Longitudinal waves in an inhomogeneous plasma

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Electrostatic waves are considered in an inhomogeneous plasma with an arbitrary degree of spatial dispersion. The problem is solved analytically for the case of an exponential dependence of the electron density on the coordinate. The wave field distribution in the plasma is found in the form of a contour integral. The absorption of energy in the plasma is calculated.

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1. INTRODUCTION

A kinetic theory of longitudinal (electrostatic) waves with a potential electric field in a monotonically inhomogeneous electron plasma is constructed in the present work (the inhomogeneity is one-dimensional, along the X axis). Division of the waves into purely longitudinal and purely transverse is generally speaking, impossible in an inhomogeneous plasma; however, under the condition $k \gg \omega/c$ (k is the characteristic wave vector, ω is the frequency), which we shall assume to be satisfied, the field of the wave is described by the electrostatic potential and, consequently, such a wave is longitudinal.

Excitation of longitudinal oscillations in a plasma can take place either as a result of the action of an external field, or under the action of foreign charges located inside the plasma itself. The classic method of external excitation is the formation of natural oscillations in a plasma placed in a homogeneous high-frequency field of a capacitor. Another example is the action on the plasma of the field created by a decelerating system. In both cases, the field is potential throughout the entire space. We also note that the longitudinal waves arise upon oblique incidence on the plasma of a transverse electromagnetic wave provided that the plasma possesses a weak spatial dispersion and the effective frequency of collisions is small. The field here is not a potential one throughout the entire space, but only in the vicinity of the singular point.

Excitation of the oscillations by foreign charges takes place upon the presence of electron or ion beams in the plasma. Theoretically, the presence of these charges is equivalent to the assumption that the electron distribution function is initially perturbed (initial-value problem). The condition that the field be potential means here that only natural longitudinal waves are considered, which are not radiated but are spatially damped outside the plasma.

Analytically, the distribution of the wave field in the plasma is determined in the present work for an exponential distribution of the density of the electrons in space, while the temperature is assumed to be uniform. Both excitation by an external field (Secs. 3, 4) and by foreign charges (Sec. 5) is considered. In both cases, the field of the wave is found in the form of a contour integral. The fourth section of the paper is devoted to the determination of the power absorbed in the plasma. It is known that the absorption takes place simultaneously with the so-called transformation of the electromagnetic waves into plasma waves (see the review, Ref. 1). In our case, the transformation is distributed in space, since arbitrary values of the parameter $a\omega/v_T$ are permitted (a is the size of the inhomogeneity, v_T is the thermal velocity of the electrons). In the limit of weak spatial dispersion of the waves, the consideration reduces to the well-known problem of transformation near the point at which the dielectric constant of the cold plasma $\epsilon(x)$ vanishes. Thus, the present work turns out to be closely related to the well-known problem of the behavior of the field of an electromagnetic wave near a singular point (see Refs. 2-5).

The investigation is carried out on the basis of the set of Vlasov equations. The results in the limiting case of weak spatial dispersion agree with the results obtained by means of the macroscopic equations.[1-5]

2. EQUATION FOR THE POTENTIAL OF THE FIELD OF THE WAVE IN AN INHOMOGENEOUS PLASMA

The initial set of equations is of the form

$$-g-i\omega\tilde{f}+i\mathbf{k}_\perp\mathbf{v}_\perp\tilde{f}+v_x\frac{\partial\tilde{f}}{\partial x}-\frac{e}{m}\frac{\partial\psi}{\partial x}\frac{\partial\tilde{f}}{\partial v_x}-\frac{e}{m}\frac{\partial\tilde{\varphi}}{\partial x}\frac{\partial f_0}{\partial v_x}-i\mathbf{k}_\perp\frac{e}{m}\frac{\partial f_0}{\partial \mathbf{v}_\perp}\tilde{\varphi}=0, \tag{2.1}$$

$$\frac{\partial^2\tilde{\varphi}}{\partial x^2}-k_\perp^2\tilde{\varphi}=-4\pi e\int\tilde{f}\,d\mathbf{v}. \tag{2.2}$$

Here $\mathbf{v}_\perp$ is the component of the velocity of the electron parallel to the (Y, Z) plane, $\tilde{\varphi}$ and $\tilde{f}$ are the respective Fourier components of the potential of the electric field $\varphi(r, t)$ and the departure from the equilibrium distribution function $f_1=\tilde{f}-f_0$:

$$\tilde{\varphi}(x, \mathbf{k}_\perp, \omega)=\int_0^\infty dt\int d^2\mathbf{r}_\perp \exp[i\omega t-i\mathbf{k}_\perp\mathbf{r}_\perp)\varphi(\mathbf{r}, t), \tag{2.3}$$

$$\tilde{f}(\mathbf{v}, x, \mathbf{k}_\perp, \omega)=\int_0^\infty dt\int d^2\mathbf{r}_\perp \exp(i\omega t-i\mathbf{k}_\perp\mathbf{r}_\perp)f_1(\mathbf{v}, \mathbf{r}, t). \tag{2.4}$$

We shall assume the equilibrium distribution function of the electrons to be equal to

$$f_0=n_0\left(\frac{m}{2\pi T_e}\right)^{3/2}\exp\left(-\frac{e\psi(x)+mv^2/2}{T_e}\right). \tag{2.5}$$

We shall also assume that the potential $\psi(x)$ is a monotonic function, and if, that $e\psi(x)\to\mp\infty$ as $x\to\pm\infty$. The function g in Eq. (2.1), which is equal to

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$$g(\mathbf{v}, x, \mathbf{k}_\perp) = \int f_1(\mathbf{v}, \mathbf{r}, 0) \exp(-i\mathbf{k}_\perp \mathbf{r}_\perp) d^2\mathbf{r}_\perp,$$

arises when the initial-value problem is considered. Equation (2.1) is solved by the method of characteristics exactly as was done in Refs. 6 and 7. As a result, we find the charge density in the plasma:

$$\bar{\rho}(x, \mathbf{k}_\perp, \omega) = e\int \bar{f}\, d\mathbf{v} = -\frac{n_0 e^2}{T_e}\left\{\varphi(x)\exp\left(-\frac{e\psi}{T_e}\right)\right.$$

$$\left. -i\left[\int_{-\infty}^{x}\Sigma(x, x') + \int_{x}^{+\infty}\Sigma(x', x)\right]\varphi(x')dx' + G(x, \mathbf{k}_\perp)\right\}. \qquad (2.6)$$

Here

$$\Sigma(x', x) = -\omega\left(\frac{m}{2\pi T_e}\right)^{1/2}\exp\left(-\frac{e\psi}{T_e}\right)\int\exp\left(-\frac{mv_\perp^2}{2T_e}\right)d^2\mathbf{v}_\perp\int_0^\infty dv_x\{v_x^2$$
$$+2e[\psi(x)-\psi(x')]/m\}^{-1/2}\{\exp[-\Phi(x, x')] + \exp[-\Phi(x^*, x') - \Phi(x^*, x)]\}, \qquad (2.7)$$

$$G(x, \mathbf{k}_\perp) = e\int d\mathbf{v}\left\{\int_{x^*} g_+(x')\exp[\Phi(x, x')]\right.$$

$$+\int_x^\infty g_-(x')\exp[\Phi(x, x')] + \int_{x^*}^\infty g_-(x')\exp[-\Phi(x^*, x')$$

$$\left.-\Phi(x^*, x)]\right\}\left[v_x^2 + \frac{2e}{m}(\psi(x) - \psi(x'))\right]^{-1/2}dx', \qquad (2.8)$$

$$g(\mathbf{v}, x, \mathbf{k}_\perp) = \begin{cases} g_+(x), & v_x > 0 \\ g_-(x), & v_x < 0 \end{cases}$$

$$\Phi(x_1, x_2) = -\int_{x_1} dx'(i\omega - i\mathbf{k}_\perp\mathbf{v}_\perp)[v_x^2 + 2e(\psi(x) - \psi(x'))]^{-1/2},$$

the quantity $x^*(v_x, x)$ is determined from the relation

$$e\psi(x^*) = e\psi(x) + {}^1\!/\!_2 m v_x^2.$$

The equation for the potential is obtained upon substitution of (2.6) in (2.4).

To test the resultant expression, we consider the transition to a homogeneous plasma: $\psi = 0, x^* = -\infty$. Calculating the Fourier transform

$$\rho(\mathbf{k}, \omega) = \int_{-\infty}^{+\infty} dx \exp(-ik_x x)\bar{\rho}(x, \mathbf{k}_\perp, \omega)$$

under these conditions, we can then prove that the result is represented in the form[8]

$$\rho(\mathbf{k}, \omega) = i\int\frac{eg(\mathbf{v}, \mathbf{k})}{\omega - \mathbf{k}\mathbf{v}}d\mathbf{v} - \frac{e^2}{m}\varphi(\mathbf{k}, \omega)\int\frac{\mathbf{k}(\partial f_0/\partial\mathbf{v})}{\omega - \mathbf{k}\mathbf{v} + i0}d\mathbf{v}.$$

In the other limiting case, which corresponds to a cold plasma $(T_e = 0)$, Eq. (2.4) reduces to

$$(1 - \omega_0^2 e^{-e\psi}/\omega^2)\left(\frac{\partial^2\varphi}{\partial x^2} - k_\perp^2\varphi\right)$$
$$+\left(\frac{\omega_0^2 e^{-e\psi}}{\omega^2}\right)\left(e\frac{d\psi}{dx}\right)\frac{\partial\varphi}{\partial x} = 0, \quad \omega_0 = \left(\frac{4\pi n_0 e^2}{m}\right)^{1/2}. \qquad (2.9)$$

The last equation is none other than $\nabla(\epsilon\nabla\bar{\varphi}) = 0$.

We shall assume that the equilibrium density of the electrons depends exponentially on the x coordinate, and substitute in (2.5)

$$-e\psi/T_e = x/a. \qquad (2.10)$$

After uncomplicated transformations, the dimensionless equation for the potential takes the form

$$\frac{d^2f}{d\xi^2} - p^2 f = f\alpha e^\xi - q_p(\xi)$$
$$+\frac{i\alpha\gamma}{2\pi^{1/2}}\int_{-\infty}^{+\infty} f(\xi')\exp\left(\frac{\xi+\xi'}{2}\right)R(\xi-\xi')d\xi', \qquad (2.11)$$

where

$$\xi = x/a, \quad f(\xi) = \varphi(a\xi), \quad \mathbf{p} = \mathbf{k}_\perp a,$$
$$\alpha = m\omega_0^2 a^2/T_e, \quad \gamma = \omega a(2m/T_e)^{1/2},$$
$$R(\xi - \xi') = \int\left(\frac{m}{2\pi T_e}\right)\exp\left(-\frac{mv_\perp^2}{2T_e}\right)d^2\mathbf{v}_\perp$$
$$\times\int_{-\infty}^{+\infty} dw \exp\left[-\frac{1}{4}e^{2w} + ile^w - \frac{1}{4}(\xi' - \xi)^2 e^{-2w}\right],$$
$$l = a(\omega - \mathbf{k}_\perp\mathbf{v}_\perp)(2m/T_e)^{1/2}, \quad q_p(\xi) = 4\pi G(a\xi, \mathbf{k}_\perp)$$

(see also Ref. 7).

Equation (2.11) is solved by the same method as the corresponding integral equation in Refs. 6, 7 and 9; it reduces to the functional equation

$$[(k+1)^2 - p^2]F(k+1) - \alpha(1 - M(k))F(k) = -Q_p(k+1). \qquad (2.12)$$

Here $F(k)$ and $Q_p(k)$ bilateral Laplace transforms corresponding to the functions $f(\xi)$ and $q_p(\xi)$,

$$M(z) = -i\int_0 dy \exp\left[iy - \frac{p^2 - z(z+1)}{\gamma^2}y^2\right]; \qquad (2.13)$$

another representation of $M(z)$ is of the form

$$M(z) = \gamma[p^2 - z(z+1)]^{-1/2}\exp\left\{-\frac{\gamma^2}{4}[p^2 - z(z+1)]^{-1}\right\}$$
$$\times\int_0^{(\gamma/2)[p^2 - z(z+1)]^{-1/2}} e^{z'^2}dz' - \frac{i\gamma\pi^{1/2}}{2[p^2 - z(z+1)]^{1/2}}\exp\left\{-\frac{\gamma^2}{4}[p^2 - z(z+1)]^{-1}\right\}.$$

3. SOLUTION OF THE HOMOGENEOUS EQUATION. THE BOUNDARY-VALUE PROBLEM

In the consideration of the boundary-value problem, the right side of Eq. (2.12) must be set identically equal to zero. The solution of the corresponding homogeneous equation, according to Ref. 7, is of the form

$$F'(k) = h(k)\frac{\alpha^k\exp(-i\pi k)}{\Gamma(k+1+p)\Gamma(k+1-p)}\frac{U(k)}{\sin\pi(k-p)}. \qquad (3.1)$$

Here

$$U(k) = \exp\left\{\frac{1}{2i}\int_{c-i\infty}^{c+i\infty}\frac{\ln(1 - M(z))\sin k\pi}{\sin z\pi\sin(z-k)\pi}dz\right\}, \qquad (3.2)$$

$h(k)$ is a function with the period of unity which should be so chosen that the necessary analytic properties of the function $F'(k)$ are satisfied. The contour integration in (3.2) is carried out in such a way that

$$-1 < \mathrm{Re}\, k < 1 \qquad (3.3)$$

the value of c is contained in the interval

$$\max(-1, \mathrm{Re}\, k - 1) < c < \min(0, \mathrm{Re}\, k). \qquad (3.4)$$

In addition to the potential, we shall need the current $\bar{j}_x(x, k_\perp)$ in what follows. This current is most simply calculated by starting out from the equation of continuity. As a result, after nondimensionalizing and carrying out the Laplace transform, we obtain

$$J(k) = \frac{i\omega}{4\pi a}\frac{(k-1)(k^2 - p^2)}{p^2 - k(k-1)}F'(k), \qquad (3.5)$$

$J(k)$ is the Laplace transformation of the function $j(\xi) = \bar{j}_x(a\xi, pa)$.

We now return to the expression (3.1). The function $M(z)$ has singularities at the points $z_{1,2} = -\frac{1}{2} \mp (\frac{1}{4}+p^2)^{1/2}$, whence we see that the function (3.2) has singularities at $k = z_1 - n$ and $k = z_2 + n + 1$ ($n = 0, 1, 2, \ldots$). As $k \to \pm i\infty$, the function $U(k)$ can be majored by an exponential. In the boundary-value problem, $F'(k)$ should be such that

$$f(\xi) \underset{\xi\to-\infty}{\longrightarrow} +\infty, \quad f(\xi) \underset{\xi\to+\infty}{\longrightarrow} 0.$$

Taking these conditions into account, and also the regularity of $F'(k)$ at $k = \pm i\infty$, we find, without ambiguity,

$$h(k) = i\frac{2\pi^2\alpha^p \exp(-i\pi p)}{[1-\exp(-2\pi i(k+p))]U(-p)\Gamma(2p)p}\bar{f}. \tag{3.6}$$

The factor that is independent of k is so chosen that the function $f(\xi)$ which is found from the inverse Laplace transform formula,

$$f(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F'(k)\exp(k\xi)dk, \quad z_1 < c < -p, \tag{3.7}$$

takes as $\xi \to -\infty$ the asymptotic form

$$f(\xi) = (\bar{f}/p)\exp(-p\xi). \tag{3.8}$$

4. ASYMPTOTIC FORMULAS. ENERGY ABSORPTION

1. We first investigate the integral (3.7) under the condition

$$\gamma \gg p+1. \tag{4.1}$$

For this, we start from the representation (3.1), and transform $F'(k)$ to the form

$$F'(k) = f'\frac{\Gamma(k-z_1)\Gamma(k-z_2)}{\Gamma(k+1+p)\Gamma(k+1-p)}$$
$$\times\left(\frac{\omega_0}{\omega}\right)^{2k}\frac{\sin\pi(k-z_2)}{\sin\pi(k-p)}\frac{1}{1-\exp[-2\pi i(k+p)]}$$
$$\times\exp\left\{\frac{1}{2i}\int_{c-i\infty}^{c+i\infty}\ln\left[\frac{\gamma^2}{2}\frac{1-M(z)}{z(z+1)-p^2}\right]\frac{\sin\pi k}{\sin\pi z\sin\pi(z-k)}dz\right\}, \tag{4.2}$$

where

$$f' = 2i\bar{f}\frac{\alpha^p\exp(-i\pi p)\Gamma(1+z_1)\Gamma(1+z_2)\sin\pi z_2}{U(-p)\Gamma(2p)p},$$

and call attention to the fact that the function (2.13) satisfies the identity

$$\frac{\gamma^2}{2}\frac{1-M(z)}{z(z+1)-p^2} = 1-\varepsilon^T[a^{-1}(p^2-z(z+1))^{1/2}], \tag{4.3}$$

in which $\epsilon^T(k)$ is the wave-vector-dependent longitudinal permittivity of a homogeneous Maxwellian electron plasma[8] under the condition that the Langmuir frequency of the oscillations is equal to ω (compare with Ref. 5). At $\gamma = \infty$, the value of ϵ^T vanishes, so that $\exp\{\cdots\} = 1$ in (4.2). The remaining expression is the Laplace transform of the potential of the field in the cold plasma. This expression could have been obtained directly by solving Eq. (2.9) under the condition (2.10). Then, calculating the integral (3.7), we could have established the fact (which was to have been expected) that it diverges logarithmically at $\xi = \xi_0 = -2\ln(\omega_0/\omega)$. In the subsequent approximation in γ^{-1}, we set [1)] $\ln(1-\epsilon^T) = -\epsilon^T$, after which we consider the behavior of the function $f(\xi)$ near ξ_0. Then the values $|\mathrm{Im}k| \gg 1$ are important in (4.2), and the expression in the curly brackets in (4.2) is equivalent to the integral

$$-\int_0^k \varepsilon^T(a^{-1}(p^2+|z|^2)^{1/2})dz.$$

Then, transforming in (4.2) from the Laplace component to the Fourier component, we find complete agreement with Ref. 5 (see Eq. (5.2) in Ref. 5). Such a result is entirely natural; near the singularity ξ_0, the condition of weak inhomogeneity of the plasma is satisfied; therefore, the behavior of the field does not depend on the specific profile of the electron concentration.

In the other limiting case, which is the opposite of (4.1),

$$\gamma \ll p+1,$$

we can neglect the integral term in the expression (2.6) in the zeroth approximation in γ, so that the problem reduces to Debye screening in an inhomogeneous plasma. For an exponential concentration profile, we find the following solution:

$$\varphi(x) = K_{2p}(2\alpha^{1/2}e^{x/2a})\frac{2\alpha^p}{p\Gamma(2p)}\bar{f}. \tag{4.4}$$

Naturally, these formulas could have been obtained by calculating the intergral (3.7), but such a method turns out to be too complicated, although it does allow us to find the corrections to the solution (4.4) to a higher order in γ. We shall not pause to set down the corresponding expressions here.

2. We now consider the asymptotic behavior of $f(\xi)$ as $\xi \to -\infty$. Calculating the residues in (3.7) at the points $-p$, $-p+1$ and p, we find

$$f(\xi) = \frac{\bar{f}}{p}\left\{e^{-p\xi} + \frac{\alpha(1-M(-p))}{1-2p}\exp(\xi-p\xi) + \frac{\alpha^{2p}U(p)\Gamma(-2p)}{U(-p)\Gamma(2p)}e^{p\xi}\right\} \tag{4.5}$$

In the particular case $p = 0$, the asymptotic form of (4.5) is linear:

$$f(\xi) = -2\bar{f}(\xi+2C+\ln\alpha+\Delta_0(\gamma)), \tag{4.6}$$

where C is the Euler constant and the function $\Delta_p(\gamma)$ is defined below. We use the expansion (4.5) for calculation of the power absorbed per unit area of the plasma

$$P = \frac{1}{2}\mathrm{Re}\int j^*E dx = \left[\frac{\omega}{8\pi}\mathrm{Im}\left(\varphi\frac{\partial\varphi^*}{\partial x}\right) + \frac{1}{2}\mathrm{Re}(\bar{j}_x^*\varphi)\right]\Big|_{x=-\infty}. \tag{4.7}$$

We first assume that

$$p < 1/2.$$

Then $\bar{j}_x$ is the plasma is proportional to $n_e(x)E_x(x)$ as $x \to -\infty$; therefore, $j_x^*\varphi \sim \exp[(1-2p)x/a)]$, which enables us to discard the second term in (4.7). Then the first term leads to the result

$$P=\frac{\omega|\tilde f|^2}{2\pi a}\left(\frac{2\alpha}{\gamma^2}\right)^{2p}\chi_p(\gamma), \tag{4.8}$$

where

$$\chi_p(\gamma)=-\left(\frac{\gamma^2}{2}\right)^{2p}\frac{\Gamma(-2p)}{\Gamma(2p)}\operatorname{Im}\Delta_p(\gamma)$$

$$\Delta_p(\gamma)=\frac{1}{2p}\left[\exp\left\{\sin(2p\pi)\int_0^\infty\frac{dx}{\operatorname{ch}\pi x+\cos 2p\pi}\right.\right. \tag{4.9}$$

$$\left.\left.\times\ln\left[1-M\left(\frac{ix-1}{2}\right)\right]\right\}-1\right].$$

Thus, the absorption is on the whole determined by the outer asymptote of (4.5). The graphs of the function $\chi_p(\gamma)$ are shown in Fig. 1.

In the case of weak spatial dispersion [condition (4.1)] it is possible to take into account in the absorption the corrections due to the collisions (see footnote 1); as a result, the formula takes the following form:

$$P=\frac{\omega|\tilde f|^2}{2a\sin 2p\pi}\left(\frac{\omega_0}{\omega}\right)^{4p}\left[\frac{\Gamma(p-z_1)}{\Gamma(1+2p)\Gamma(-p-z_1)}\right]^2$$

$$\times\left(1+\frac{4p+8p^3}{\gamma^2}\right)\sin 2p(\pi-\nu_{eff}/\omega-I), \tag{4.10}$$

where

$$I=\frac{2^{5/3}(\pi\gamma)^{4/3}}{3^{1/2}}\frac{\sin 2p\pi}{p}\exp\left[-3\left(\frac{\pi\gamma}{2}\right)^{2/3}+p^2\left(\frac{2\pi^2}{\gamma}\right)^{2/3}\right]. \tag{4.10'}$$

This formula is valid under the condition $\gamma^{2/3}\gg p+1$. In the absence of collisions and dispersion ($\nu_{eff}=\gamma^{-1}=0$) the expression (4.10) remains finite. The presence of absorption in this case is connected with the excitation of plasma waves at the singularity. If $p=0$, then the formulas (4.8) and (4.10) transform to

$$P=\frac{\omega a}{8\pi}|E_x^0|^2\chi_0(\gamma);\quad \nu_{eff}=0,\quad p=0; \tag{4.11}$$

$$P=\frac{\omega|E_x^0|^2 a}{8}\left\{1-\frac{\nu_{eff}}{\pi\omega}-\frac{2^{5/3}(\pi\gamma)^{4/3}}{3^{1/2}}\exp\left[-3\left(\frac{\pi\gamma}{2}\right)^{2/3}\right]\right\};\tag{4.12}$$

$$\gamma\gg 1,\ p=0.$$

respectively. In these formulas, $E_x^0=2\tilde f/a$ is the uniform field outside the plasma.

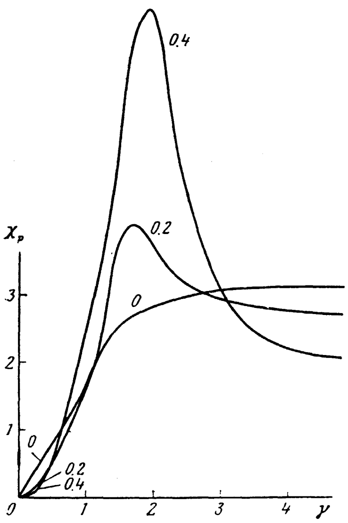

FIG. 1. Dependence of χ_p on $\gamma=\omega a(2m/T_e)^{1/2}$. The values of $p=k_\perp a$ are indicated by the numbers on the graph.

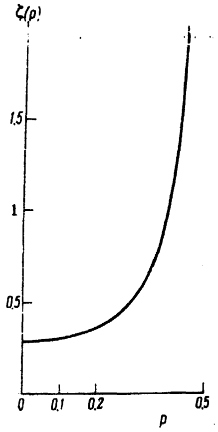

FIG. 2.

In the other limiting case, when $\gamma\ll 1$, we find

$$P=\frac{\omega a\pi^{1/2}(\omega_0/\omega)^{4p}\gamma^{1+4p}}{2^{2p+3}\Gamma^2(2p+1)}\zeta(p)|E_x^0|^2, \tag{4.13}$$

where

$$\zeta(p)=\int_0^\infty\frac{dx}{(4p^2+x^2+1)^{1/2}(\operatorname{ch}\pi x+\cos 2p\pi)}$$

(see Fig. 2). At $p=0$, (4.13) transforms into

$$P=0.127\frac{\omega^2a^2}{v_T}|E_x^0|^2,\quad v_T=\left(\frac{2T_e}{m}\right)^{1/2}. \tag{4.14}$$

We discuss in detail the behavior of the absorption curves in a collision-free plasma (Fig. 1). [It is convenient to set the origin of the coordinates at the point ξ_0, when $\omega_0=\omega$ and the dependence of the absorption on the wave vector $k_\perp$ is entirely determined by the function $\chi_p(\gamma)$.] As is seen from Fig. 1, in a cold plasma ($\gamma=\infty$) the absorption decreases with increase in $k_\perp$. This is explained by the fact that, upon an increase in the parameter $p=k_\perp a$, the wave is damped more rapidly in the interior of the plasma and therefore the amplitude of the electric field in the vicinity of the singularity decreases. We note that at $\gamma^{-1}=\nu_{eff}=0$ the absorption, which is determined from formulas (4.10) and (4.12), agrees entirely with the value of the absorption given in Refs. 1 and 5.

We now consider large but finite values of γ. It is seen from this same drawing that the absorption at $p=0$ falls off with increase in γ while the absorption increases for the remaining values of p. Such a qualitative difference is explained by the appearance at weak spatial dispersion, of two competitive effects, which have different affects on the absorption.

Actually, it is clear first of all that upon decrease in γ the resonance conditions deteriorate in the vicinity of ξ_0. In the case at hand, the spatial dispersion, as well as the collisions, decreases the coefficient of transformation of the electromagnetic wave into a plasma wave, thus leading to a corresponding decrease in the absorption. At weak dispersion, the corrections that are obtained are exponentially small. They are connected with the excitation of plasma waves with k_x

$\sim a^{-1}\gamma^{2/3}$ (see Ref. 2. p. 339), the value of the correction is given in formulas (4.10)–(4.10′) and (4.12).

Another effect of spatial dispersion is the increase in the plasma frequency, which is connected with the spatial inhomogeneity of the field: $\Delta\omega_{\rm corr} \sim (\frac{3}{2})\omega_0(kr_D)^2 \sim \omega_0(1+p^2),/\gamma^2$. In other words, the density of electrons in the plasma effectively increases, which, in agreement with (2.5), is equivalent to a displacement of the entire plasma to the left, closer to the source of the external field. If $k_\perp \neq 0$, than the amplitude of the field increases as a result in the vicinity of ξ_0, and consequently the absorption becomes greater. The corresponding correction has a power-law dependence; it is easy to estimate it by adding the quantity $\Delta\omega_{\rm corr}$ to ω_0 in (4.8). Exact calculation leads to the factor $(1+(4p+8p^3)\gamma^2)$, see the formula (4.10). As is seen from these considerations, the absorption increment is connected with the spatial change of the field outside the plasma. It vanishes if $p=0$. At $p>0$ this increment is more important than the exponentially small correction (4.10).

In the other limiting case, at small γ, the coefficient of transformation of the wave into a plasma wave should be obviously proportional to γ. Moreover, the field in the plasma, according to (4.4), begins to differ appreciably from the field in the vacuum at $x \sim -2a\ln\gamma$ and $\gamma^2 \sim \alpha$: therefore, the amplitude of the potential in the absorption region is of the order of $\tilde{f}\gamma^{2p}/p$ [see Fig. 1 and formula (4.13)].

The shape of the curves in Fig. 1 becomes understandable from the discussions above. If $k_\perp \neq 0$, then the absorption has a maximum at $\gamma \approx 1.5$–2. But if $k_\perp = 0$, then the absorption curve is monotonic.

We call attention to the fact that if $\nu_{\rm eff}=0$, then the expression (4.12) for the absorbed power remains entirely valid in a plasma with an arbitrary concentration profile. In fact, Eq. (2.9) at $k_\perp=0$ describes a cold plasma placed in a uniform field; therefore the solution at the given point $\tilde{x}$ does not depend on the distribution of $\epsilon(x)$ throughout all space, but is completely determined by the local dielectric constant $\epsilon(\tilde{x})$. For the same reason, the behavior of the field near the singularity is determined by the derivative $|d(\ln\epsilon)/dx| = a^{-1}$ at the singularity. The exponential correction in (4.12), which is connected with the presence of dispersion, is also valid for an arbitrary density profile, since the region of space in which are excited, the waves that contribute to this correction (the transformation region) is much smaller than the dimension of the inhomogeneity ($k_x^{-1} \sim a\gamma^{-2/3} = |d\ln\epsilon/dx|^{-1}$). As to the collision correction in (4.12), it is valid only for the given case of an exponential distribution of the electrons in space. This is connected with the fact that the collisions influence the absorption at all points in space where the electric field is significant. Consequently, the region of collision absorption turns out always to be of the order of the size of the inhomogeneity. The corresponding correction to the absorption for another concentration profile can even have a sign opposite to (4.12). This is the case, for example, if the electron density increases in the positive x direction in power-law fashion.

We note that all the formulas relating to the wave with $k_\perp=0$, (4.6), (4.11), (4.12) and (4.14), can be used for the case of excitation of a longitudinal field of the plasma of a plane wave incident obliquely to the surface, provided that $k_\perp < \omega/c \ll a^{-1}$ and, consequently, the field in the plasma is a potential one in the zeroth approximation in ω/c. The transverse wave that is generated upon oblique incidence cannot of course be described by a potential field. The problem of the transverse wave is a problem of the skin effect, and was considered in Refs. 6, 7 and 9. The absorption in the case of oblique incidence is determined by the total absorption of the longitudinal wave (4.11) and of the transverse wave.[7] 2)

3. We shall now assume that

$$p \geqslant 1/2.$$

In this case, the results obtained by us, which refer to the energy absorption, are incorrect. Actually, calculation according to (4.7) shows that the terms we have discarded now become divergent at $\xi \to -\infty$. This means that the basic contribution to the absorption is made by the "tail" of the spatial distribution of the electrons in the plasma at $-\xi \gg 1$, and not by the vicinity of the singularity in the interior of the plasma. Let the potential of the wave be specified at a point located at a distance L to the left of the coordinate origin (for example, the wall of the waveguide is located at this point). If L is sufficiently large,

$$\exp(L/a) \gg \alpha + (\omega_0/\omega)^2, \tag{4.15}$$

then the field near the wall does not differ from the field in the vacuum, since the effect of the plasma can be neglected. 3) The problem of the absorption can be solved as before by the aid of the asymptotic form of (4.5) at any p.

Before we proceed to the calculation of the absorption from (4.7), we must find the asymptotic behavior of the current j_x at $-x \gg a$. Applying the inverse Laplace transform to the function (3.5), we find that as $\xi \to -\infty$

$$j(\xi) = \frac{i\omega}{4\pi a p}\tilde{f}\alpha(1-M(-p))\exp(\xi - p\xi). \tag{4.16}$$

The calculation according to (4.7) must now be carried out with $-\infty$ substituted for $-L$, and we obtain 4)

$$P = \frac{\omega}{4\pi a}\frac{|\varphi_0|^2}{p}\left\{\frac{e^{-\lambda}}{1-2p}\,\mathrm{Im}[\alpha - \alpha M^*(-p)] + 2p\left(\frac{2\alpha}{\nu^2}\right)^{2p} e^{-2p\lambda}\chi_p(\gamma)\right\}. \tag{4.17}$$

Here $\lambda = L/a$ and $\varphi = \tilde{f}e^{p\lambda}$ is the fixed potential at the point $x=-L$. In (4.17), we have kept as the second term the already known term that determines the absorption near the singularity. It is necessary to take into account both terms in Eq. (4.17) in the case in which $|2p-1|\lambda \lesssim 1$; if $(2p-1)\lambda \gg 1$ or $(1-2p)\lambda \gg 1$, then it suffices to limit ourselves to only the first or second term (at $p > \frac{1}{2}$, this is valid if the spatial dispersion is not too weak, so that the factor in front of $e^{-\lambda}$ in (4.17) is not anomalously small). In this case, when $|2p-1|\lambda \ll 1$, the absorption contains the factor x, since (4.17) transforms to

$$P = \frac{|\tilde{f}|^2}{2}\left(\frac{m^3}{\pi T_e^3}\right)^{1/2}(\omega\omega_0)^2 x a \exp\left(-\frac{m\omega^2 a^2}{T_e} - \frac{x}{a}\right).$$

If $2p = 2, 3, 4, \ldots$, then the calculation with (4.17) gives erroneous results, since we have omitted in the derivation all terms that fall off more rapidly than $e^{-\lambda}$. Analysis shows that these terms have singularities at $2p = 2, 3, 4, \ldots$ and such that they cancel the singularities in (4.17). We can write down the formula for the power that is suitable for all p_2, by artificially canceling the singularities of the function $\Gamma(-2p)$:

$$P=\frac{\omega}{4\pi ap}|\varphi_0|^2\left\{\frac{\alpha\gamma}{2(2p-1)}\left(\frac{\pi}{p}\right)^{1/2}\exp\left(-\lambda-\frac{\gamma^2}{4p}\right)\right.$$
$$\left.-2p\alpha^{2p}e^{-2p\lambda}\frac{\Gamma(-2p)}{\Gamma(2p)}\operatorname{Im}\Delta_p(\gamma)+\sum_{n=2}^{\infty}\frac{(-1)^n e^{-n\lambda}}{(n-2p)[(n-1)!]^2}\alpha^n\operatorname{Im}\Delta_{n/2}(\gamma)\right\}. \tag{4.18}$$

In this case, if the dispersion in the collision can be entirely neglected, then the tail of the electron distribution makes no contribution to the absorption. The entire absorption, as in the case $p<\frac{1}{2}$, is determined by the neighborhood of the singularity. This is also seen from the fact that the expression (4.10) will not have a singularity at any finite p if we set $\nu_{eff}=\gamma^{-1}=0$.

4. The power absorbed in the plasma can be connected with the energy flux of the wave outside the plasma $P=k_dS$. Here k_d is the damping coefficient of the wave per unit length (in the direction of $\mathbf{k}_\perp$), S is energy flux, taken in the direction perpendicular to x and $\mathbf{k}_\perp$). This flux is equal to[10]

$$S=\frac{\omega k_\perp|E_\perp(-L)|^2}{16\pi\varkappa^3};\quad \varkappa^2=k_\perp^2-\frac{\omega^2}{c^2},\quad \varkappa L\gg 1.$$

It is not difficult to note that in our case

$$E_\perp(-L)=-i(f\varkappa/k_\perp a)\exp(\varkappa L),$$

whence, at $p<\frac{1}{2}$, we find the damping coefficient (in cm^{-1}) from Eq. (4.8):

$$k_d=8k_\perp\varkappa a(\omega_0/\omega)^{4k_\perp a}e^{-2\varkappa L}\chi_p(\gamma),\quad k_\perp>\omega/c\ll a^{-1}.$$

We can find k_d at $p>\frac{1}{2}$ also in similar fashion.

In the case in which $k_\perp=0$, the absorption can be described conveniently with the help of the specific impedance

$$Z=\int_{-L}^{+\infty}\frac{E_x(x)\,dx}{j_x^0}=\frac{\varphi(-L)}{j_x^0}. \tag{4.19}$$

Here $j_x^0=-i\omega(4\pi)^{-1}(\partial\varphi/\partial x)=i\omega(4\pi)^{-1}E_x^0$ is the current density in the interior of the plasma at $x=+\infty$, $\varphi(-L)$ is the potential of the capacitor plate or of the probe which produce a uniform field E_x^0 outside the plasma at $-x\gg a$. Equation (4.19) describes the impedance of the plasma of unit cross section in the (Y, Z) plane. The real part of the impedance, as is easily seen [see (4.6)], does not depend on L, and the resistivity is equal to

$$\rho=\operatorname{Re}Z=\frac{4\pi a}{\omega}\chi_0(\gamma) \tag{4.20}$$

while at $\gamma\ll 1$, it turns out to be independent of frequency:

$$\rho=40.07a^2/v_T,\quad v_T=(2T_e/m_e)^{1/2}.$$

The absorption (4.11) can be represented in the form $P=\rho|j_x^0|^2/2$.

Equation (4.20) can be used in the study of the absorption in a plasma layer with diffuse boundaries, located in a uniform high-frequency field inside a parallel-plate capacitor. In this case, the frequency should be sufficiently low that the field does not penetrate into the interior of the plasma and the absorption takes place on the boundaries of the layer with the assumed exponential decrease of the density. The total active resistance of the plasma capacitor is determined by the sum of expressions of the form (4.20) corresponding to each boundary of the layer.

In the problem of oblique incidence, on the plasma, of a wave with vector $\mathbf{E}$, lying in the plane of incidence, the absorption, as has already been pointed out, is determined by the sum of the "transverse" and "longitudinal" absorptions:

$$\mathscr{P}=\frac{R|I_\perp|^2}{2}+\frac{\rho|j_x^0|^2}{2}=\frac{|E_0|^2}{8\pi^2}(c^2R+\omega\pi a\chi_0\sin^2\beta).$$

Here β is the angle of incidence, R is the "transverse" surface resistance, which is determined by the expression (3.1) in Ref. 7, while the "transverse" current $I_\perp$ is equal to

$$I_\perp=-c(4\pi)^{-1}H|_{x=-\infty}=-c(2\pi)^{-1}E_0$$

($\mathbf{E}_0$ is the amplitude of the field of the wave).

5. In conclusion, we write down the asymptotic expression for the field in the plasma as $\xi\to+\infty$:

$$f(\xi)=\begin{cases}C_1(p)\alpha^{p+z_1}\gamma e^{z_1\xi}\xi^{-1/2}, & \gamma\xi^{1/2}\ll p+1\\ C_2(p)\alpha^{p+z_1}\gamma^2e^{z_1\xi}, & \gamma\xi^{1/2}\gg p+1\end{cases}.$$

We shall not give the formulas for the coefficients C_1 and C_2.

5. FIELD EXCITED BY THE FOREIGN CHARGES

We shall seek a solution of the inhomogeneous equation (2.12) in the form

$$F(k)=F'(k)F''(k),$$

where $F'(k)$ is an arbitrary solution of the corresponding homogeneous equation. As a result, we obtain the equation

$$F''(k+1)-F''(k)=-\frac{Q_p(k+1)}{[(k+1)^2-p^2]F'(k+1)}$$

Solving the above equation (see Ref. 7), we find

$$F(k)=F'(k)\left\{r(k)-\frac{1}{2i}\int_{c-i\infty}^{c+i\infty}dy\frac{Q_p(y+1)}{[(y+1)^2-p^2]F'(y+1)}\frac{\sin\pi k}{\sin\pi y\sin(y-k)\pi}\right\}.$$

The location of the contour is determined as before by the conditions (3.3) and (3.4), $r(k)$ is a periodic function with a period of unity, $F'(k)$ is determined by the formula (3.1), also with accuracy to an arbitrary periodic factor. It is necessary to find $h(k)$ and $r(k)$ from the conditions imposed on the analytic properties of $F(k)$. We shall seek the characteristic plasma oscillations brought about by the presence of the foreign charges; we therefore require that the field be damped in space as $\xi\to\pm\infty$. We shall also assume that $Q_p(k)$ is an entire function. As a result, we obtain the formula

$$F(k)=-F'(k)\cdot\frac{1}{2i}\int\limits_{c-i\infty}^{c+i\infty}dy\frac{Q_p(y+1)}{[(y+1)^2-p^2]F'(y+1)}[\operatorname{ctg}(y-k)\pi-i]. \quad (5.1)$$

Here $F'(k)$ is determined by the equation (3.1) with $h(k)$ = const. As $\xi\to-\infty$, we can determine the asymptotic form of the field from Eq. (5.1):

$$f(\xi)=\frac{\alpha^p e^{-i\pi p}U(p)}{\pi\Gamma(1+2p)}\exp(p\xi)\frac{1}{2i}\int\limits_c dy\frac{Q_p(y+1)}{[(y+1)^2-p^2]F'(y+1)}[\operatorname{ctg}\pi(y-p)-i].$$

The integration is carried out along the contour C, which coincides with the contour determined by the conditions (3.4) at $k=p<1$ or $p>1$, is obtained by distorting the initial contour in such a way that the locations of the singularities relative to it remain unchanged. The asymptotic form as $\xi\to+\infty$ can also be found and has a form similar to (4.19). Equation (5.1) allows us to consider plasma waves which develop at $\gamma\gg1$ near the point ξ_0. The basic contribution to the integral of the inverse Laplace transform in the calculation of the field in the vicinity of ξ_0 is made by the values $|\mathrm{Im}k|\gg1$. If the characteristic size b of the perturbation $G(x,k_\perp)$ is much greater than the zone of diffuseness of the field near the singularity: $b\gg a\gamma^{-2/3}$, then Eq. (5.1) reduces to the form

$$F(k)=F'(k)B(p)\theta(-\mathrm{Im}\,k), \quad (5.2)$$

where $\theta(x)$ is the Heaviside theta function,

$$B(p)=-\int\limits_c dy\frac{Q_p(y+1)}{[(y+1)^2-p^2]F'(y+1)}.$$

Then, transforming in (5.2), as was done in the previous section, to large values of $|\mathrm{Im}k|\gg1$, and replacing the Laplace transform by the Fourier transform, we obtain complete agreement with Eq. (5.2) of Ref. 5.

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[1)]We can take the collisions into account here by replacing ϵ^T by $\epsilon^T+i\nu_{\mathrm{eff}}/\omega$, which leads to the following correction in (4.2): $\omega_0^2/\omega^2\to\omega_0^2[\omega^2(1+i\nu_{\mathrm{eff}}/\omega)]^{-1}$.

[2)]Under the condition $\omega/c\ll a^{-1}$, the two polarizations can be regarded as independent.

[3)]It is also necessary that the distance from the wall d, at which the density distribution of the electrons is significantly distorted, be small: $(2p-1)\,d\ll a$.

[4)]Equation (4.17) is valid if $|2p-n|X\geq1$, where $n=2, 4, 4, \ldots$. In addition, see (4.18).

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