Vortex motion in uniformly and nonuniformly rotating helium II

L. V. Kiknadze and Yu. G. Mamaladze

Physics Institute, Georgian Academy of Sciences
(Submitted 8 February 1978)
Zh. Eksp. Teor. Fiz. 75, 607–616 (August 1978)

The equations of motion of the vortices produced in the superfluid component of helium II and interacting with its normal component are used for the analysis of the self-acceleration of a decelerating vessel with helium or of a pulsar. It is shown that in this case the vortices transfer the angular momentum to the vessel gradually, during the course of their motion to the walls, a process that ends in annihilation of the already "exhausted" vortex; the annihilation speeds up sharply immediately before the acceleration, but its integrated value is small. Computer experiments have refined the detail of this process, and confirmed as well as the Khalatnikov formula for the width of the irrotational region in a uniformly rotating vessel.

PACS numbers: 67.40.Vs

1. In Tsakadze's experiments,[1] a freely suspended vessel with helium II underwent, against the background of weak damping of the rotation, a number of spontaneous accelerations. The interpretation of this phenomenon is connected with the dependence of the equilibrium number of vortices N_0 on the angular velocity ω_0 of the uniform rotation[1)]

$$N_0=m\omega_0R^2/\hbar \tag{1}$$

(m is the mass of the helium atom and R is the radius of the vessel). In the case of decelerated rotation, N_0 should decrease. However, in view of the tendency of helium II to preserve nonequilibrium regimes of motion for a long time (see, for example, the reviews in Refs. 2 and 4), it is assumed that the superfluous vortices continue to exist for some time, after which they decay in relatively large groups and transfer the corresponding angular momentum (and energy) to the vessel. On the basis of these assumptions, it was proposed that the considered phenomenon is the possible cause of the acceleration of superfluid neutron stars—pulsars[5] (see also Ref. 6). It will be shown below that a more detailed quantitative analysis, confirming in general the proposed interpretation of the self-acceleration of a superfluid liquid when it is decelerated, reveals somewhat unexpected qualitative features of this phenomenon (see Secs. 7 and 9).

2. When a superfluid liquid rotates uniformly, the vortices are distributed in it with an equilibrium density close to the value $m\omega_0/\pi\hbar$ determined by formula (1), but not over the entire cross section of the vessel. Near the walls, an irrotational region of width $d\equiv R-R_t$ (R_t is the radius of the circle containing the vortices) is produced and is determined, according to Khalatnikov,[7] by the formula

$$d=\frac{s}{\sqrt{2}}\left(\ln\frac{b}{a}-1\right)^{1/2}, \tag{2a}$$

and according to Staufer and Fetter[8] by the formula

$$d=\frac{s}{\sqrt{2\pi}}\left(\ln\frac{b}{a}\right)^{1/2}. \tag{2b}$$

where $s=(\pi\hbar/m\omega_0)^{1/2}$ and is according to (1) of the order of the distances between the vortices (in an ideal triangular lattice the distance between the vortices is $s_\Delta=(2s^2\sqrt{3})^{1/2})$, $b\sim s$ is the effective radius of the vortex and $a\sim3\cdot10^{-8}$ cm is the radius of the vortex core.

In addition to the behavior of the vortices as they are slowed down, we investigate also the situation in a uniformly rotating vessel and show that the Khalatnikov formula (2a) is more accurate than formula (2b) (see Sec. 8).

3. Returning to the case of retarded motion, we note first that it does not admit to a quasi-equilibrium analysis. In fact, if the slowing down of the vessel ($\omega=\omega(t)$) were accompanied by a corresponding deceleration of the rotation of the normal component ($\mathbf{v}_n=\boldsymbol{\omega}\times\mathbf{r}$), of the averaged rotation of the superfluid component ($\langle\mathbf{v}_s\rangle=\boldsymbol{\omega}\times\mathbf{r}$), and of the rotation of the vortices themselves ($v_L=\boldsymbol{\omega}\times\mathbf{r}$), then the distances between the vortices in the region where they exist should increase ($s\propto\omega^{-1/2}$). However, according to formula (2) the width of the irrotational region d should increase as well, and this is incompatible.

4. A more realistic although not fully justified (see Sec. 10) is the following idealization. In the case of weak slowing down of the rotation, the normal component follows the vessel:

$$v_{nr}=0,\quad v_{n\alpha}=\omega r \tag{3}$$

(we are using the polar coordinates r and α). As to the motion of the vortices, it is assumed that they remain straight, and that the vessel is cylindrical and infinitely long (along the rotation axis) or, equivalently, that free slippage of the vortices over the bottom and top of the vessel is permissible. Then the flow of the superfluid component and the motion of the vortices can be fully calculated (in principle) without any other simplifying assumptions, on the basis of the following equations. The vanishing of the sum of the forces acting on each of the vortices is, in the case of straight vortices, of the form[9]

$$\rho_s\Gamma_0\mathbf{k}\times(\mathbf{v}_s-\mathbf{v}_L)+B_L\Gamma_0\frac{\rho_s\rho_n}{2\rho}\mathbf{k}\times[\mathbf{k}\times(\mathbf{v}_n-\mathbf{v}_L)]+B'_L\Gamma_0\frac{\rho_s\rho_n}{2\rho}\mathbf{k}\times(\mathbf{v}_n-\mathbf{v}_L)=0, \tag{4}$$

where the first term is the Magnus force, and the next

 0038-5646/78/080305-06$02.40

two comprise the force of the mutual friction between the vortex and the superfluid component, $\Gamma_0 = 2\pi\hbar/m$ is the circulation quantum, **k** is a unit vector along the z axis ($\omega = \mathbf{k}\omega$, $\Gamma_0 = \mathbf{k}\Gamma_0$); B_L and B'_L are the coefficients of the mutual friction; $\mathbf{v}_s$ and $\mathbf{v}_n$ are the velocities of the superfluid and normal components at a vortex point moving with velocity $\mathbf{v}_L$. Equation (4) is valid both in nonequilibrium situations and under equilibrium rotation when $\mathbf{v}_s = \mathbf{v}_n = \mathbf{v}_L = \omega \times \mathbf{r}$.

5. Under the same assumptions (i.e., independently of the equilibrium of the rotation, but only for planar low, when the vortices are straight), the following equation is valid and determines the velocity of the superfluid component at the j-th vortex point (with the exception of the contribution of the j-th vortex):

$$v_{sr}^{(j)} - iv_{s\alpha}^{(j)} = \lim_{z\to z_j}\left[e^{i\alpha}\left(\frac{dw}{dz} - \frac{\Gamma_0}{2\pi i}\frac{1}{z-z_j}\right)\right], \qquad (5)$$

where w is the complex potential of the flow of the superfluid component. In the absence of mutual friction (in a fully superfluid or in a classically ideal liquid) formula (5) determines the velocity of the vortex directly ($\mathbf{v}_L^{(j)} = \mathbf{v}_s^{(j)}$). In equilibrium rotation, this formula yields $v_{sr}^{(j)} = 0, v_{s\alpha}^{(j)} = \omega_0 r_j$ (see, for example, Ref. 10 and also Sec. 8 of this article). In the general case, the motion of the vortices is determined jointly by formulas (4) and (5). In a cylindrical vessel we have

$$w = \frac{\Gamma_0}{2\pi i}\sum_k \ln\frac{z-z_k}{z-z_k'}, \qquad (6)$$

where $z_k = r_k \exp(i\alpha_k)$ are the coordinates of the vortices and $z'_k = \exp(i\alpha_k)R^2/r_k$ are the coordinates of their reflections (inversions).

Substituting v_L in the form

$$v_{Lr}^{(j)} = dr_j/dt, \quad v_{L\alpha}^{(j)} = r_j d\alpha_j/dt$$

and substituting (3) in (4), we obtain a system of vortex-motion equations corresponding to our assumptions:

$$\begin{aligned} r_j\, d\alpha_j/dt - \omega r_j &= A_1(v_{s\alpha}^{(j)} - \omega r_j) - A_2 v_{sr}^{(j)}, \\ dr_j/dt &= A_2(v_{s\alpha}^{(j)} - \omega r_j) + A_1 v_{sr}^{(j)} \end{aligned} \qquad (7)$$

(two equations for each vortex). The coefficients A_1 and A_2 can be easily expressed in terms of the coefficients B_L and B'_L, which in turn can be expressed in terms of the more frequently employed coefficients of mutual friction between the superfluid and normal components B and B' (for the relations between B, B', and B_L, B'_L see Ref. 9):

$$A_1 = 1 - \frac{\rho_n}{2\rho}B', \quad A_2 = \frac{\rho_n}{2\rho}B. \qquad (8)$$

6. In the case of a freely suspended vessel, the system (7) must be supplemented by the equation of motion of the vessel with the liquid, which we express in terms of the moments

$$\frac{d\omega}{dt} = \frac{2I_s\Gamma_0}{\pi I R^4}\sum_j r_j\frac{dr_j}{dt} - \gamma\omega. \qquad (9)$$

We have taken account here of the fact that the angular momentum of the superfluid component, due to the j-th vortex, is equal to $\rho_s\Gamma_0(R^2 - r_j^2)H/2$ (Ref. 2), where H is the height of the vessel, $I\gamma\omega$ is the moment of the external friction force experienced by the vessel, is the damping, $I_s = \pi\rho_s R^2 H/2$ is the moment of inertia of the superfluid component, and I is the sum of the moments of inertia of the vessel and of the normal component. We write out also the formula for the angular moment of the superfluid component:

$$L_s = \frac{1}{2}\rho_s\Gamma_0 H\sum_j (R^2 - r_j^2), \qquad (10)$$

which we shall need below.

The system (7) (9) with Eqs. (5) and (6) solves completely the problem of the motion of the vortices and of the vessel itself, and determines the time variation of r_j, α_j, and ω.

7. According to the second equation of (7), the vortex is at equilibrium if $v_{s\alpha}^{(j)} = \omega r_j$ and moves towards the wall or away from it, depending on the sign of the difference $v_{s\alpha}^{(j)} - \omega r_j$. The decrease of ω leads to the appearance of a positive velocity dr_j/dt. Exceptions can occur for vortices whose initial positions are not in equilibrium and can therefore have a velocity lower than ωr_j, and move towards the axis of the vessel. It is easy to verify, however, that with decreasing ω any vortex should ultimately move towards the wall. Simultaneously with r_j, a change takes place also in α_j. Consequently, the vortices move along a spiral to the irrotational region, where at equilibirum their presence is forbidden, but at $v_{s\alpha}^{(j)} > \omega r_j$ they penetrate deeper and deeper into this region.[2)]

We consider the concluding stage of this process, in which the vortex is much closer to the wall than to another vortex or to the center of the vessel ($R - r_j \ll R$, $R - r_j \ll s$). Then the principal contribution to the sum (5) and to Eqs. (7) and (9) is made by a reflected vortex, and this contribution tends to infinity as $r_j \to R$:

$$\frac{dr_j}{dt} \approx -\frac{A_2\Gamma_0}{2\pi}\frac{1}{r_j - R^2/r_j}. \qquad (11)$$

The quantities $d\alpha_j/dt$ and ω/dt also increase without limit. Consequently there are no equilibrium (quasi-equilibrium) positions for a vortex that approaches the wall. It must annihilate.

The singularity in $dr_j/dt(d\omega/dt)$ of the type (11) leads to the following singularities in the quantities r_j, ω, and L:

$$r_j \approx R - \left[\frac{A_2\Gamma_0}{2\pi}(t_0 - t)\right]^{1/2}, \qquad (12)$$

$$\omega \approx \omega(t_0) - \frac{2I_s\Gamma_0}{\pi I R^3}\left[\frac{A_2\Gamma_0}{2\pi}(t_0 - t)\right]^{1/2}, \qquad (13)$$

$$L_s \approx \sum_{k\neq j} L_{sk} + \rho_s\Gamma_0 RH\left[\frac{A_2\Gamma_0}{2\pi}(t_0 - t)\right]^{1/2}, \qquad (14)$$

where t_0 is the instant of annihilation of the vortex with its reflection. The formulas are valid at $t_0 - t$ (at $t_0 - t \ll mR^2/A_2\hbar$).

Formulas (13) and (14) show that at infinite rates of change of the quantities ω and L these changes themselves are quite insignificant during the period immediately preceding the annihilation. The vessel velocity has a small maximum, and the angular momentum of the superfluid liquid has a small minimum with kinks on the $\omega(t)$ and $L(t)$ curves (see also Sec. 9 and Figs. 3 and 4). The point is that the contribution of this vortex to L decreases mainly not during the period when $(t_0 - t)$ is small, but earlier, as r_j increases from its initial value. The vortex arrives at the wall already "exhausted," and its annihilation should possibly more correctly be called "extinction." Thus, the initial idea that the "decay of metastable vortices" causes the self-acceleration is developed in greater detail by the concept of gradual transfer of the angular momentum of the superfluid component to the vessel as a result of the motion of the vortices to the wall. The self-acceleration occurs in this case when a definite group of vortices approaches the wall (these vortices lie all on a single circle), the momentum transfer is already small, and the process is accelerated without limit. Therefore the "jump" of the velocity of the vessel $(\omega_{max} - \omega_{min})$ does not make it possible to estimate directly the number of "decaying" vortices (see also Sec. 9 and Fig. 3).

8. A more detailed calculation of the considered phenomena, on the basis of Eqs. (7) and (9), calls for the use of a computer. We have performed preliminary numerical calculations to determine the equilibrium positions of the vortices. The rotation velocity ω_0 was specified and maintained constant, and the vortices, to which certain initial positions were assigned, were allowed to move in accordance with Eqs. (7) until their velocity became equal to zero. In some cases, simultaneously with the motion of the vortices, we varied also the rotation velocity of the vessel (even before all the dr_j/dt, $\omega - d\alpha_j/dt$, $d\omega/dt$ reached zero values). Finally, we used also a method in which we solved the system of algebraic equations obtained by equating to zero the right-hand sides of Eqs. (7). The initial positions of the vortices were specified to be either the lattice points of a triangular lattice with a spacing increased enough to fill the entire cross section of the vessel, or else by three random radii of circles with six vortices on each. The results were the same in all the variants.

Figure 1 shows the equilibrium (in the sense of stable immobility in a rotating coordinate frame) configurations of nineteen vortices and sixty-one vortices. The points of the grids marked on the figures correspond to positions of the vortices in the triangular lattice at a corresponding speed of rotation. The vortices are displaced relatively little from these positions. The displacement increases from the center towards the boundary $r = R_t$ of the irrotational region, where the density of the vortices, which heretofore was close to the density determined by formula (1), vanishes abruptly. These calculations constitute a computer experiment aimed at a determination of the width of the irrotational region d and confirm Khalatnikov's formula (2a) even when the initial configuration and of the velocity are chosen by the formula of Staufer and Fetter, which corresponds to formula (2b): $N \approx N_0(1 - 5/\sqrt{N_0})$ (Ref. 8), where N_0 is determined by formula (1) and N is the actual number of the vortices. These are precisely the cases shown in Fig. 1. At $\omega_0 = 56.25\hbar/mR^2$ the computer experiment yields $d/s \approx 2.2$, and at $\omega_0 = 115\hbar/mR^2$ it yields $d/s \approx 2.1$, whereas formulas (2a) and (2b) give also values that depend little on ω_0, and range accordingly, when $\ln(b/a)$ is varied in the range 12–15, from 2.4 to 2.6 according to (2a) and from 1.4 to 1.6 according to (2b). Staufer and Fetter have suggested that their results should be more accurate, since they have stipulated that the velocity be continuous at $r = R_t$. Khalatnikov purposely neglected this stipulation, having in mind a small jump in the velocity. Actually, however, vortices are present on a circle $r = R_t$ and the smoothed-out picture with a continuous distribution of the velocities is also approximate. The results of our calculation show that Khalatnikov's approximation is a rougher one.[3]

9. Figures 2–4 show the result of a computer calculation of the time variation of the quantities r, ω, and L in a successive departure, from a decelerating vessel, of eighteen vortices that have occupied at the initial instant of time equilibrium positions shown in Fig. 1a, at an initial velocity $\omega(0) = 56.25\hbar/mR^2$. The vortices are grouped on three circles with radii r_1, r_2, and r_3. The central vortex remained immobile (as shown in Ref. 3) its position is unstable at $\omega < \hbar/mR^2$.

Contributing to the emergence of the vortices from the vessels are the "notches" on Figs. 3 and 4. Attention must be called to the difference $\Delta_1\omega$ between the successive maximal values of the angular velocity (Fig. 3), and also to the difference $\Delta_2\omega$ between its maximal and the preceding minimal values. The former depends

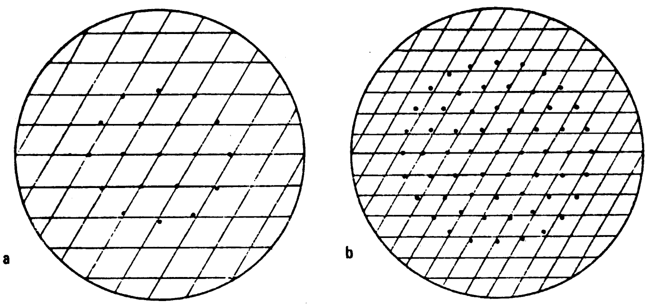

FIG. 1. Equilibrium configuration of vortices in a cylindrical vessel at $\omega_0 = 56.25\ \hbar/mR^2$ (a) and at $\omega_0 = 115\ \hbar/mR^2$ (b). The grid corresponds to a triangular lattice of vortices in an infinite liquid at the same rotational velocity.

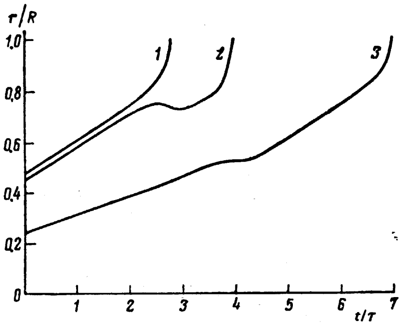

FIG. 2. Variation with time ($\tau = mR^2/\hbar$) of the radii of three circles carrying six vortices each, whose initial configuration is shown in Fig. 1a. Curve 1 corresponds to the outer circle, and curves 2 and 3 to the following two; $\gamma = 0.5\ \hbar/mR^2$ ($I_s/I = 0.5$; $A_1 = 0.94$; $A_2 = 0.2$).

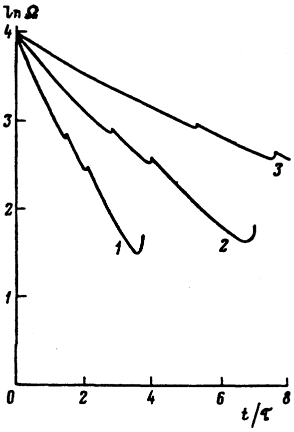

FIG. 3. Change with time ($\tau = mR^2/\hbar$) of the angle of velocity of the vessel ($\Omega = \omega(\hbar/mR^2)^{-1}$) at different dampings: $\gamma = \hbar/mR^2$ (curve 1), $\gamma = 0.5\hbar/mR^2$ (curve 2), $\gamma = 0.25\,\hbar/mR^2$ (curve 3). The initial state corresponds to Fig. 1a, ($I_s/I = 0.5$; $A_1 = 0.94$; $A_2 = 0.2$).

strongly on the damping (on the moment of the external friction force). This means that not only the entire considered process, but also the states produced at the end of each self-acceleration, are not in quasi-equilibrium, for otherwise $\Delta_1\omega$ would depend only on the number of departing vortices. Actually the vortices overtake the quasi-equilibrium process in the case of weak damping and lag this process in strong damping. The smallness of the ratio $\Delta_2\omega/\Delta_1\omega$ in Fig. 3 is a measure of the fraction of the angular momentum transferred by the vortices to the vessel prior to the start of the self-acceleration. If the start of the self-acceleration ($\omega = \omega_{\min}$) were the instant when the "vortex decay" begin, then the ratio $\Delta_2\omega/(\Delta_1\omega + \Delta_2\omega)$ would be I_s/I (in our case $I_s/I = 0.5$). Figure 2 also offers evidence that it is precisely in the self-acceleration process that a strong nonequilibrium state is produced, as a result of which the vortices are sometimes even directed towards the axis of the vessel, deceleration of the latter notwithstanding.

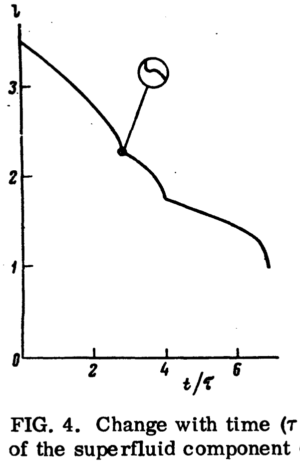

FIG. 4. Change with time ($\tau = mR^2/\hbar$) of the angular momentum of the superfluid component ($l = L_s(2I_s\hbar/mR^2)^{-1}$) in the process shown in Fig. 3 (curve 2) and Fig. 2. The region of the instantaneous shock that accompanies the departure of the vortices from the vessel is shown in enlarged scale ($I_s/I = 0.5$; $A_1 = 0.94$; $A_2 = 0.2$).

Thus, the results agree fully with the preliminary analysis made in Sec. 7, and provide additional qualitative information. The slowing down of the vessel leads to a nonequilibrium situation, in which the vortices move towards the walls, constantly interacting with the normal component and with the vessel, and in the period of the self-acceleration the exchange of the angular momentum of the motion is relatively small, but is effected quite rapidly. It can be stated that at instants of time t_0 there occurs an infinitely strong instantaneous shock ($dL_s/dt = -\infty$) with zero transfer of angular momentum (L_s and $I\omega$ do not experience a jump).

10. It is natural to raise the question of the extent to which the idealization used in our calculation corresponds to real processes in helium II or in pulsars. Unfortunately, the number of vortices close to those obtained in the experiments of Dzh. and S. Tsakadze or in pulsars is too large for the computational techniques available at our disposal. Therefore a direct quantitative comparison of the available data with our curves on Fig. 3 is impossible. However, some conclusions are possible also via a qualitative comparison, and call for a discussion.

A rough agreement is in fact observed: On the $\ln\omega(t)$ curves one can see a number of self-accelerations at unequal time intervals. In a more detailed comparison, however, there is a striking difference between the character of the experimental and calculated curves, particularly on the left side of the "notch." On the calculation curve the spike begins smoothly and ends abruptly, while on the experimental curves it begins more abruptly and ends smoothly. It seems to us that the reason for this discrepancy, as well as the weakest point of the idealization used by us of real phenomena, is the neglect of the pinning of the vortices by roughnesses on the bottom and on the top of the vessel (concerning the slipping of the vortices, which is never perfectly free, see, for example, the reviews in Refs. 2 and 4). A vortex that is partially pinned on its ends will bend, and one result of this fact will be an effective decrease of the contribution of (11) to Eq. (9), in which it is assumed that the distance from the vortex to the wall is the same over the entire length H. Then the derivative $d\omega/dt$ wil have, during the self-acceleration, not a singularity of the type $(t_0 - t)^{-1/2}$ but of the form $(t_0(t) - t)^{-1/2}\chi(t)$, where $\chi(t)$ is a dimensionless quantity connected with the relative length of the vortex segment that interacts most strongly with the wall at the instant of time t_0. This quantity becomes different from zero when the increase of $\chi(t)$ is bounded, since the already annihilated segment is eliminated from the interaction, and $\chi(t)$ again becomes equal to zero by the instant when the annihilation is completed. If in the concluding stage of this process $\chi(t)$ tends to zero more rapidly than $(t_0 - t)^{1/2}$, then the limiting value of $d\omega/dt$ will be not infinity but zero.

In addition, the bending of the vortices, the propagation of elastic waves and Tkachenko waves along the vortices, and the difference between the slippage coefficients on different sections of the solid surfaces should inevitably produce a nonuniformity of the displacements of the vortices towards the vessel wall, thus eliminating from the curves of $\ln\omega(t)$ the smoothness demonstrated in Fig. 3 but missing from the experimental curves.[4)]

The assumption that the angular velocities of the vessel and the normal component is more justified, especially for pulsars, where the normal charged liquid is additionally linked with the core by the magnetic field. However, during the periods of the abrupt change of the rotation velocity, even this assumption may not be correct. The moments of the internal-friction force which must be introduced in this case, will also contribute to the smoothing out of the "notches" and the turbulence, if it does appear, complicates the phenomena in question even more.

Summarizing the foregoing, we can apparently conclude that the model developed in this article, for all its incompleteness, nevertheless describes accurately the main features of the self-acceleration phenomenon. The remarks on the shortcomings of the model do not pertain to cases when it is used to determine the equilibrium configuration.

11. By smoothing out the details of the distribution of the velocities produced by the vortices and determined by the complex potential (6), we can calculate v_s by starting from the equality of the circulation Γ_r over a circle of radius r to the sum of the circulations of the vortices contained in it: $v_{sr}=0, v_{s\alpha}=\Gamma_r/2\pi r$. These equations are accurate enough and can be used to determine the $v_s^{(j)}$ velocities of the superfluid component at the vortex points, provided only that there is no other vortex or wall next to them (in the general case $v_{sr}\neq 0$ and the proximity of a vortex, including a reflected one, creates a large difference between the local values of $v_{s\alpha}$ and its value averaged over the contour, as a result of which $\Gamma_r \neq 2\pi r v_{s\alpha}$. Assuming the possibility of such an approximation and recognizing that Γ_r does not change with changing number of vortices in the given circle, we obtain $v_{s\alpha}^{(j)}=\omega_0 r_{j0}^2/r_j$, where r_{j0} is the initial position of the vortex at the initial rotation velocity ω_0, when the density of the vortices is determined by formula (1). Equations (7) then take the form

$$r_j^2 d\alpha_j/dt-\omega r_j^2=A_1(\omega_0 r_{j0}^2-r_j^2\omega),$$
$$r_j dr_j/dt=A_2(\omega_0 r_{j0}^2-\omega r_j^2). \tag{15}$$

These equations practically coincide with those obtained by Krasnov.[11] It follows from the second of them, as noted in Ref. 11, that the function $\theta(t)\equiv r_j^2/r_{j0}^2$ does not depend on j. We note that the same property is possessed also by $\alpha(t)\equiv\alpha_j$:

$$d\theta/dt=2A_2(\omega_0-\omega\theta), \quad (d\alpha/dt-\omega)\theta=A_1(\omega_0-\omega\theta). \tag{16}$$

The situation is more complicated with the equation for ω. In the course of its derivation in Ref. 11, an expression was obtained for the angular momentum, $L_s(t)=L_{s0}/\theta(t)$, which contradicts formula (10) since the latter means that so long as the number of the vortices in the vessel remains unchanged, the variation of L_s with time should follow the law $L_s(t)=C_1-C_2\theta(t)$, where C_1 and C_2 are constants. The calculation of the values is possible if the sum in (10) is replaced by an integral

$$L_s=\int_0^{R_i}\pi\rho_s\frac{\hbar H}{m}(R^2-r^2)\frac{m\omega_0}{\pi\hbar}\frac{\pi R_{i0}^2}{\pi R_i^2}2\pi r\,dr=I_s\omega_0\frac{R_{i0}^2}{R^2}\left[2-\frac{R_{i0}^2}{R^2}\theta(t)\right], \tag{17}$$

where $R_{i0}=R_i/\sqrt{\theta(t)}$ is the equilibrium value of the radius R_i of the boundary of the irrotational region (it is assumed that it has not yet approached too close to the wall).

Substitution of (17) in the equation $I d\omega/dt+dL_s/dt=-\gamma\omega$ for the moments yields the expression

$$\frac{d\omega}{dt}=\frac{I_s}{I}\frac{R_{i0}^4}{R^4}\omega_0\frac{d\theta}{dt}-\gamma\omega, \tag{18}$$

which differs from Eq. (20) of Ref. 11. The system of Eqs. (16) and (18) can be used at a fixed number of vortices, and also before and after the number changes, for example, in the case of small changes of the vessel velocity during the initial or concluding stages of a prolonged but slow change. The assumptions on which the derivation of this system is based are violated every time (at least for some of the vortices) that vortices leave the vessel or enter in it.[5)]

On the other hand, if the process is initiated by an abrupt jump of the vessel rotation, as in the experiment considered in Ref. 11, where, simulating a starquake, Dzh. and S. Tsakadze increased the velocity from $\omega_0=5$ sec^{-1} to $\omega(0)=5.66$ sec^{-1}, then it cannot be described by the smoothed equations at all. In addition to the already mentioned causes, in this situation it would also be wrong to assume that the normal component is fully dragged.

For one reason or another, the experimental curve in its initial part diverges strongly from the solutions of these equations (the curve constructed in Ref. 11 was obtained by empirically fitting the parameters, and not by substituting the experimental parameters into the theoretical expressions for the coefficients). The empirical values of the initial slope and of the argument of the exponential differ from those predicted by the theory by approximately two orders of magnitude. Only the final slope of the $\omega(t)$ curve, which is established approximately 100 seconds after the initial jump, is in satisfactory agreement with the predictions of the theory, which in this asymptotic limit are identical for Eqs. (16), (18), and the operations of Ref. 11.

[1)]In Eq. (1) we neglect the presence of the irrotational region (see Secs. 2 and 8). Its accuracy increases with increasing ω_0 (Refs. 2 and 3).

[2)]The idea of the possibility of quantitatively considering the phenomenon of spontaneous acceleration as a result of the emergence of the vortices from the vessel (and not of the process somewhat incorrectly called ("decay of metastable vortices") was advanced by Yu. K. Krasnov. In the discussion following his paper, however, it turned out that the equations used by him (see Sec. 11) cannot serve as a basis for such an investigation. Self-acceleration was hence not considered in Ref. 11, and we have therefore undertaken the work reported in the present article.

[3)]In this connection, to estimate the equilibrium number of vortices N at a velocity ω_0, a preferable estimate is the one based on formula (2a), $N=N_0(1-9,4\sqrt{N_0})$, where N_0 is determined by formula (1).

[4)]The authors are grateful to S. Dzh. Tsakadze for the opportunity of becoming acquainted with new as yet unpublished experimental curves that contain a large number of successive self-accelerations, as well as data used in Sec. 11.

[5)]In our computer experiments, r_j^2/r_{j0}^2 is initially the same for all j, but as the outer vortices approach the wall it becomes more strongly dependent on j. The relation $L_s=I_s\omega_0/\theta(t)$ is obtained by replacing in (17) the upper limit of the integral by R, i.e., when $\theta(t)$ loses its meaning of the universal ratio r_j^2/r_{j0}^2.

[1]J. S. Tsakadze and S. J. Tsakadze, Zh. Eksp. Teor. Fiz. **64**, 1816 (1973) [Sov. Phys. JETP **37**, 918 (1973)].

[2]E. L. Andronikashvili and Yu. G. Mamaladze, Prog. in Low Temp. Phys., Amsterdam, 1967, vol. 5, Chap. 3, § 2.

[3]O. B. Hess, Phys. Rev. **161**, 189 (1967).

[4]E. L. Andronikashvili and Yu. G. Mamaladze, Rev. Mod. Phys. **38**, 567 (1966).

[5]R. E. Packard, Phys. Rev. Lett. **28**, 1080 (1972).

[6]Dzh. S. Tsakadze and S. Dzh. Tsakadze, Usp. Fiz. Nauk **115**, 503 (1975) [Sov. Phys. Usp. **18**, 242 (1975)].

[7]I. M. Khalatnikov, Teoriya sverkhtekuchesti (Theory of Superconductivity), Nauka, 1971, p. 107.

[8]D. Staufer and A. L. Fetter, Phys. Rev. **168**, 156 (1968).

[9]W. F. Vinen, Progr. in Low Temp. Phys., Amsterdam, 1961, vol. 3, p. 21.

[10]L. V. Kiknadze and Yu. G. Mamaladze, Fiz. Nizk. Temp. **2**, 1501 (1976) [Sov. J. Low Temp. Phys. **2**, 731 (1976)].

[11]Yu. K. Krasnov, Zh. Eksp. Teor. Fiz. **73**, 348 (1977) [Sov. Phys. JETP **46**, 181 (1977)].

Translated by J. G. Adashko

Effect of shock wave on the conductivity of *n*-type germanium

A. V. Polyaninov, K. P. Gurov, and V. A. Yanushkevich

A. A. Baĭkov Metallurgy Institute, USSR Academy of Sciences

(Submitted 15 February 1978)
Zh. Eksp. Teor. Fiz. **75**, 617–627 (August 1978)

We investigate the relaxation of the electric conductivity of *n*-Ge after the passage of shock waves generated by laser pulses. The initial concentration of the structure point defects and the probability of electron capture by donor impurity center are determined. Two models are considered in the discussion of the experimental results: narrowing of the forbidden band under uniaxial compression, and ionization of Shockley-Read traps. Residual effects are also considered.

PACS numbers: 62.50. + p, 72.80.Cw, 72.20.Jv, 79.20.Ds

1. INTRODUCTION

Investigations of the influence of a shock wave (SW) on the electrophysical properties of germanium and silicon are of considerable interest to modern microelectronics. The most convenient and easily controlled method of producing SW in materials is to act on them by a laser pulse. A laser can be used to generate SW of small amplitude without damaging the material.

The action of SW excited by ruby-laser radiation of pulse duration 50 nsec and energy flux density 10^8–10^9 W/cm^2 on the electric conductivity of germanium and silicon whisker crystals is described in Ref. 1. At a SW pressure in the front 1.2–4.8 kbar ((1.2–4.8) × 10^8 Pa), the resistance of *n*- and *p*-type silicon decreased by a factor of 2; the corresponding decrease in germanium was by a factor of 2 for *n*-type and by more than an order of magnitude for *p*-type. Similarly, the relaxation times of the excess conductivity in silicon of both types were comparable in magnitude, whereas in *n*-Ge the relaxation time was smaller by approximately one order of magnitude than in *n*-Ge. Thus, from the scientific point of view, particular interest attaches to a detailed study of the electrophysical properties of germanium.

We report here a detailed investigation of the influence of SW on the conductivity of *n*-Ge, as well as of the relaxation and residual effects. In addition, the following particular question was posed: are the observed effects properties peculiar to whiskers?

2. EXPERIMENTAL PROCEDURE

The bombardment procedure is described in Ref. 1. The laser pulse duration was 30 or 50 nsec, and the flux density range from 1.8 to 4.4 × 10^8 W/cm^2.

In contrast to Ref. 1, the samples were prepared by the traditional procedure, in the form of parallelepipeds of area 1.2 × 3.6 mm and with variable thickness from 0.25 to 1.4 mm. The material used was commercial *n*-type germanium with resistivity 40 Ω-cm. Electric contacts of tin with antimony admixture were deposited on the end faces and checked for linearity of the current-voltage characteristic. The samples were glued to a quartz substrate and a copper foil was glued on their top surface to protect them from the direct action of the laser radiation. An estimate of the depth of SW formation, using the formulas of Refs. 2 and 3, has shown that in our experiments copper foil 55 μm thick was sufficient. The germanium single-crystal samples were so oriented that the SW propagated in the [111] direction, which was perpendicular to the di-

0038-5646/78/080310-06$02.40