Divergence of the perturbation-theory series and the quasi-classical theory

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We use the quasi-classical method to find an exact asymptotic formula for the expansion coefficients of the Gell-Mann–Low function in renormalized scalar theories.

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1. INTRODUCTION

The Feynman diagram technique makes it possible to find the Green function and scattering amplitude in quantum field theory in the form of an expansion in a series in the coupling constant g.[1] However, even in quantum electrodynamics, where the coupling constant is small, the effective expansion parameter becomes for some problems of the order of unity[2] so that one needs evaluate Feynman diagrams of arbitrarily high order. The necessity of a good estimate for various observed quantities in high orders of perturbation theory is felt also in those cases where the experimental accuracy of their measurements is sufficiently high. The anomalous magnetic moment of an electron is an example of this; the accuracy of the measurements of this quantity exceeds the accuracy of the theoretical calculations up to sixth order perturbation theory and it becomes necessary to take into account in the calculations a huge set of eighth order diagrams.[3] The methods for estimating Feynman diagrams in higher order which exist at the present time[4] cannot claim to have the accuracy desired.

In the present paper we use as an example the calculation of the asymptotic behavior of the expansion coefficients in the Gell-Mann–Low function in scalar theories to show that one can construct for a high order k of perturbation theory a computational scheme which acts upon the solution of the corresponding classical equations and which gives us the possibility to find the physical quantities as an expansion in powers of $1/k$. Although we carry out our discussion in the framework of scalar theories, the method for the calculations can be carried over to theories of greater practical interest (quantum electrodynamics, Yang-Mills theory, and so on).

2. STATEMENT OF THE PROBLEM

We consider a set of scalar models for field theory in a Euclidean space with an interaction Hamiltonian

$$H(\varphi, g_\mu)=\int d^Dx\left[\frac{(\partial_\mu\varphi)^2}{2}+g_\mu\frac{\varphi^n}{n!}+\mathscr{H}'(\varphi, g_\mu)\right]. \quad (1)$$

In order that the theory be renormalizable (but not hyperrenormalizable) we connect the dimensionality of the space D with the index n of the power of the non-linearity of the theory by the following formula:

$$D=2n/(n-2), \quad n=4,\ 6,\ 8\dots \quad (2)$$

In the class (2) of theories models with an interaction Hamiltonian

$$H_{int}=g\int\frac{\varphi^4}{4!}d^4x, \quad H_{int}=g\int\frac{\varphi^6}{6!}d^3x,$$

are of physical interest, but we shall in what follows give the discussion for the general case of arbitrary even $n\geqslant 4$.

In Eq. (1) g_μ is the renormalized coupling constant which is equal to the value of the invariant charge in the normalization point μ (see below). The additional term $\mathscr{H}'(\varphi, g_\mu)$ in the energy density is a polynomial of degree n in the field $\varphi(x)$, introduced to compensate the ultraviolet divergences in the subdiagrams.[1] One can easily find its lower expansion coefficients in the series in the renormalized charge g_μ:

$$\mathscr{H}'(\varphi, g_\mu)=\frac{\varphi^n}{n!}\left[g_\mu^2\frac{1}{2}\frac{n!}{((n/2)!)^3}a^{n/2}(n)\int\frac{d^Dx}{x^D}\exp\left\{i\frac{n(p^\mu x)}{2(n-1)}\right\}+O(g_\mu^3)\right]+g_\mu\left[\exp\left\{-\frac{1}{2}\int d^Dx_1\,d^Dx_2\,\Delta^0(x_1-x_2)\frac{\delta}{\delta\varphi(x_1)}\frac{\delta}{\delta\varphi(x_2)}\right\}-1\right]\times\frac{\varphi^n}{n!}+P_{n-2}(\varphi)g_\mu^2+\dots; \quad (3)$$

the first term is intended here to remove the logarithmic divergences connected with the renormalization of the charge, $(p^\mu)^2=\mu^2$, and the other terms remove the power-law ultra-violet divergences in the diagrams with number of tails $m\leqslant n-2$. The quantity $\Delta^0(x)$ is the free-particle Green function:

$$\Delta^0(x)=\int\frac{d^Dp}{(2\pi)^D}\frac{e^{ipx}}{p^2}=a(n)|x|^{2-D}, \quad a(n)=\frac{1}{4}\Gamma\left(\frac{2}{n-2}\right)\pi^{-n/(n-2)}. \quad (4)$$

The additional term $\mathscr{H}'$ may also contain an arbitrary polynomial $P_{n-2}(\varphi)$ with finite coefficients which is a generalization of the mass term $\frac{1}{2}m^2\varphi^2$ in the theory

$$g\int\frac{\varphi^4}{4!}d^4x,$$

but if we choose the normalization momentum μ much larger than the characteristic mass parameter which is connected with these dimensional coefficients, the Gell-Mann–Low function will not depend on them.

We define the invariant charge in the model considered by means of the formula

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$$g\left(\frac{p^2}{\mu^2}, g_\mu\right) = g_\mu \Gamma_n\left(\frac{p^2}{\mu^2}, g_\mu\right)\left[d\left(\frac{p^2}{\mu^2}, g_\mu\right)\right]^{n/2},$$

$$\Gamma_n(1, g_\mu) = d(1, g_\mu) = 1, \qquad (5)$$

where Γ_n and $\Delta(p^2) = d/p^2$ are the renormalized vertex function and the Green function for large momenta p. In a real pseudo-Euclidean world we can in all Feynman diagrams perform a Wick rotation for the time component of the virtual momenta $p_0 \to ip_4$, if we choose the external invariants in the vertex and in the Green function spatiall scaled. Using such a procedure we can evaluate the invariant charge directly in a Euclidean theory with the Hamiltonian (1). We further choose the external invariants for n-tails in a symmetrical spatially scaled point:

$$p_i^2 = p^2 > 0, \quad p_i p_j|_{i\neq j} = -p^2/(n-1) \qquad (6)$$

and correspondingly we assume the invariants in the normalization point to be equal to

$$(p_i^\mu)^2 = \mu^2 > 0, \quad p_i^\mu p_j^\mu|_{i\neq j} = -\mu^2/(n-1). \qquad (7)$$

The Gell-Mann–Low function can then be evaluated through the formula

$$\psi\left(g\left(\frac{p^2}{\mu^2}, g_\mu\right)\right) = \partial g\left(\frac{p^2}{\mu^2}, g_\mu\right) \Big/ \partial \ln\frac{p^2}{\mu^2}, \qquad (8)$$

where by virtue of the renormalizability the right-hand side is independent of p^2/μ^2 under the condition that it is expressed as a function of the invariant charge.

In the present paper we evaluate the asymptotic behavior as $k \to \infty$ of the expansion coefficients $C_k(n)$ of the Gell-Mann–Low function in the perturbation theory series:

$$\psi(g) = \sum_{k=2}^{\infty} (-1)^k C_k(n) g^k. \qquad (9)$$

It is convenient to change to coordinate space. We have for the Green function of a scalar particle

$$\Delta_c(p^2) = \int d^D x\, e^{-ipx} \Delta_c(x), \qquad (10)$$

where for $\Delta_c(x)$ the representation in the form of a Feynman functional integral is valid:

$$\Delta_c(x_1 - x_2) = s_0^{-1} \int \prod_x d\varphi(x) \varphi(x_1) \varphi(x_2) e^{-H},$$

$$s_0 = \int \prod_x d\varphi(x) \exp\left\{-\frac{1}{2}\int d^D x (\partial_\mu \varphi)^2\right\}. \qquad (11)$$

The ultra-violet divergences which are connected with the renormalization of the mass and the wavefunction are cancelled in each given order of the perturbation theory due to the counter terms (3) in the Hamiltonian H and the finite corrections to $\mathscr{H}'$ must be chosen such that

$$\Delta_c(p^2)|_{p^2=\mu^2} = 1/\mu^2. \qquad (12)$$

Similarly we have for the vertex function Γ_n

$$-g_\mu \Gamma_n\left(\frac{p^2}{\mu^2}, g_\mu\right)\left\{\Delta_c\left(\frac{p^2}{\mu^2}, g_\mu\right)\right\}^n$$

$$= \int \prod_{t=1}^{n} d^D x_t [G_n(x_1, \ldots, x_n) - G_n'] \delta^D\left(\frac{1}{n}\sum x_t\right) \exp\left(i\sum_t p_t x_t\right), \qquad (13)$$

where

$$G_n(x_1, x_2, \ldots, x_n) = \frac{1}{s_0}\int \prod_x d\varphi(x) \varphi(x_1) \ldots \varphi(x_n) \exp\{-H(\varphi, g_\mu)\}. \qquad (14)$$

In (13) G_n' is the contribution of unconnected diagrams and contains products of lower-order Green functions.

In higher perturbation-theory orders the contribution to the invariant charge of corrections to $\Delta_c(p^2)$ (see (11)) are negligibly small compared to the contribution from the corrections to Γ_n, and, moreover, Eq. (14) contains only connected diagrams in the limit $k \to \infty$ as is clear from the further discussion. We thus have approximately

$$g\left(\frac{p^2}{\mu^2}, g_\mu\right) = \sum_{k=1}^{\infty} A_k(p) (-g_\mu)^k, \qquad (15)$$

$$A_k(p) \approx p^{2n} \int \prod_{t=1}^{n} d^D x_t G_n^{(k)} \delta^D\left(\frac{1}{n}\sum x_t\right) \exp\left(i\sum_t p_t x_t\right), \qquad (15a)$$

$$G_n^{(k)}(x_1, x_2, \ldots, x_n) = \frac{1}{s_0}\int \prod_x d\varphi(x) \int \frac{dg_\mu}{(-g_\mu)^{k+1}} \frac{\varphi(x_1) \ldots \varphi(x_n)}{2\pi i} \exp[-H(\varphi, g_\mu)], \qquad (16)$$

where the integration over g_μ is over a closed contour around the origin. We find the asymptotic behavior of the integral (16) as $k \to \infty$ in the following sections.

3. THE IDEA OF THE CALCULATION METHOD AND ITS ILLUSTRATION FOR EXACTLY SOLUBLE MODELS

To find the asymptotic behavior of the integral (16) for large k we apply the saddle-point method. As the functional to be varied we choose the quantity (see (1))

$$J(\varphi, g_\mu, k) = \int d^D x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + g_\mu \frac{\varphi^n}{n!}\right] + k \ln(-g_\mu), \qquad (17)$$

expanding all other terms in the integrand in (16) in a series near the saddle-point values $\tilde{\varphi}(x)$ and $\tilde{g}_\mu$ (this will be justified by the course of the subsequent calculations). The Euler-Lagrange equations for the functional (17) have the form

$$k = -\tilde{g}_\mu \int d^D x \frac{\tilde{\varphi}^n}{n!}, \qquad (18a)$$

$$-\partial_\mu^2 \tilde{\varphi} + \tilde{g}_\mu \tilde{\varphi}^{n-1}/(n-1)! = 0. \qquad (18b)$$

Equations (18b) are the usual classical equations for the field $\tilde{\varphi}(x)$, but by virtue of (18a) we must find their solutions which decrease at infinity under the condition that the coupling constant is negative ($\tilde{g}_\mu < 0$). The theory with the Hamiltonian (1) becomes unstable for a negative coupling constant, as one can take arbitrarily large φ which are independent of x. However, the condition (18a) means that the minimum of the Hamiltonian (1) must be looked for for a fixed potential energy, i.e.,

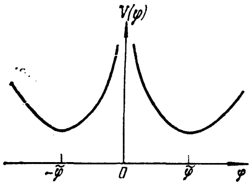

FIG. 1.

one must discard functions φ independent of x and consider only functions which decrease as $|x|\to\infty$. For such functions, which satisfy condition (18b), we have

$$J(\tilde{\varphi},\tilde{g}_\mu,k)=-k(1-n/2)+k\ln(-\tilde{g}_\mu). \qquad (19)$$

There is a set of spherically symmetric solutions of Eqs. (18) which differ in the position of the center x_0 and the scale y:

$$\tilde{\varphi}(x)=\pm\sqrt{k}\left(\frac{y}{y^2+|x-x_0|^2}\right)^{2/(n-2)}\frac{n-2}{(8\pi)^{1/2}}\pi^{-1/(n-2)}\left\{\frac{\Gamma(D)}{\Gamma(D/2)}\right\}^{1/2},$$
$$-\tilde{g}_\mu=\frac{n!}{k^{(n-2)/2}}\left[\frac{8\pi}{(n-2)^2}\right]^{n/2}\left\{\frac{\Gamma(D/2)}{\Gamma(D)}\right\}^{(n-2)/2}. \qquad (20)$$

The fact that $\tilde{\varphi}$ is proportional to $\sqrt{k}$ justifies the use of the quasi-classical method for evaluating the functional integral (16).

In the next section we use the set of solutions (20) to evaluate the integral (16). Here we illustrate the possibility to apply the saddle-point method for calculating the asymptotic behavior of the expansion coefficients in the perturbation theory series by two exactly soluble models. We consider firstly instead of the functional integral a normal integral ("zero-dimensional" theory):

$$G_{2m}^{(k)}=\frac{1}{s_0}\int_{-\infty}^{\infty}d\varphi\int\frac{dg}{(-g)^{k+1}}\frac{\varphi^{2m}}{2\pi i}\exp\left[-\left(\frac{\varphi^2}{2}+g\frac{\varphi^n}{n!}\right)\right],\quad s_0=\sqrt{2\pi}. \qquad (21)$$

Evaluating the integral over g we get

$$G_{2m}^{(k)}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\varphi\,\varphi^{2m}\frac{e^{-V(\varphi)}}{k!},\quad V(\varphi)=\frac{\varphi^2}{2}+k\ln\frac{\varphi^n}{n!}. \qquad (22)$$

The effective potential $V(\varphi)$ has the form shown in Fig. 1 which means that for large k there is a saddle-point (cf. (20)) in the integral (21) when

$$\tilde{\varphi}=\pm(nk)^{1/2},\quad -\tilde{g}=\frac{n!}{n^{n/2}}k^{-n/2+1}. \qquad (23)$$

After finding the coefficient of the exponent by means of evaluating the Gaussian integral over small fluctuations near the two saddle points (23) we get for the asymptotic behavior of (21)

$$G_{2m}^{(k)}|_{k\to\infty}=\left(\frac{k}{e}\right)^{k(n/2-1)}\left(\frac{n^{n/2}}{n!}\right)^k\frac{(kn)^m}{(\pi k)^{1/2}}, \qquad (24)$$

i.e., the perturbation theory series diverges as $(k!)^{(1/2)n-1}$. Under those conditions the asymptotic behavior of the integral (21), which is the contribution of all possible k-th order diagram for the $2m$ ends, is, as we already noted earlier when deriving the expression (16), the same as the asymptotic behavior of the contribution of the connected diagrams as can easily be verified by finding the generating functional for the connected Green functions

$$W=\ln\sum_{m,k}J^{2m}G_{2m}^{(k)}(-g)^k$$

and finding the coefficients of its expansion in J and g. One also checks easily that Eq. (24) is the same as the asymptotic behavior of the integral (22) which can be expressed explicitly in terms of the gamma function.

The second exactly soluble example which we study is very close to the real case considered in the following sections. In fact, it arises in the limiting transition $n\to\infty$, $D\to 2$ in the class (2) of the theories. In this limit we can evaluate the Gell-Mann–Low function in any perturbation-theory order[5]:

$$\psi(g)=\sum_{k=2}^{\infty}(-g)^kC_k(n),$$
$$C_k(n)|_{n\to\infty}=\frac{1}{k!}\left(\frac{e}{8\pi}k^{(k+1)/(k-1)}\right)^{n(k-1)/2}$$
$$\times\left(2\pi n\frac{k-1}{k}\right)^{-k/2}\exp[-(k-1)(1+\ln\pi+c_E)]\sqrt{2}C_k, \qquad (25)$$

where $c_E=0.577$ is the Euler constant. The factor C_k in Eq. (25) is given by the following integral:

$$C_k=\int\prod_{r=1}^{k}d^2x_r\prod_{r<r'}|x_r-x_{r'}|^{-4/k}\delta^2\left(\frac{1}{k}\sum x_r\right)\delta\left(\frac{1}{k}\sum\ln(x_r^2)\right). \qquad (26)$$

It can be set in correspondence with a Feynman diagram in which k points are connected by lines with propagators $|x_r-x_{r'}|^{-4/k}$. The exact value of the integral (26) is equal to[5]

$$C_k=\frac{\pi^{k-1}}{\Gamma(k-1)}\left\{\frac{\Gamma(1/k)}{\Gamma(1-1/k)}\right\}^k, \qquad (27)$$

whence

$$\bar{C}_k\equiv C_k|_{k\to\infty}=\frac{(\pi e)^kk^{3/2}}{(2\pi)^{1/2}}\frac{\exp(-2c_E)}{\pi}\left(1+O\left(\frac{1}{k}\right)\right). \qquad (27a)$$

We get an asymptotic expression for C_k directly from Eq. (26) by a statistical method—by scanning the saddle-point density of the interaction point distribution $\tilde{\rho}(x)$. This allows us to interpret the saddle-point method of calculating the functional integral (16) in the framework of the diagram method of perturbation theory. We assume that as $k\to\infty$ the main contribution to the integral (26) is collected from a limited space volume Ω. In that case the density of the number of interaction points increases and we apply the quasi-classical method to evaluate it. We divide the space Ω into N cells $\Delta\Omega_r\ll 1$ with k_r points in each cell, and we shall assume that $1\ll N\ll k$, i.e., $1\ll k_r\ll k$. Averaging the integrand in Eq. (26) by means of integrating over the region $\prod_r(\Delta\Omega_r)^{k_r}$ we can rewrite (26) as

$$C_k=\sum_{\{k_r\}}\delta\left(k-\sum_r k_r\right)\delta^2\left(\sum x_r\frac{k_r}{k}\right)$$
$$\times\delta\left(\sum\frac{k_r}{k}\ln x_r^2\right)\frac{k!}{\prod_r(k_r!)}\prod_r(\Delta\Omega_r)^{k_r}e^{-J}, \qquad (28)$$

where

$$J(k_r)=\frac{1}{2}\sum_{r,r'}k_rk_{r'}v_{rr'}-\frac{1}{2}\sum_r k_rv_{rr}$$

$$+\frac{1}{2}\sum_{r,r',r''}k_rk_{r'}k_{r''}[\overline{v_{rr'}v_{rr''}}-\overline{v_{rr'}}\,\overline{v_{rr''}}]+\dots; \qquad (28a)$$

$$v_{rr'}=\frac{1}{\Delta\Omega_r\,\Delta\Omega_{r'}}\int_{\substack{x\in\Delta\Omega_r\\ y\in\Delta\Omega_{r'}}}d^2x\,d^2y\,\frac{4}{k}\ln|x-y|,$$

$$\overline{v_{rr'}v_{rr''}}\equiv\frac{(4/k)^2}{\Delta\Omega_r\Delta\Omega_{r'}\Delta\Omega_{r''}}\int_{\substack{x_1\in\Delta\Omega_r\\ x_2\in\Delta\Omega_{r'}\\ x_3\in\Delta\Omega_{r''}}}\prod_{i=1}^{3}d^2x_i\ln|x_1-x_2|\ln|x_1-x_3|. \qquad (28b)$$

The terms left out of Eq. (28a) are unimportant under the conditions that the dimensions of an elementary cell $\Delta\Omega_r$ is sufficiently small: $\Delta\Omega_r\ll(1/k)^{2/3}$. We shall assume, nonetheless, that $\Delta\Omega_r\gg 1/k$ so that we can use Stirling's formula for $k_r!$ in Eq. (28):

$$\frac{(\Delta\Omega_r)^{k_r}}{k_r!}\approx\frac{1}{(2\pi k_r)^{1/2}}\exp\left(-k_r\ln\frac{k_r}{\Delta\Omega_r e}\right).$$

Under this condition the third term in Eq. (28a) is of the order $k\Delta\Omega_r\gg 1$. One can check that its role, when we evaluate the sum in (28) using the saddle-point method, reduces to extending the minimum of the functional

$$\Phi(\rho)=\frac{2}{k}\int\rho(x)\rho(y)\ln|x-y|d^2x\,d^2y+\int\rho(x)\ln\frac{\rho(x)}{e}d^2x \qquad (29)$$

in the class of the step functions

$$\rho(x)=\sum_r\frac{k_r}{\Delta\Omega_r}\theta(x\in\Delta\Omega_r)$$

to its minimum in the class of all functions. One must find the minimum of the functional (29) with the following additional conditions:

$$\int d^2x\,\rho(x)=k,\qquad\int d^2x\,\rho(x)x=0,\qquad\int d^2x\,\rho(x)\ln(x^2)=0. \qquad (30)$$

The solution of this variational problem in the class of continuous functions is

$$\bar\rho(x)=k/\pi(x^2+1)^2. \qquad (31)$$

It is convenient when summing in (28) over the fluctuations Δk_r near the saddle-point number of the particles $\bar k_r=\bar\rho\Delta\Omega_r$ to change to a new variable $y(x)\equiv(2\pi k_r)^{-1/2}\times\Delta k_r$. We have

$$C_k|_{k\to\infty}=(\pi e)^k\frac{k}{4}(2\pi k)^{1/2}\int\prod_x dy(x)\,\delta\left(\sum_x\frac{y(x)}{x^2+1}\sqrt{d^2x}\right)$$

$$\times\delta^2\left(\sum_x\frac{y(x)x}{x^2+1}\sqrt{d^2x}\right)\delta\left(\sum_x\frac{y(x)\ln x}{x^2+1}\sqrt{d^2x}\right)\exp\Bigg[-\pi\sum_x y^2(x)$$

$$-4\sum_{x,x'}\frac{y(x)y(x')\sqrt{d^2x\,d^2x'}}{(x^2+1)(x'^2+1)}\ln|x-x'|+\frac{2}{\pi}\int\frac{d^2x\ln 0}{(x^2+1)^2}\Bigg]. \qquad (32)$$

The counterterm

$$\frac{2}{\pi}\int d^2x\frac{\ln 0}{(x^2+1)^2},$$

resulting from the second term in Eq. (28a) removes the logarithmic divergence which appears when integrating over $y(x)$. When we diagonalize the quadratic form in $y(x)$ in (32) there arise dangerous directions along which it does not change. The integration over the parameters which correspond to these directions is performed by means of the δ-functions. Using the method of the following sections we can easily integrate over $y(x)$ and obtain Eq. (27a). Another way of calculating the asymptotic behavior of the integral (26) is reducing it to a functional integral of the form (16) with an exponential interaction

$$H_{int}=g\int d^2x\exp[-(8\pi/k)^{1/2}\varphi(x)]$$

and the evaluation of that functional integral by means of the saddle-point method gives the same result (27a). We have thus verified through the exactly soluble model (25) that the saddle-point method of evaluating the integral (16) is in perturbation theory equivalent to looking for the saddle-point density of the interaction-point distribution.

4. ASYMPTOTIC BEHAVIOR OF THE EXPANSION OF THE GELL-MANN-LOW FUNCTION

We have shown in the beginning of the preceding section that there is in the limit $k\to\infty$ in the integral (16) a set of saddle-point values for $\varphi(x)$ and g_μ which differ in the position of the center of the distribution and the scale y (see (20)). All saddle points found give the same contribution when we evaluate the integral (16) because the functional J of (17), which is varied, is the same in those points (see (19)). In order to separate explicitly the integrations over x_0 and y we apply Faddeev's method.[6] We introduce in the functional integral (16) the following expansion of unity:

$$1=\left[\int(-g_\mu)\frac{\varphi^n}{n!}d^Dx\right]^{D+1}\int_{-\infty}^{\infty}d\ln y^2\int d^Dx_0\delta^D\left(-g_\mu\int d^Dx\frac{\varphi^n(x)}{n!}(x-x_0)_\nu\right)$$

$$\times\delta\left(-g_\mu\int d^Dx\frac{\varphi^n(x)}{n!}\ln\left(\frac{x-x_0}{y}\right)^2\right). \qquad (33)$$

After that, using the formulae

$$x\to yx+x_0,\quad\varphi(yx+x_0)\to y^{-(D-2)/2}\varphi(x) \qquad (34)$$

to transform the parameters x and the integration variables φ, and using the fact that the functional (17) is invariant under those transformations we can put (16) into the form

$$G_n^{(k)}(x_1,\dots,x_n)=\int d^Dx_0\int_0^\infty\frac{dy^2}{y^{2+D}}\frac{1}{J_0}\int\prod_x d\varphi(x)$$

$$\times\int\frac{dg_\mu}{(-g_\mu)\cdot 2\pi i}\left[-g_\mu\int d^Dx\frac{\varphi^n}{n!}\right]^{D+1}\prod_{r=1}^{n}y^{1-D/2}\varphi\left(\frac{x_r-x_0}{y}\right)$$

$$\times\exp\left(-J-\int\mathscr{H}_y'\,d^Dx\right)\delta^D\left(-g_\mu\int d^Dx\frac{\varphi^n(x)}{n!}x_\nu\right)\delta\left(-g_\mu\int d^Dx\frac{\varphi^n(x)}{n!}\ln x^2\right),$$

$$(35)$$

where $\mathscr{H}_y'(\varphi,g_\mu)$ differs from $\mathscr{H}'(\varphi,g_\mu)$ (see (3)) by the substitution

$$p^\mu \to yp^\mu \tag{36}$$

and the dropping (for sufficiently small y) of the polynomial in φ of degree $n-2$ which might contain arbitrary finite dimensional coupling constants (such as the renormalized mass).

If we put

$$\rho(x) = -g_\mu \varphi^n(x)/n!, \tag{37}$$

the δ-functions in Eq. (35) will be completely analogous to the δ-functions in the integral (26), i.e., they fix the center and scale in the density of the interaction point distribution in k-th order. There are thus in the integral (35), in contrast to (16), only two saddle-point values for $\varphi(x)$ (cf. (20)):

$$\tilde{\varphi} = \pm\sqrt{k}(1+x^2)^{-2/(n-2)} \frac{n-2}{(8\pi)^{1/2}} \pi^{-1/(n-2)} \left\{\frac{\Gamma(D)}{\Gamma(D/2)}\right\}^{1/2}. \tag{38}$$

The corresponding saddle-point interaction-point distribution is given by the formula

$$\tilde{\rho}(x) = \frac{k\pi^{-n/(n-2)}}{(x^2+1)^{2n/(n-2)}} \frac{\Gamma(D)}{\Gamma(D/2)}. \tag{39}$$

In particular, as $n \to \infty$ we get Eq. (31).

Bearing in mind that Eq. (18b) holds at the saddle point, we can make the following substitution in the integral (35):

$$y^{-(D-2)/2}\tilde{\varphi}\left(\frac{x_r - x_0}{y}\right) \to \int d^D z_r \Delta^0(x_r - z_r) \frac{-\tilde{g}_\mu}{(n-1)!}\left[y^{-2/(n-2)}\tilde{\varphi}\left(\frac{z_r - z_0}{y}\right)\right]^{n-1}$$
$$z_0 = x_0. \tag{40}$$

The enables us to "break the ends" of $G_n^{(k)}(x_1, \ldots, x_n)$ and change to the vertex function $\Gamma_n^{(k)}(x_1, \ldots, x_n)$ (see (13)):

$$\Gamma_n = \sum_{k=1}^{\infty} (-g)^{k-1}\Gamma_n^{(k)}(x_r),$$

$$\Gamma_n^{(k)}(x_r)|_{k\gg1} = \int d^D x_0 \int_0^\infty \frac{dy^2}{y^{2+D}}(-\tilde{g}_\mu)^{-k} \exp\left[k\left(1-\frac{n}{2}\right)\right]$$
$$\times \prod_{r=1}^{n} \frac{(-\tilde{g}_\mu)}{(n-1)!}\left[y^{-2/(n-2)}\tilde{\varphi}\left(\frac{x_r - x_0}{y}\right)\right]^{n-1} k^{D/2}\gamma_n(y\mu). \tag{41}$$

Here γ_n is an integral over small fluctuations $\Delta\varphi$ close to the two saddle points (38):

$$\gamma_n = 2k^{1+D/2}\tilde{g}_\mu^{-1} \int_{i\infty}^{-i\infty} \frac{d(\delta g_\mu)}{2\pi i} J_0^{-1} \int \prod_x d(\Delta\varphi(x))$$
$$\times \delta^D\left(-\tilde{g}_\mu \int d^D x \frac{\tilde{\varphi}^{n-1}(x)}{(n-1)!}\Delta\varphi(x)x_\nu\right)\delta\left(-\tilde{g}_\mu \int d^D x \frac{\tilde{\varphi}^{n-1}(x)}{(n-1)!}\Delta\varphi(x)\ln x^2\right)$$
$$\times \exp\left[-\int d^D x\, \mathscr{H}'(\tilde{\varphi}, \tilde{g}_\mu, p^\mu y) + \Psi(\Delta\varphi, \delta g_\mu)\right], \tag{42}$$
$$\Psi(\Delta\varphi, \delta g_\mu) = -\Phi(\Delta\varphi) + \frac{k(\delta g_\mu)^2}{2\tilde{g}_\mu^2} - \delta g_\mu \int d^D x \frac{\tilde{\varphi}^{n-1}(x)}{(n-1)!}\Delta\varphi(x).$$

The functional $\Phi(\Delta\varphi)$ in Eq. (42) is quadratic in $\Delta\varphi$:

$$\Phi(\Delta\varphi) = \int d^D x\left[\frac{(\partial_\mu\Delta\varphi)^2}{2} + \tilde{g}_\mu \frac{\tilde{\varphi}^{n-2}}{(n-2)!}\frac{(\Delta\varphi)^2}{2}\right]. \tag{43}$$

We have included some k-dependent factors in front of the integral in Eq. (42), in order that the quantity γ_n be independent of k as $k \to \infty$. When calculating with an accuracy up to a constant it is sufficient to retain in $\mathscr{H}'$ only two terms (see (3)):

$$\mathscr{H}' = -\tilde{g}_\mu \frac{n(n-1)}{2n!}\Delta^0(0)\tilde{\varphi}^{n-2}(x)$$
$$+\tilde{g}_\mu^2 \frac{\tilde{\varphi}^n}{n!}\frac{1}{2}\frac{n!}{((n/2)!)^3}a^{n/2}(n)\int \frac{d^D x}{x^D}\exp\left[i\frac{n(p^\mu x)y}{2(n-1)^{1/2}}\right]. \tag{44}$$

We use Eq. (41) to express the expression for the k-th order of perturbation theory in the expansion of the invariant charge in a series in the physical charge g_μ in terms of γ_n. To do this we change to the renormalized vertex part in the momentum representation:

$$\Gamma_n\left(\frac{p^2}{\mu^2}, g_\mu\right) = \sum_{k=1}^{\infty}(-g_\mu)^{k-1}\Gamma_n^{(k)}\left(\frac{p^2}{\mu^2}\right), \tag{45}$$

$$\Gamma_n^{(k)}\left(\frac{p^2}{\mu^2}\right) = \int \prod_r d^D x_r \left[\exp\left(i\sum p_r x_r\right)\right.$$
$$\left.-\exp\left(i\sum p_r^\mu x_r\right)\right]\delta^D\left(\sum\frac{x_r}{n}\right)\Gamma_n^{(k)}(x_1, \ldots, x_n), \tag{45a}$$

where the momenta p_r and p_r^μ are chosen such that Eqs. (6) and (7) are satisfied. Substituting Eq. (41) into Eq. (45a) and changing the variables $x_r \to yx_r + x_0$ we get after integrating over x_0

$$\Gamma_n^{(k)}\left(\frac{p^2}{\mu^2}\right) = \int_0^\infty \frac{dy^2}{y^2}\int \prod_r d^D x_r \frac{-\tilde{g}_\mu}{(n-1)!}[\tilde{\varphi}(x_r)]^{n-1}\left[\exp\left(iy\sum p_r x_r\right)\right.$$
$$\left.-\exp\left(iy\sum p_r^\mu x_r\right)\right]k^{D/2}(-\tilde{g}_\mu)^{-k}e^{k(1-n/2)}\gamma_n(y\mu). \tag{46}$$

For large k the main contribution to the invariant charge (5) comes from the vertex part (see, for instance the splitting off of the free particle Green function Δ^0 in Eq. (40)). The expansion of the invariant charge is thus the same as the expansion of Γ:

$$g\left(\frac{p^2}{\mu^2}, g_\mu\right) = \sum_{k=1}^{\infty}(-g_\mu)^k A_k\left(n, \frac{p^2}{\mu^2}\right), \quad A_1\left(n, \frac{p^2}{\mu^2}\right) = -1, \tag{47a}$$

$$A_k(n, p^2/\mu^2)|_{k\to\infty} = -\Gamma_n^{(k)}(p^2/\mu^2). \tag{47b}$$

To find the series expansion of the Gell-Mann–Low function (9) we must according to the definition (8) differentiate (47a) with respect to $\ln(p^2/\mu^2)$ and express the result as a function of $g(p^2/\mu^2, g_\mu)$. Since the coefficients $A_k(n, p^2/\mu^2)$ increase fast with the number k (see $\tilde{g}_\mu$ in Eq. (20)) the main role in such a re-expansion is played by the terms in (47a) closest to a given number k, and in them it is sufficient to retain only two terms in the expansion of g_μ in $g(p^2/\mu^2, g_\mu)$. We thus get for the coefficients $C_k(n)$ in (9) the following expression:

$$C_k(n) = A_k'\left(n, \frac{p^2}{\mu^2}\right) + (k-1)A_2 A_{k-1}' + \frac{(k-2)(k-3)}{2!}A_2^2 A_{k-2}' - \ldots,$$
$$A_i' = \partial A_i/\partial\ln(p^2/\mu^2). \tag{48}$$

The separate terms on the right-hand side of Eq. (48) may depend on p^2/μ^2, but the sum does not depend on it by virtue of the renormalizability of the theory.

Bearing in mind that the coefficients $A_k(n, p^2/\mu^2)$ for the case $n \geq 6$ increase faster than $(k!)^{2-\epsilon}$, we find from

(48) that the $C_k(n)$ must the same as the A'_k, i.e., the A_k for $n \geqslant 6$ contain only the single-logarithmic contribution. This in turn means that for $n \geqslant 6$ the coefficients $\gamma_n(y\mu)$ in Eq. (46) are independent of y, i.e.,

$$C_k(n)\big|_{n\geqslant 6}=\left[\int d^Dx\frac{(-\tilde{g}_\mu)}{(n-1)!}\tilde{\varphi}^{n-1}(x)\right]^n(-\tilde{g}_\mu)^{-k}e^{k(1-n/2)}k^{D/2}\gamma_n. \tag{49}$$

The case $n=4$ in the four-dimensional space $D=4$ is singular, for in that case the coefficients $A_k(n)$ increase as a factorial (see (20) and (46)). By virtue of Eq. (48) this leads to a more complicated formula expressing the C_k of (4) in terms of $\gamma_4(y\mu)$:

$$C_k(4)=-(-\tilde{g}_\mu)^{-k}e^{-k}k^2\exp(16\pi^2A_2)\int\limits_0^\infty\frac{dy^2}{y^2}p^2\frac{\partial}{\partial p^2}\left[\int\frac{d^4x}{3!}\tilde{g}_\mu\tilde{\varphi}^3e^{ipxy}\right]^4\gamma_4(y\mu). \tag{50}$$

Since

$$A_2\left(\frac{p^2}{\mu^2}\right)=\frac{3}{2\cdot 16\pi^2}\ln\frac{p^2}{\mu^2}, \tag{51}$$

$\gamma_4(y\mu)$ must appropriately depend on $y\mu$ in order that there does not remain a p^2-dependence in the product (50). In the next section we show that this is, in fact, the case. The cause of the $y\mu$-dependence of γ_4 is the fact that in the $g\varphi^4/4!$-theory in higher orders of the perturbation theory comparable contributions to the invariant charge come from both the skeleton diagrams and diagrams with the simplest single-loop insertions which have an ultra-violet divergence before renormalization. At the same time diagrams with radiative corrections to the Green functions turn out to be important only in terms of order $1/k$ relative to the contribution from (50).

5. CALCULATION OF CORRECTIONS CONNECTED WITH QUANTUM FLUCTUATIONS TO THE ASYMPTOTIC FORMULA

For the calculation of γ_n (see (42)) it is convenient to change from the Euclidean D-dimensional space to the unit sphere S_{D+1} in the $D+1$-dimensional z space[1)]:

$$z_\mu=\frac{2x_\mu}{1+x^2},\quad z_{D+1}=\frac{x^2-1}{x^2+1},\quad z_\mu^2+z_{D+1}^2=1. \tag{52}$$

If we use the formula

$$\Delta\varphi(x)=\left(\frac{2}{1+x^2}\right)^{D/2-1}Y(z) \tag{53}$$

to introduce a new integration variable $Y(z)$ and use the relations

$$-\frac{1}{2}\int d^Dx(\partial_\mu\Delta\varphi)^2=-\frac{1}{2}\int dS_{D+1}\left[(\nabla_nY)^2+\frac{D}{2}\left(\frac{D}{2}-1\right)Y^2\right],$$

$$(z,\nabla_n)=0,\qquad -\frac{1}{2}\int\tilde{g}_\mu\frac{\tilde{\varphi}^{n-2}}{(n-2)!}(\Delta\varphi)^2d^Dx=\frac{1}{2}\int dS_{D+1}\frac{D}{2}\left(\frac{D}{2}+1\right)Y^2,$$

$$\frac{1}{\sqrt{k}}\int(-\tilde{g}_\mu)\frac{\tilde{\varphi}^{n-1}}{(n-1)!}\Delta\varphi\,d^Dx=\int dS_{D+1}Y\frac{D}{\sqrt{2}}Y_0, \tag{54}$$

$$Y_0=(S_{D+1})^{-1/2}=\left[\Gamma\left(\frac{D+1}{2}\right)\Big/2\pi^{(D+1)/2}\right]^{1/2},$$

we can write Eq. (42) for γ_n after integration over Δg_μ in the form

$$\gamma_n=\frac{1}{\sqrt{2\pi}}\left(\int\prod_z dY(z)\exp\left\{-\frac{1}{2}\int dS_{D+1}\left[(\nabla_nY)^2+\frac{D}{2}\left(\frac{D}{2}-1\right)Y^2\right]\right\}\right)^{-1}$$
$$\times\exp\left(-\int\mathscr{H}_y'\,d^Dx\right)\int\prod_z dY(z)\exp\left\{-\frac{1}{2}\int dS_{D+1}[(\nabla_nY)^2-DY^2]\right.$$
$$\left.-\frac{D^2}{4}Y_0^2\left(\int dS_{D+1}Y\right)^2\right\}\delta^D\left(\frac{D}{\sqrt{2}}Y_0\int dS_{D+1}x_\sigma Y\right)\delta\left(\frac{D}{\sqrt{2}}Y_0\int dS_{D+1}\ln|x|Y\right). \tag{55}$$

The counterterm $-\int\mathscr{H}'_y d^Dx$ is intended to remove the ultra-violet divergence which occurs when one evaluates the functional integral (55). One discovers this divergence most simply in the old variables x_μ and $\Delta\varphi$ (see (42)). If we consider the term $\frac{1}{2}\tilde{g}_\mu\,\tilde{\varphi}^{n-2}(\Delta\varphi)^2/(n-2)!$ in perturbation theory (to do this we introduce a factor ε), the calculation of the functional integral

$$e^F=J_0^{-1}\int\prod_x d\Delta\varphi(x)\exp\left[-\int d^Dx\frac{(\partial\mu\Delta\varphi)^2}{2}\right.$$
$$\left.+\varepsilon\int(-\tilde{g}_\mu)\frac{\tilde{\varphi}^{n-2}}{(n-2)!}\frac{(\Delta\varphi)^2}{2}d^Dx\right], \tag{56}$$
$$F=\varepsilon F^{(1)}+\varepsilon^2F^{(2)}+\ldots$$

is equivalent to summing for F the contribution from closed loops in the field $\varepsilon(-\tilde{g})\tilde{\varphi}^{n-2}/(n-2)!$. In first order in the external field we get

$$F^{(1)}=\int d^Dx(-\tilde{g}_\mu)\frac{\Phi^{n-2}}{(n-2)!}\frac{\Delta^0(0)}{2}, \tag{57}$$

where $\Delta^0(x)$ is the Green function of the scalar particle in the coordinate representation. It is clear from (4) that (57) contains an ultra-violet divergence. This divergence is removed for $\varepsilon=1$ due to the first term in $\mathscr{H}'_y$ of (44) corresponding to the infinite renormalization of Γ_{n-2}. The role of the first term in $\mathscr{H}'$ thus reduces to multiplying the result of calculating the integral (55) by the factor

$$A_1=\exp(-\partial F/\partial\varepsilon)\big|_{\varepsilon=0}. \tag{58}$$

In the next order of the perturbation theory in ε we have

$$F^{(2)}=\int d^Dx_1d^Dx_2\left(-\tilde{g}_\mu\frac{\tilde{\varphi}^{n-2}(x_1)}{(n-2)!}\right)\left(-\tilde{g}_\mu\frac{\tilde{\varphi}^{n-2}(x_2)}{(n-2)!}\right)\frac{\{\Delta(x_1-x_2)\}^2}{4}. \tag{59}$$

The integral converges for $n\geqslant 6$ ($D\leqslant 3$). When $n=4$ ($D=4$) there is a logarithmic divergence in the integral

$$F^{(2)}\big|_{x_1\to x_2}\sim\int d^4x\left(-\tilde{g}_\mu\frac{\tilde{\varphi}^2(x)}{2}\right)^2\frac{1}{4}\int\frac{d^4(x_1-x_2)}{|x_1-x_2|^4}\left(\frac{1}{2\pi}\right)^4. \tag{60}$$

One checks easily that the second counterterm in $\mathscr{H}'_y$ (see (44)) corresponding to the renormalization of the charge removes that divergence. The role of the second term in $\mathscr{H}'$ thus reduces to multiplying the result of the integration in (55) by the factor

$$A_2=\exp\left[-\frac{1}{2}\frac{\partial^2F}{\partial\varepsilon^2}\Big|_{\varepsilon=0}+F^{(2)}-\frac{3}{2}\frac{\tilde{g}_\mu^2}{(2\pi)^4}\int\frac{\tilde{\varphi}^4}{4!}d^4x\int\frac{d^4x_{12}}{x_{12}^4}\exp\left(i\frac{2}{\sqrt{3}}p_\mu xy\right)\right]. \tag{61}$$

It is convenient, when evaluating the integral (56), to rewrite it in terms of the variables (52) and (53):

$$e^F = \left(\int \prod_z dY(z) \exp\left\{-\frac{1}{2}\int dS_{D+1}\left[(\nabla_n Y)^2 + \frac{D}{2}\left(\frac{D}{2}-1\right)Y^2\right]\right\}\right)^{-1}$$

$$\times \int \prod dY(z) \exp\left\{-\frac{1}{2}\int dS_{D+1}\left[(\nabla_n Y)^2 + Y^2\left(\frac{D}{2}\left(\frac{D}{2}-1\right) - \varepsilon\frac{D}{2}\left(\frac{D}{2}+1\right)\right)\right]\right\}. \qquad (62)$$

We must diagonalize the quadratic form in Y. This calls for the solution of the following equation for eigenvalues and eigenfunctions in the $(D+1)$-dimensional space:

$$-\nabla_n^2 Y_\lambda = \lambda Y_\lambda, \qquad (63)$$

which is a generalization of the well known equation for the spherical harmonics $Y_l^m e^{im\varphi}$.

The eigenvalue spectrum of Eq. (63) is discrete:

$$\lambda_m = m(m+D-1), \quad m=0, 1, \ldots, \qquad (64)$$

and the eigenfunctions corresponding to a given eigenvalue (64) are

$$Y_{\mu_1\ldots\mu_m} = C_m\{z_{\mu_1}z_{\mu_2}\ldots z_{\mu_m}\}, \quad z_\mu^2 = 1, \qquad (65)$$

where the curly brackets $\{\ldots\}$ denote separation of the terms and symmetrization; we note that the spectrum (64) can easily be obtained if we bear in mind that

$$-\frac{\partial^2}{\partial z_\mu^2} Y_{\mu_1\ldots\mu_m} = 0, \quad \frac{1}{|z|^D}\frac{\partial}{\partial |z|}|z|^D\frac{\partial}{\partial z} Y_{\mu_1\ldots\mu_m} = \frac{m(m+D-1)}{z^2} Y_{\mu_1\ldots\mu_m}.$$

Each eigenvalue of (64) corresponds to

$$N_m = \frac{\Gamma(D+m-1)}{\Gamma(D)\Gamma(m+1)}(D+2m-1) \qquad (66)$$

linearly independent eigenfunctions (65).

We can choose the solutions of Eq. (63) to be orthonormalized on the sphere:

$$\int dS_{D+1} Y_\lambda(z) Y_{\lambda'}^*(z) = \delta_{\lambda\lambda'}.$$

If, therefore, we expand the integration variable in (62) in terms of these functions

$$Y(z) = \sum_\lambda C_\lambda Y_\lambda(z), \qquad (67)$$

the functional integral (62) will factorize when we change to the new variable C_λ:

$$e^F = \prod_\lambda \frac{\int dC_\lambda \exp\{-{}^1/{}_2 C_\lambda^2[\lambda + {}^1/{}_2 D({}^1/{}_2 D - 1 - \varepsilon({}^1/{}_2 D+1))]\}}{\int dC_\lambda \exp\{-{}^1/{}_2 C_\lambda^2[\lambda + {}^1/{}_2 D({}^1/{}_2 D-1)]\}}$$

$$= \exp\left[-\frac{1}{2}\sum_{m=0}^\infty N_m \ln\frac{\lambda_m + {}^1/{}_2 D[{}^1/{}_2 D - 1 - \varepsilon({}^1/{}_2 D+1)]}{\lambda_m + {}^1/{}_2 D({}^1/{}_2 D - 1)}\right]. \qquad (68)$$

We thus get for the expansion in ε the following coefficients, which are necessary for evaluating A_1 and A_2 (see (58) and (61)):

$$\frac{\partial F}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{1}{2}\frac{D}{2}\left(\frac{D}{2}+1\right)\sum_{m=0}^\infty \frac{N_m}{\lambda_m + {}^1/{}_2 D({}^1/{}_2 D-1)},$$

$$\frac{1}{2}\frac{\partial^2 F}{\partial \varepsilon^2}\Big|_{\varepsilon=0} = \frac{1}{2}\left[\frac{D}{2}\left(\frac{D}{2}+1\right)\right]^2 \frac{1}{2}\sum_{m=0}^\infty \frac{N_m}{[\lambda_m + {}^1/{}_2 D({}^1/{}_2 D-1)]^2}. \qquad (69)$$

It is clear from Eqs. (64) and (66) that the series for $(\partial F/\partial\varepsilon)_{\varepsilon=0}$ and for $(\partial^2 F/\partial\varepsilon^2)_{\varepsilon=0}$ diverge at the upper limit like $\sum m^{D-3}$ and $\sum m^{D-5}$, respectively, i.e., there is a power-law divergence in $(\partial F/\partial\varepsilon)_{\varepsilon=0}$ for all $D>2$, and a logarithmic divergence only for $D=4$ in $(\partial^2 F/\partial\varepsilon^2)_{\varepsilon=0}$.

We now turn to the calculation of the functional integral for γ_n (see (55)). Using Eq. (67) to introduce the variables C_λ instead of the integration variables $Y(z)$, we can write the result in the form (cf. (68))

$$\gamma_n = \frac{A_1 A_2}{\sqrt{2\pi}}\prod_\lambda \frac{\int dC_\lambda \exp\{-{}^1/{}_2 C_\lambda^2(\lambda - D) - {}^1/{}_4 D^2\delta_{\lambda 0}C_0^2\}\delta^D(R_\sigma)\delta(R)}{\int dC_\lambda \exp\{-{}^1/{}_2 C_\lambda^2[\lambda + {}^1/{}_2 D({}^1/{}_2 D-1)]\}} \qquad (70)$$

$$R_\sigma = \frac{D}{\sqrt{2}} Y_0 \int dS_{D+1} x_\sigma Y, \quad R = \frac{D}{\sqrt{2}} Y_0 \int dS_{D+1} \ln|x| Y. \qquad (71)$$

The quadratic form in the C_λ in the exponent in the integrand in Eq. (70) is not negative-definite. Indeed, for $m=1$ we have $\lambda_m = D$, i.e., the coefficients of the squares of the $m+1$ parameters C corresponding to the orthonormalized eigenfunctions (see (65))

$$Y^i = \frac{z^i}{(D+1)^{1/2} S_{D+1}^{1/2}}, \quad S_{D+1} = 2\pi^{(D+1)/2} \Big/ \Gamma\left(\frac{D+1}{2}\right), \qquad (72)$$

vanish. This fact is a consequence of the Goldstone theorem. Indeed, the initial equations in the theory (18) were symmetric with respect to translational and scale transformations. When we choose the solution with a fixed center and scale (see (38)) there must occur in the quantum spectrum of the problem constants with zero energy. Moreover, the corresponding eigenfunctions $\Delta\varphi^i$ (see (72) and (53)) can be obtained by means of differentiating the set of classical solutions (20) with respect to the parameters x_0 and y.

One can use the δ-functions to get rid of the integration in Eq. (70) over the parameters C_i, employing the formulae

$$\frac{D}{\sqrt{2}} Y_0 \int dS_{D+1} x_\sigma Y^i = \left(\frac{D+1}{2}\right)^{1/2}\delta_{i\sigma},$$

$$\frac{D}{\sqrt{2}} Y_0 \int dS_{D+1} \ln|x| Y^i = \left(\frac{D+1}{2}\right)^{1/2}\delta_{i,D+1}. \qquad (73)$$

Thus we have (cf. (68))

$$\gamma_n = \frac{A_1 A_2}{2\sqrt{\pi}}\left\{\frac{D(D+2)}{4\pi(D+1)}\right\}^{(D+1)/2} \exp\left[-\frac{1}{2}\sum_{m=2}^\infty N_m \ln\frac{\lambda_m - D}{\lambda_m + {}^1/{}_2 D({}^1/{}_2 D-1)}\right]. \qquad (74)$$

Substituting Eq. (74) into Eqs. (49) and (50) we get after some elementary calculations of the integrals which we encounter the following expressions for the asymptotic behavior of the expansion coefficients of the Gell-Mann–Low function (9):

$$\tilde{C}_k(n)=\lim_{k\to\infty} C_k(n)=(-\tilde{g}_\mu(k,n))^{-k}\exp[k(1-n/2)]k^{(n+D)/2}\tilde{C}(n),$$

$$\tilde{C}(n)=\left\{2^{1+D/2}\pi^{(D-1)/4}\left[\Gamma\left(\frac{D}{2}\right)\right]^{-1}\left[\Gamma\left(\frac{D+1}{2}\right)\right]^{1/2}\right\}^n\frac{1}{(4\pi)^{1/2}}$$

$$\times\left\{\frac{D(D+2)}{4\pi(D+1)}\right\}^{(D+1)/2}\exp\left\{-\frac{1}{2}\sum_{m=2}^{\infty}\frac{\Gamma(D+m-1)}{\Gamma(D)\Gamma(m+1)}(D+2m-1)\right.$$

$$\left.\times\left[\ln\left(1-\frac{D(D+2)}{(D+2m)(D+2m-2)}\right)+\frac{D(D+2)}{(D+2m)(D+2m-2)}\right]-\frac{D^2/2}{D-2}\right\} \tag{75}$$

for the case $n\geqslant 6$ and

$$\tilde{C}_k(4)=\lim_{k\to\infty}C_k(4)=\left(\frac{k}{16\pi^2 e}\right)^k k^4\tilde{C}(4),$$

$$\tilde{C}(4)=\exp(3c_E-8)\frac{2^{19/2}\cdot 3^4}{5^{5/2}}\pi\int_0^\infty dy\, y^6[K_1(y)]^4 \tag{76}$$

$$\times\exp\left\{-\frac{1}{2}\sum_{m=2}^{\infty}\frac{2m+3}{\alpha_m}\left[\ln(1-\alpha_m)+\alpha_m+\frac{\alpha_m^2}{2}\right]\right\}\approx 2.75;$$

$$\alpha_m=\frac{6}{(m+1)(m+2)},\quad K_1(y)=\frac{1}{y}\int_0^\infty\frac{dx}{(x^2+1)^{3/2}}\cos xy$$

for the case of a theory with the interaction $H_{int}=g\int d^4x\,\varphi^4/4!$.

It is interesting to compare Eq. (75) with the calculations in an exactly soluble model. In the limit $n\to\infty$ the sum in Eq. (75) can be found in analytic form and we get

$$\lim_{n\to\infty}\lim_{k\to\infty}C_k(n)=\left(\frac{ke}{8\pi}\right)^{(k-1)n/2}\frac{1}{\pi}\exp[-c_E(k+1)]k^{n+1}\left(\frac{k}{e}\right)^{-k}\frac{e^{3/2}}{(2\pi n)^{k/2}\sqrt{2}}. \tag{77}$$

One sees easily that this expression is the same as the asymptotic form for large k of the exact Eq. (25) obtained in a $g\varphi^n/n!$ theory in the limit of large n[5] (see (27a)). Equation (77) proves the statement given in a previous paper[5] that the coefficients $C_k(n)$ can, at large n, be evaluated for fixed k and that such a calculation leads to their correct asymptotic form even when $k\gg n$.

The problem arises: how fast is the asymptotic form approached and what is the accuracy of the asymptotic formulae for small k? The exact calculation of the first coefficients C_2, C_3, C_4, and C_5 (see (27)) for the integral C_k occurring in an exactly soluble model[5] (see (26)) and their approximate calculation, using the asymptotic formula (27a) give, respectively,

$$C_2=\pi\approx 3.14,\ C_3\approx 74.5,\ C_4\approx 1.18\cdot 10^3,\ C_5\approx 1.51\cdot 10^4,$$
$$\tilde{C}_2\approx 8.25,\ \tilde{C}_3\approx 129.5,\ \tilde{C}_4\approx 1.70\cdot 10^3,\ \tilde{C}_5\approx 2.02\cdot 10^4. \tag{78}$$

For the case of a theory with interaction $H_{int}=g\varphi^4/4!$ only the first three expansion coefficients of the Gell-Mann–Low function are known.[9] A comparison of exact calculations and calculations using the approximate Eq. (76) gives, respectively ($\tilde{C}_k(4)\equiv\tilde{A}_k(16\pi^2)^{-k+1}$),

$$A_2={}^3/_2,\quad A_3={}^{17}/_6\approx 2.83,\quad A_4\approx 19.2;$$
$$\tilde{A}_2\approx 0.15,\quad \tilde{A}_3\approx 1.90,\quad \tilde{A}_4\approx 20.9. \tag{79}$$

The approximation to the true coefficients of the Gell-Mann–Low function given by the asymptotic Eqs. (75) and (76) is thus a good one. Moreover, it is clear from the whole preceding discussion that we can make the asymptotic Eqs. (75) and (76) more exact by developing a perturbation theory in the small parameter $1/k$ (which even for the first coefficient of the Gell-Mann–Low function is equal to $\frac{1}{2}$). The exact asymptotic formulae thus obtained could be used to evaluate coefficients $C_k(n)$ with complex k. If the correct analytical continuation of the asymptotic perturbation theory series (9) is to write it as a Watson-Sommerfeld integral

$$\psi(g)=-\frac{1}{2i}\int_{\sigma-i\infty}^{\sigma+i\infty}dk\frac{g^k}{\sin\pi k}C_k(n),\quad 1<\sigma<2, \tag{80}$$

as was suggested in a previous paper[5] the exact asymptotic formulae would give the possibility for an approximate calculation of the integral (80), i.e., would enable us to find the Gell-Mann–Low function approximately. Such a calculation would be of particularly actual interest in the asymptotic-free theories where an infrared catastrophe occurs.[10] The calculation of the Gell-Mann–Low function would also be a great interest in quantum electrodynamics.

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