac{\Delta}{T}\right)\int_0^\infty k\,dk\,((k+\beta)^3-|k-\beta|^3)B_2(\alpha-\varepsilon). \qquad (29)$$

To estimate the energy smearing of the impurity excitations it suffices to know $\eta(\alpha)\cong\eta(\alpha,\beta)$, where $\alpha=\varepsilon(\beta)$. We change over in (29) to integration with respect to the energy $\varepsilon=p^2/2m$. Recognizing that $B_2(-\omega)=\exp(-\omega/T)\times B_2(\omega)$, we obtain

$$\eta(\alpha)=\mu\Gamma^2m^2\exp(-\Delta/T)B(\alpha)/6\pi^3, \qquad (30)$$

where

$$B(\alpha)=\int_0^\alpha d\varepsilon\,(4\alpha-\varepsilon)(1-\varepsilon/\alpha)^{1/2}B_2(\varepsilon)+\int_0^\infty d\varepsilon\,(3\varepsilon+4\alpha)B_2(\varepsilon)\exp(-\varepsilon/T). \qquad (31)$$

The results of the numerical integration by formula (31) at $T=1$ are shown for different values of g in Fig. 5a.

COMPARISON WITH EXPERIMENT

The considered model is valid in the temperature region 0.6–1.4 °K; it is bounded from below by the statification curve of the superfluid He^3–He^4 solutions, and from above by effects connected with the presence of roton-roton interaction, which becomes appreciable as the number of rotons increases. We assume that in the indicated temperature regions the impurity excitations obey Boltzmann statistics, a fact that limits the applicability of the model, to (molar) concentrations x of the He^3 atoms up to approximately 0.2, in which case the effective mass of the impurity excitations depends little on their concentration.[15]

Recalling the expression for g, we can conclude that if we neglect the dependence of the effective mass of the impurity excitations and of the particle density in the solution on the concentration, then g becomes proportional to x. Accordingly, the proportional coefficient $\xi=g/x$ can be determined from various experiments.

Direct information on the quasiparticle spectrum is obtained from experiments on the scattering of various particles by He^3–He^4 solutions. However, the interpretation of the neutron scattering cross sections, which are usually employed for this purpose, is made complicated by the large probability of neutron absorption by the He^3 nuclei. In[8] are cited results from which we can estimate the damping of the rotons in the region above the threshold (at $x=0.05$ and $T=1.6$ °K). Figure 5b shows a comparison of the experimental data of[8] and the curve obtained from the considered model at $\zeta=3$ $K^{1/2}$ (corresponding to $gT^{-1/2}=0.12$).

There are also experimental results on Raman scattering.[6,7] In the energy-transfer region investigated in these experiments, a major role is played by the two-roton scattering cross section, which is determined within the framework of the considered model by formula (26), since the photon momentum is small (and the momentum transfer is correspondingly small). For optical photons the frequency shift at the considered Raman scattering is small in comparison with its frequency, therefore $p_2\approx p_1$. In addition, it is reasonable to assume that γ depends weakly on the energy transfer near $\chi=2\Delta$, therefore the energy dependence of the cross section in this region should be determined by the factor $B_3(\varepsilon)$, $\varepsilon=\chi-2\Delta$. However, the cross section receives also contributions from processes in which a roton pair is produced,[16] so that the measured cross sections must be interpreted as sums of cross sections corresponding to these two mechanisms. Figure 6 shows the experimental data[6] for $x=0.059$ and $x=0.109$, the plots determined by $B_3(\varepsilon)$ at $\zeta=3$ $K^{1/2}$ and $\Delta=8.5$ °K, and also the fit curves that are obtained by adding to the aforementioned the Lorentz cross section corresponding to the production of a roton pair with zero momentum, a binding energy $E_B=0.3$ °K, and half-widths $\gamma=0.7$ °K and 1.1 °K. We note that at such a conversion coefficient, the relation $g/T^{1/2}\lesssim 1$, which was used when we

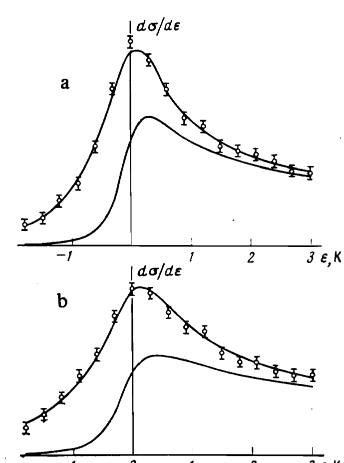

FIG. 6.

established the criteria for the applicability of the asymptotic expressions, does indeed hold in the considered region of He^3 temperatures and concentrations. Recalling the definition of ζ, we can estimate Γ from the formula $\zeta = \nu_0\Gamma^2 m(2\mu)^{1/2}/4\pi$, where ν_0 is the density of the particles in pure He^4. Substituting the numerical values of the constants, we find

$$\Gamma = 300\ \mathrm{K}\ \mathrm{\AA}^3. \qquad (32)$$

Substitution of this value in (30) and the use of Fig. 5a shows that in the entire energy region of interest to us the impurity-excitation damping due to the interaction with the rotons is much less than the temperature, and this justifies the assumption made from the very outset, that we can neglect the impurity-excitation Green's-function distortion due to collision with the rotons.

The interpretation of experiments in which the normal density was measured by the method of an oscillating stack of disks[1–3] and was estimated in terms of the velocity of the fourth sound[4] is hindered by the fact that ρ_n consists of roton and impurity components ρ_r and ρ_i. Within the framework of a quadratic dispersion law, ρ_i does not depend on the temperature, but this independence does not obtain if the dispersion of the impurity excitations deviates from quadratic. Esel'son *et al.*[10] have considered the general form of the expansion of the energy of the impurity excitations in terms of the momentum and sought to find the first terms of this expansion by reducing the experimental data. The corrections to ρ_r due to the interaction of the rotons with the impurity excitations were taken into account phenomenologically in accord with the paper of Reut and Fisher.[11] The model considered by us makes it possible to find ρ_r directly from formula (15). The reduction of the data given in[3] for the impurity molar concentrations $x=0.11$ and $x=0.2$ allows us to conclude that the $\rho_n(T)$ dependences can be explained within the framework of the concepts described in[10], but to obtain quantitative results it is necessary to know more accurately the shift δ of the roton gap.

In addition to those mentioned, experiments were performed to determine the mobility of heavy ions in He^3–He^4 solutions. The effective mass of the ions is large, so that their reciprocal mobility, which is connected with scattering by the rotons, is determined by the cross section for scattering with zero energy transfer. At $x=0$ this cross section is determined mainly by rotons having momenta close to k_0. The presence of impurity excitations, which leads to an effective smearing of these states, should decrease the cross section for the scattering of ions by rotons with decreasing concentration. Within the framework of the considered model, the concentration dependence of the cross section for scattering by rotons, as seen from (23), is determined by the factor $B_2(0)$. In accordance with the described general considerations, $B_2(0)$ decreases with increasing g, and in the range of values in which the numerical integration was carried out, this dependence can be approximated by the formula $B_2(0) \propto g^{-1/2}$.

The roton contribution to the reciprocal ion mobility in the presence of impurity excitation should at any rate be smaller than in pure He^4. On this basis, the data given in[5] lead to the conclusion that in the considered temperature region, for $x \gtrsim 0.05$, the mobility of the heavy ions in He^3–He^4 solutions should be determined by scattering by impurity excitations.

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