Parametric excitation of electrons on the surface of liquid helium in a magnetic field

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It is shown that parametric excitation of electrons is possible in a quasistationary magnetic field $\mathbf{H}+\mathbf{h}(t)(\mathbf{h}\|\mathbf{H},\ h<H)$ which is homogeneous over the volume of the solenoid. The frequencies ω_n of the parametric instability depend on the geometry of the vortical electric field induced by the field $\mathbf{h}(t)$. In the cylindrically symmetrical case the frequencies $\omega_n=\Omega/n$, where Ω is the cyclotron frequency and $n=1,2,3,\ldots$. In the one-dimensional variant of the problem, which is realized in a rectangular solenoid XY with $X<Y$, the corresponding frequencies $\omega_n=2\Omega/n$, $n=1,2,3,\ldots$. The observed difference between the spectra of the parametric excitation of an electron in the cylindrical and one-dimensional problem is preserved in the quantum treatment. For electrons whose average position is shifted relative to the symmetry axis of the solenoid, the parametric instability develops against the background of the electron motion typical of cyclotron resonance in the vortical electric field averaged over the electron orbit. Some singularities of the classical threshold of the parametric instability are discussed for a system of electrons localized on the surface of liquid helium. A scheme is proposed for using a cylindrically symmetrical vortical electric field to excite transverse waves in a hypothetical cylindrically symmetrical Wigner crystal made up of surface electrons in helium.

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A characteristic feature of a system of electrons localized over the surface of liquid helium is the weakness of its interaction with the liquid substrate. Thus, the electron mobility μ determined experimentally by a number of workers[1] in the region $T\approx 0.5$ K and at clamping fields $E_\perp\approx 20-50$ V/cm is of the order of $\mu\sim 10^7$ cm^2/V-sec, which is larger by two or three orders of magnitude than the record carrier mobilities in semiconductors. Theoretical calculations of the mobility, which are in good agreement with experiment in the ohmic region, lead to the conclusion that when the temperature is decreased to $T\sim 0.1$ K and the clamping field is decreased to $E_\perp\lesssim 1$ V/cm the electron mobility along the helium surface can be increased by another two or three orders of magnitude. An interesting possibility of decreasing the contact between the electrodes and the liquid substrate was demonstrated recently by Édel'man.[2] It turned out that it is fairly easy in experiment to heat the system of surface electrons with an RF field and cause some of the electrons to "evaporate" from the ground level to quasiclassical levels that exist in a weak clamping field. Despite some increase in the average electron energy, the probability of their collision with the surface of the helium decreases sharply, thus contributing to additional conservation of the energy accumulated by the electron system in the electromagnetic field.

Taking the foregoing into account, we can conclude that an electron sheet on the surface of helium is apparently a convenient object for the observation of various subtle classical and quantum effects, known to exist in principle but easily suppressed by dissipative phenomena in the system. Included among these effects is parametric excitation of the electrons in a magnetic field. A description of this and some related phenomena in a system of electrons over the surface of helium at low temperatures is presented in the present paper. Principal attention is paid to a discussion of the mechanism of the effect in the nondissipative regime. A consistent calculation of the corresponding relaxation times is a major problem in itself and will be carried out separately.

It should be noted that parametric excitation of the electrons can occur spontaneously in real experiments on cyclotron resonance and can influence the cyclotron-resonance line width. This circumstance must be borne in mind in the interpretation of data on cyclotron resonance on surface electrons in helium and other systems of charged particles. In particular, it is most desirable to accompany the precision measurements of the cyclotron-resonance parameters with control experiments on the excitation of the electron system at the frequencies $\omega_n=\Omega/n$, where ω and Ω are the external and cyclotron frequencies, and $n=2,3,4,\ldots$. The absence of such resonances guarantees that a pure cyclotron situation exists. On the other hand, if such resonances do appear, then when working at the frequency $\omega_1=\Omega$ an uncontrollable parametric instability may become superimposed on the cyclotron absorption.

1. CLASSICAL DESCRIPTION OF INSTABILITY

1. The question of parametric excitation of an oscillator was investigated quite fully in the classical and quantum limiting cases. There are many real physical systems that admit of parametric excitation. One of them is an electron in a magnetic field. The presence of parametric instability in the equations of motion of a magnetized electron is discussed, for example, in Lehnert's book.[3] The feasibility of resonance, in principle, is implied also in the general solution of the quantum problem of electron motion in alternating fields (Malkin, Man'ko[4]). Nonetheless, a number of features of this phenomenon for an electron in a magnetic field have not been sufficiently well explained. It is meaningful therefore to repeat the solution of the problem with attention focused on these singularities.

Consider a free electron in a stationary magnetic field

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directed along the z axis. Assume that besides the main field the system has a weak alternating field $\mathbf{h}(t)$ collinear with the main field: $\mathbf{h}\|\mathbf{H}$, $h\ll H$. The cyclotron frequency of the electron is then a periodic function of the time

$$\Omega(t)=\Omega[1+\delta(t)], \quad \Omega=eH/mc,$$
$$\delta(t)=h(t)/H\ll1, \quad h(t)=h_0\cos\omega t, \tag{1}$$

e and m are the charge and mass of the free electron, and c is the speed of light in vacuum.

The appearance of a periodic time dependence of the oscillator frequency in the oscillator problem is sufficient for parametric excitation of the oscillator. In the case of an electron revolving in a magnetic field, this statement is incorrect. A substantial additional factor that determines the transfer of the RF-field energy to the electron and must be taken into account in the equations of motion is the vortical electric field $\mathbf{E}(\mathbf{r}, t)$ induced by $\mathbf{h}(t)$. A real field combination $\mathbf{H}+\mathbf{h}(t)$ can be obtained, for example, in the volume of a long solenoid of radius R under the condition $c\omega^{-1}\gg R$, where ω is the frequency of the alternating magnetic field. In this case the vortical electric field $\mathbf{E}(\mathbf{r}, t)$ takes the following form in a Cartesian coordinate system connected with the z axis of the solenoid[1)]:

$$E_x=\frac{\dot{h}}{c}\frac{y}{2}, \quad E_y=-\frac{\dot{h}}{c}\frac{x}{2}, \quad \dot{h}=\frac{\partial h}{\partial t}. \tag{2}$$

Taking into account the presence of the fields h and E_i, we write down the system of equations of motion of the electron in the combined field $H+h(t)$:

$$m\ddot{x}=eE_x+\frac{e}{c}\dot{y}(H+h),$$
$$m\ddot{y}=eE_y-\frac{e}{c}\dot{x}(H+h). \tag{3}$$

Proceeding in analogy with Ref. 5 (p. 80 of the original), we multiply the second equation of (3) by i and add it to the first

$$\ddot{z}+i\Omega(t)\dot{z}+\frac{i}{2}\Omega\dot{\delta}z=0, \quad z=x+iy. \tag{4}$$

Equation (4) explains the role of the electric field in this problem. In fact, if we confine ourselves in the equations of motion to allowance for the field $h(t)$ alone, then we get in place of (4)

$$\ddot{z}+i\Omega(t)\dot{z}=0, \tag{5}$$

whose solution is

$$\dot{z}=\text{const}\cdot\exp\left[-i\int\Omega(t)\,dt\right]. \tag{5a}$$

In this solution, the small increment h to the magnetic field H leads to a small perturbation $\dot{z}$ at any ratio of Ω and ω. In other words, there is no parametric resonance in the solution (5).

Turning to Eq. (4), we rewrite it in a form convenient for comparison with (5a):

$$\dot{z}=\exp\left\{-i\int\Omega(t)\,dt\right\}\left[-\frac{i}{2}\Omega\int\dot{\delta}z\exp\left\{i\int\Omega(t)\,dt\right\}dt+c\right]. \tag{6}$$

According to (6), when account is taken of the vortical electric field, the equation of motion acquires an integral term in which the oscillating part z in combination with $\dot{\delta}$ can produce upon integration a small frequency denominator, which is a characteristic attribute of the instability. The noted formal difference between (5a) and (6) has a simple physical meaning: The magnetic field by itself cannot perform work on the electron. Usually the change of the energy of the system under the influence of the alternating magnetic field is due to the onset of the electric field induced by the change of the magnetic field. A particular example of this general law is the difference between relations (5a) and (6).

2. To obtain concrete results with respect to the behavior of the electron in the combined field $H+h(t)$, we consider first a special case, when the center of the electron orbit is situated exactly on the solenoid axis. In this case it is convenient to transform Eq. (4) in the following manner[3]:

$$z=\xi\exp\left[-\frac{i}{2}\int_0^t\Omega(t)\,dt\right], \quad \ddot{\xi}+\lambda(t)\xi=0,$$

or

$$\ddot{\xi}+\omega_0^2[1+2\delta_0\cos\omega t]\xi=0,$$
$$\omega_0=\tfrac{1}{2}\Omega, \quad \delta_0=h_0/H,$$
$$\lambda(t)=\frac{\Omega^2(t)}{4}+\frac{i}{2}\Omega\dot{\delta}-\frac{i}{2}\dot{\Omega}(t)=\frac{\Omega^2(t)}{4}. \tag{7}$$

It is useful to note that the transformation (7), performed in accordance with Eq. (5), leads to the equality

$$\ddot{\xi}+\lambda^*(t)\xi=0, \quad \lambda^*=\tfrac{1}{4}\Omega^2(t)-\tfrac{1}{2}i\dot{\Omega}(t). \tag{7a}$$

Equation (7a) for ξ has an exact solution

$$\xi=\text{const}\cdot\exp\left\{-\frac{i}{2}\int\Omega(t)\,dt\right\},$$

which returns us to the definition (5a) of $\dot{z}$. Thus, allowance for the vortical field $E_i(r, t)$ and the difference, due to this field, between the functions $\lambda(t)$ from (7) and $\lambda^*(t)$ from (7a) are indeed of fundamental significance in our problem.

Equation (7) for ξ takes the form of the standard equation of the theory of parametric resonance in the one-dimensional case[6] and the basic frequency in it is $\omega_0=\Omega/2$. This means that parametric excitation of an electron located exactly on the axis of the solenoid does indeed take place and occurs at frequencies of the alternating field $h(t)$ which are defined by the equation $\omega_n=2\omega_0/n$, $n=1, 2, \ldots$, or, in terms of Ω,

$$\omega_n=\Omega/n, \quad n=1, 2, 3, \ldots. \tag{8}$$

3. Relations (7) and (8) obtained for an electron located on a solenoid axis point to the existence of parametric instability in the discussed situation. However, most electrons located, for example, on the surface of liquid helium in a plane normal to the solenoid axis have Larmor-orbit centers that are noticably displaced from the solenoid axis. To study the dynamics of these electrons it is necessary to take into account the fact that the quantity z contains a large constant z_0, which determines the position of the center of the Larmor orbit of the given electron relative to the axis of the solenoid. Under such conditions the use of the transformation (7), which is convenient for oscillating variables, turns out to be ineffective. More suitable is an approximate method of solving the problem, based on separating the rapidly oscillating part of the electron coordinate. We put

$$z=\zeta(t)+z_0(t), \tag{9}$$

where $z_0(t)$ is the running coordinate of the center of the electron orbit, and ζ is a rapidly oscillating increment, whose average value over the period of variation of the field $h(t)$ is close to zero. Substituting (9) in (4) and averaging the result over the period of the oscillation of the field $h(t)$, it is easy to obtain an equation that determines the time variation of the coordinate of the center of the electron orbit:

$$\begin{gathered}\dot{z}_0+\langle\delta\dot{\zeta}\rangle+{}^1/_2\langle\dot{\delta}\zeta\rangle=0,\\ \langle\delta\dot{\zeta}\rangle=\frac{1}{T}\int_0^T\delta(t)\dot{\zeta}(t)\,dt.\end{gathered} \tag{10}$$

The definition (10) shows that the drift velocity of the center of the orbit is proportional to the convolution $\langle\cdots\rangle$ of the quantity δ with the amplitude ζ of the oscillatory motion of the electrons (which in turn is proportional to δ). Thus, $z_0(t)$ varies with time only in second order in δ and is quite small if $\delta\ll 1$. This circumstance makes it possible to split the general equation (4) for z into two equations for z_0 and ζ. The first takes the form (10). The second, which is linear in δ, is

$$\ddot{\zeta}+i\Omega(1+\delta)\dot{\zeta}+{}^1/_2 i\Omega\dot{\delta}\zeta=-{}^1/_2 i\Omega\dot{\delta}z_0-i\Omega\delta\dot{z}_0, \tag{11}$$

where $z_0(t)$ is defined by (10). The solution of (11) consists of a homogeneous part and inhomogeneous part. The homogeneous part is of greater importance for the parametric resonance and coincides with the already investigated equation (7). As to the inhomogeneous solution, in the approximation linear in δ it takes the form

$$\begin{gathered}\zeta_1=b_1\cos\omega t+c_1\sin\omega t,\\ b_1=i\Omega c_1/\omega,\quad c_1=i\omega\Omega z_0\delta_0/2(\Omega^2-\omega^2).\end{gathered} \tag{12}$$

This solution, which usually appears in the theory of cyclotron resonance, corresponds to motion of a magnetized electron in a locally homogeneous alternating field equal to the average value of the vortical field on the orbit of the given electron. In the next higher orders of perturbation theory in δ_0, the amplitude ζ in Eq. (11) acquires increments ζ_n that contain resonant denominators at the frequencies $\omega_n=\Omega/n$. However, the amplitudes of the corresponding resonances contain the small quantity δ_0^n at $\omega=\omega_n$, $n=1, 2, 3, \ldots$.

Having the solution (12), we can return to an investigation of (10). In this equation, in the calculation of the mean values $\langle\delta\dot{\zeta}\rangle$ and $\langle\dot{\delta}\zeta\rangle$, it is necessary to use for ζ the complete expression, which includes both the homogeneous and the inhomogeneous parts. In the case $z_0 \gg R_H$ (R_H is the Larmor radius of the electron), however, i.e., for electrons far enough from the solenoid axis and at sufficiently short instability development times, the inhomogeneous part of ζ contains the large factor $z_0/R_H\gg 1$ compared with the solution of the homogeneous equation. As a result we can confine ourselves to the inhomogeneous solution in the calculation of $\langle\delta\dot{\zeta}\rangle$ and $\langle\dot{\delta}\zeta\rangle$ for such orbits.

In the actual averaging it is necessary to take the slowly varying quantity $z_0(t)$ in (10) outside the sign of integration over the period of the field $h(t)$. As a result we get

$$\langle\delta\dot{\zeta}\rangle+{}^1/_2\langle\dot{\delta}\zeta\rangle={}^1/_2 c_1\omega \tag{13}$$

and Eq. (10) takes the form

$$\dot{z}_0+\frac{i}{4}\delta_0^2\Omega\frac{\omega^2}{\Omega^2-\omega^2}z_0=0,\quad z_0|_{t=0}=z_0(0). \tag{14}$$

The solution of this equation has an oscillating character with a characteristic period

$$T^{-1}=\delta_0^2\Omega\left|\frac{\omega^2}{\Omega^2-\omega^2}\right|,\quad \omega^2\neq\Omega^2, \tag{15}$$

which is much longer than Ω^{-1} if $\omega^2\neq\Omega^2$. In the case $\delta=\delta_0\sin\omega t$, the quantity $\langle\delta\dot{\zeta}\rangle+\frac{1}{2}\langle\dot{\delta}\zeta\rangle$ has a structure similar to (19), but with the imaginary unity. As a result, the equation for $z_0(t)$ has a solution that varies exponentially in time, with a time scale (15).

In the general case of an arbitrary initial phase of $\delta(t)$, the solution for $z_0(t)$ contains oscillations and an exponential dependence on the time.

Thus, the appearance of an inhomogeneity in the equation of motion of an electron displaced from the solenoid axis leads to two additional effects (compared with the situation on the solenoid axis). First, the center of the electron orbit drifts, and the more so the farther the average position of the electron from the solenoid axis and the closer the external frequency ω to the cyclotron frequency Ω. Second, the inhomogeneous solution contains resonant denominators at the frequencies $\omega_n=\Omega/n$, which coincide with the frequencies ω_n (8) of the parametric instability. The latter means that one of the main characteristic features of the parametric instability—that it has a threshold in the presence of friction in the system—turns out to be somewhat blurred in the case of a magnetized electron. When one of the resonant frequencies ω_n is approached at arbitrarily small amplitude of the alternating signal and in the case when the electron absorbs energy from the RF field, one of the resonant denominators of the inhomogeneous solution should come into play and lead to the onset of a thresholdless singularity in the absorption at this frequency. However, the influence of the inhomogeneous resonant denominators on the total absorption of the RF field energy by the electron decreases rapidly with increasing number of the harmonic of the resonance, since this effect is proportional to δ_0^n, whereas the contribution of the parametric instability for all n remains proportional to δ_0. All that changes is the threshold on the harder side, and the time of stability development increases.

4. Let now the cross section of the solenoid be a rectangle with sides X and Y with $X\ll Y$. This change of the solenoid geometry, which is insignificant from the point of view of producing a homogeneous magnetic field in the interior of the solenoid, exerts a surprising influence on the parametric instability of an electron in a magnetic field.

Formally this change of the solenoid geometry allows us to speak of a change in the geometry of the vortical electric field, which in this case takes the form

$$E_y=-\dot{h}x/c,\quad -{}^1/_2X\leqslant x\leqslant{}^1/_2X. \tag{16}$$

The corresponding equations of motion of the electron in the magnetic field are no longer symmetrical within the variables x and y, so that they cannot be written in terms of a single complex quantity $x = x + iy$:

$$m\ddot{x}=\frac{e}{c}\dot{y}(H+h),$$
$$m\ddot{y}=-\frac{e}{c}\dot{x}(H+h)-\frac{e}{c}\dot{h}x. \tag{17}$$

It is easy to verify, however, that the second equation of (17) has as its first integral

$$m\dot{y}=-\frac{e}{c}x(H+h)+\frac{e}{c}Hx_0(0). \tag{18}$$

The integration constant in (18) is chosen such that in the absence of an alternating magnetic field the oscillations of the electron along the x axis occur near the specified position of the center $x_0(0)$ of the electron orbit. Considering for simplicity an electron located at the center of the solenoid, i.e., having $x_0(0)=0$, and substituting $\dot{y}$ from (18) in (17), we again arrive at the Mathieu equation

$$\ddot{x}+\Omega^2(1+\delta)^2x=0, \tag{19}$$

but with frequency $\omega_0=\Omega$ and not $\Omega/2$ as in (7). This means that at the chosen geometry of the vortical electric field the frequencies of the parametric instability are determined by the relations

$$\omega_n=2\Omega/n, \quad n=1,\ 2,\ 3,\ldots, \tag{20}$$

which differ from (8). In particular, the first resonant frequency corresponds in this case to double the Larmor frequency, $\omega_1=2\Omega$.

In the general case $x_0 \neq 0$ the equation of motion for x, written in terms of $\eta = x = x_0$, is of the form

$$\ddot{\eta}+\Omega^2(1+\delta)^2\eta+2\Omega^2\delta x_0+\Omega^2\delta^2x_0=0. \tag{19a}$$

Just as in the cylindrical variant, Eq. (19a) contains inhomogeneous terms proportional to x_0, which correspond to the influence exerted on the electron motion by the vortical electric field averaged over the electron orbit.

5. The results (8) and (20) for the spectra of the parametric excitation call for some comments that indicate the region of existence of such a difference. The structure of the vortical electric field in the interior of the solenoid is described by expression (2) or (16) under the conditions when the electromagnetic problem is quasistationary, i.e., under conditions when the characteristic electromagnetic wavelength $\lambda = c/\omega$ is much larger than the radius of the solenoid. Only under these conditions can the alternating magnetic field $h(t)$ be regarded as spatially homogeneous in the interior of the solenoid (accurate to terms $R/\lambda \ll 1$), and we can confine ourselves in the calculation of the fields to a solution of one equation, curl $\mathbf{E} = -\dot{\mathbf{h}}/c$. An attempt to calculate the dimensions of the solenoids when the magnetized electron is located near the surface of the solenoid, with an aim at reducing the problem in both cases to a semi-three-dimensional one, and consequently to obtain coincidence of the spectra of the parametric excitation, leads automatically to violation of the quasistationarity condition. It is obvious that the quasistationary solutions discussed above do not admit of such a limiting transition, although in principle it does exist.

2. CLASSICAL THRESHOLD OF PARAMETRIC INSTABILITY FOR ELECTRONS IN A MAGNETIC FIELD

1. Assume first for the sake of argument that we are dealing with an electron in vacuum. The equation of motion (4) with allowance for the radiation friction takes the form (Ref. 5, p. 265 of the original)

$$\ddot{z}+i\Omega(t)\dot{z}+\frac{i}{2}\Omega\dot{\delta}z-\frac{2}{3}\tau\dddot{z}=0, \quad \tau=\frac{e^2}{mc^3}. \tag{21}$$

The transformation (7) of this equation yields the relation

$$\ddot{\xi}+\frac{\Omega^2}{4}(1+2\delta_0\cos\omega t)\xi$$
$$-\frac{2}{3}\tau\left[\dddot{\xi}-\frac{3}{2}i\Omega(t)\ddot{\xi}-\dot{\xi}\left(\frac{3}{2}i\dot{\Omega}-\frac{1}{2}\Omega(t)\right)-\xi\left(\frac{i}{2}\ddot{\Omega}+\frac{1}{4}\dot{\Omega}\Omega\right)\right]=0. \tag{21a}$$

Assuming in the vicinity of the first resonant frequency

$$\omega={}^1/_4\Omega+\varepsilon, \quad \varepsilon\ll\Omega,$$

and carrying out the standard calculations for the theory of paramagnetic resonance, it is easy to determine the condition for the onset of excitation:

$$-\left(\delta_0{}^2\frac{\Omega^2}{4}-\frac{\Omega^4}{16}\tau^2\right)^{1/2}<\frac{2\varepsilon}{10^{1/2}}<\left(\delta_0{}^2\frac{\Omega^2}{4}-\frac{\Omega^4}{16}\tau^2\right)^{1/2}. \tag{22}$$

For comparison, in the case of an oscillator with friction $\gamma\dot{\xi}$, the condition similar to (22) is

$$-({}^1/_4\delta_0{}^2\Omega^2-4\gamma^2)^{1/2}<\varepsilon<({}^1/_4\delta_0{}^2\Omega^2-4\gamma^2)^{1/2}. \tag{22a}$$

The numerical value of τ is very small: $\tau \approx 10^{-23}$ sec. As a result the limitation on the amplitude of the alternating magnetic field for parametrix excitation of an electron in vacuum is negligible: $\delta_0 \gg \Omega\tau \approx \Omega \times 10^{-23}$.

2. In the case of electrons on the surface of liquid helium in a plane normal to the magnetic field, the problem of the dissipative phenomena that accompany the parametric instability has a number of specific singularities.

Assume that we are dealing with a small concentration of surface electrons, when the electron–electron interaction can be neglected. In this case the odd part of the distribution function of the electrons turns out to be

$$f_1=-\mathbf{u}\frac{\partial f_0}{\partial \mathbf{v}}. \tag{23}$$

Here $\mathbf{v}$ is the random velocity of the electrons along the helium surface, $f_0(\mathbf{v})$ is the spherical part of the distribution function, and $\mathbf{u}$ is the directional velocity of the electrons. The quantity $\mathbf{u}(t)$ is determined by solving the equation

$$\dot{\mathbf{u}}+\nu(\mathbf{v})\mathbf{u}=-\frac{e\mathbf{E}}{m}-\frac{e}{mc}[\mathbf{u}\times(\mathbf{H}+\mathbf{h})], \tag{24}$$

where $\nu(\mathbf{v})$ is the effective frequency of the collision of the electron with ripplons, and depends on $\mathbf{v}$.

Equation (24) is classical in form, but the frequency ν depends on the velocity $\mathbf{v}$. This means that the solution of the problem of parametric instability for Eq. (24) in the region of one of the resonance frequencies (for example, the frequency $\omega_1 = \Omega$) leads to the following characteristic dependence of the velocity u on the time[6]:

$$u(t, v)\propto\exp[{}^1/_4\delta_0\Omega-\nu(v)]t. \qquad (25)$$

In other words, when describing the ensemble of electrons by means of a distribution function in the velocities $f_0(\mathbf{v})$, we arrive at the conclusion that if ν depends substantially on $\mathbf{v}$ then, with increasing δ_0 the parametric instability can initially occur only in individual regions of the electron distribution function, which satisfy the inequality

$$\delta_0\Omega/4>\nu(v). \qquad (26)$$

For this reason, the concept of a threshold, which is clearly defined in a mechanical problem with friction, loses its literal meaning for an ensemble of electrons. Nonetheless, the parametric instability leads in this case to significant effects, since it is physically clear that the absorption of the energy of the RF field even by a small group of electrons, defined by the condition (26), should influence the properties of the entire distribution function, meaning also the macroscopic properties of the system of electrons.

Concrete results are easier to obtain from the inequality (26) for a system of electrons localized on the surface of liquid helium because of the specific properties of the function $\nu(\mathbf{v})$. As shown in a number of papers[7-9] devoted to the calculation of $\nu(\mathbf{v})$ in collisions of electrons with the ripplons, the function $\nu(\mathbf{v})$ has in the presence of a clamping electric field $E_\perp$ a minimum $\nu_{\min}$ determined from the condition $\partial\nu/\partial\mathbf{v}|_{\mathbf{v}=\mathbf{v}_{\min}}=0$.

The position of the minimum, i.e., the value of $\mathbf{v}_{\min}$, depends on the intensity of the clamping field $E_\perp$ and can vary in a wide range. Obviously, the inequality (26) begins to be satisfied primarily for electrons with random velocities $\mathbf{v}\sim\mathbf{v}_{\min}$, for which $\nu\approx\nu_{\min}$. Consequently, the determination of the threshold of the parametric instability in this case yields differential information on the value of $\nu_{\min}$.

We note that the width of the cyclotron-resonance line also contains information on the function $\nu(\mathbf{v})$. This information, however, is integral, averaged over the electron distribution function. In this sense, the parametric and cyclotron resonances complement each other well.

The concrete form of the function $\nu(p)$, $p=m|v|$, which contains the contributions of the clamping field and of the polarization interaction of the electrons with the ripplons, can be taken from Ref. 9:

$$\nu(p)=\frac{mT}{2\hbar\alpha}\left[\frac{(eE_\perp)^2}{p^2}-\frac{eE_\perp\Lambda}{\hbar^2}\left(2\ln\frac{p}{2\hbar\gamma}+3\right)+\frac{p^2\Lambda^2}{\hbar^4}A(p)\right],$$

$$A(p)=\frac{1}{8}\pi^2+\frac{41}{16}-\frac{19}{4}\ln 2+\frac{3}{2}(\ln 2)^2+\left(\frac{19}{4}-3\ln 2\right)\ln\frac{p}{2\gamma\hbar}+\frac{3}{2}\left[\ln\left(\frac{p}{2\gamma\hbar}\right)\right]^2, \qquad (27)$$

$$\Lambda=e^2(\varepsilon-1)/4(\varepsilon+1),\quad \gamma=m\Lambda/\hbar^2,$$
$$\varepsilon-1=0.06,\quad \gamma\approx7.6\cdot10^{-7}\,\text{cm},$$

where T is the temperature, α is the coefficient of surface tension, and ε is the permittivity of the liquid helium.

In the region of weak clamping fields, the equation $\partial\nu/\partial p=0$ for the determination of $k_{\min}=p_{\min}/\hbar$ is of the form

$$k_{\min}^{-2}=\frac{\Lambda}{eE_\perp}\left(1.23+\frac{3}{2}\ln^2\frac{k_{\min}}{2\gamma}+3\ln\frac{k_{\min}}{2\gamma}\right). \qquad (28)$$

An approximate solution of this transcendental equation can be obtained by successive substitutions. Thus, in the zeroth approximation we have

$$k_{\min}^{(0)}\approx(eE_\perp/1.23\Lambda)^{1/2}. \qquad (29)$$

Substituting this value of $k_{\min}^{(0)}$ into the argument of the logarithms, we have the more accurate value $k_{\min}^{(1)}$

$$k_{\min}^{(1)}\approx\left\{\frac{eE_\perp}{\Lambda[1.23+{}^3/_2\ln^2(k_{\min}^0/2\gamma)+3\ln(k_{\min}^0/2\gamma)]}\right\}^{1/2}. \qquad (29a)$$

The succeeding refinement of $k_{\min}$ is obvious. The numerical values of some of the first approximations for $k_{\min}$ at $E_\perp\approx1$ V/cm are $k_{\min}^{(0)}\approx2.74\times10^5\,\text{cm}^{-1}$, $k_{\min}^{(1)}\cong2.08\times10^5\,\text{cm}^{-1}$, $k_{\min}^{(2)}\approx1.68\times10^5\,\text{cm}^{-1}$. The corresponding value is $\nu_{\min}\equiv\nu(k_{\min})\approx10^{-7}$ sec. The inequality (26) is then satisfied at an intensity $h_0\gtrsim1$ Oe. As to the field H, by virtue of $\delta_0\ll1$ it should be of the order of $H\gtrsim10^2$ Oe.

3. QUANTUM TREATMENT

1. In the quantum limiting case the presence of parametric instability should follow from the solution of the Schrödinger equation. As is clear from the classical analysis presented above, interest attaches in this case to both the cylindrical-symmetry and the one-dimensional variants of the problem. Both variants admit of an exact solution in the sense that the solution of the quantum problem reduces to a solution of the corresponding equation for the classical oscillator.

We begin with the simpler one-dimensional case, assuming, as above, that a similar situation can be realized in a rectangular solenoid XY, $X\ll Y$. Choosing the vector potential to be

$$A_y=-[H+h(t)]x,\quad A_x=A_z=0, \qquad (30)$$

writing down the electron wave function in the form

$$\psi=\exp\{i(p_yy+p_zz)/\hbar\}\varphi(x) \qquad (31)$$

and omitting the spin part of the problem, we have an equation for φ:

$$i\hbar\frac{\partial\varphi}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x^2}+\frac{1}{2m}\left\{\left[\hat{p}_y+\frac{e}{c}x(H+h)\right]^2+\hat{p}_z^2\right\}\varphi=0, \qquad (32)$$

$cp_y/eH=x_0$ is the average position of the oscillating electron at $h=0$. Assume, as above, that initially $x_0=0$. As a result we get

$$i\hbar\frac{\partial\varphi}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x^2}-\frac{m\Omega^2}{2}(1+\delta)^2x^2\varphi-\frac{p_z^2}{2m}\varphi=0. \qquad (33)$$

Equation (33) has the standard form used in quantum theory of parametric resonance for a one-dimensional oscillator.[10] It is typical that the natural frequency of this oscillator is Ω, i.e., the spectrum of the frequencies at which parametric excitation of the given oscillator is possible coincides with the classical result (20).

In the case $x_0\neq0$ we have in the approximation linear in δ:

$$i\hbar\frac{\partial\varphi}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x^2}-\frac{m\Omega^2}{2}[(x-x_0)^2(1+2\delta)+2\delta x_0(x-x_0)]\varphi-\frac{p_z^2}{2m}\varphi=0. \qquad (33a)$$

The term $2\delta x_0(x-x_0)$, in this equation has the meaning of the local homogeneous electric field that acts on an electron oscillating about the point x_0. We can thus conclude on the basis of (33) and (33a) that a complete one-to-one correspondence exists between the classical and quantum variants of the one dimensional problem of parametric excitation of an electron in a magnetic field: The resonance frequencies are determined by (20); for the electrons whose average position is shifted relative to the plane $x_0=0$, a periodic field averaged over the electron orbit appears and introduces cyclotron-resonance elements into the parametric resonance.

A similar solution is obtained for the cylindrically symmetrical variant of the problem. An investigation of the properties of the Schrödinger equation for the electron in an alternating magnetic field with a vector potential chosen in the form

$$\mathbf{A}={}^1/_2[\mathbf{H}\times(t)\mathbf{r}] \qquad (34)$$

was carried out by Malkin and Man'ko.[4] Just as in the one-dimensional case (30), the solution of the Schrödinger equation can be reduced here to a solution of the equation for the classical oscillator. The details of this analysis can be found in Ref. 4. We note only the detail of greatest importance for our study, namely that the characteristic frequency ω_0 of the indicated classical oscillator is equal to half the cyclotron frequency, $\omega_0=\Omega/2$. This means that when the vector potential is chosen in the form (34) the spectrum of the frequencies of the parametric resonance is determined by relation (8) rather than by (20), in full accord with the classical predictions obtained above.

The noted difference between the parametric-resonance frequency spectra, due to the choice of the explicity form of the vector potential in the form (30) or (34), does not contradict the known requirement of gauge invariance of the physical results. In this case this difference is preserved also in the classical limit, when the problem is solved in terms of $H+h(t)$. The real cause of the produced frequency difference is the difference in the structure of the vortical electric field.

4. EXCITATION OF TRANSVERSE WAVES IN A TWO-DIMENSIONAL WIGNER CRYSTAL

The possibility of producing vortical closed electric fields on a helium surface, which was noted above, may prove to be useful for the observation of some collective phenomena in a system of surface electrons. In particular, these fields can excite transverse sound oscillations in a hypothetical Wigner crystal on the surface of the helium.

We consider first free, transverse, radial vibrations of an electron crystal in the form of a flat disk. According to the prevailing concepts[11,12] the low-frequency transverse vibrations of the electron crystal have an acoustic character or else contain a threshold frequency ω_d brought about by deformation. The phenomenological description of such vibrations of $\mathbf{u}(r,t)$ is by means of the equation

$$\frac{\partial^2 u_\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial u_\varphi}{\partial r}+\frac{\omega^2}{c_\perp^2}u_\varphi=0 \qquad (35)$$

or

$$\frac{\partial^2 u_\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial u_\varphi}{\partial r}+\frac{\omega^2-\omega_d^2}{c_\perp^2}u_\varphi=0, \qquad (35a)$$
$$\omega_d^2\approx(eE_\perp)^2/2\pi\hbar\alpha,$$

where $c_\perp$ is the speed of the transverse sound in the electron crystal. To determine the spectrum of the vibrations (35) and (35a) it is necessary also to specify the boundary conditions. These conditions are that the amplitudes of the displacements on the disk axis be finite and that the elastic stresses on the outer boundary of the disk vanish; this corresponds to free slippage of the disk relative to the liquid surface of the helium. The last condition is equivalent in our case to the requirement

$$\left(\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}\right)\bigg|_{r=R}=0, \qquad (36)$$

where R is the radius of the disk.

A solution of (35) or (35a) which satisfies the imposed boundary conditions leads to the following relations for the spectrum of the electron crystal:

$$J_0(x)-xJ_1(x)=0,\quad x=\omega^* R/c_\perp, \qquad (37)$$

$J_n(x)$ is a Bessel function, while ω^* is equal to ω in the case (35) and to $\omega-\omega_d$ in the case of (35a).

Thus, the spectrum of the oscillations of the electron crystal takes the form

$$\omega_n^* R/c_\perp=\lambda_n, \qquad (38)$$

where λ_n are the roots of the reduced transcendental equation. The first of these roots are $\lambda_1=1.25$, $\lambda_2=4.05, \ldots$.

Assume now that a small inductance coil of an LC circuit, with radius $r_0\ll R$ is coaxially placed on the disk. Flow of alternating current through the coil produces a magnetic field whose lines are normal to the electron disk. The smallness of the coil compared with the radius of the disk and the rather rapid decrease of the coil field with increasing distance between the coil and the surface allow us to assume that the net magnetic flux through the surface of the disk is equal to zero. In other words, all the magnetic flux lines of the coil pass through the electron disk before they are closed in the exterior of the coil. Under these conditions the net vortical electric field induced on the surface of the disk is equal to zero, so that the action of this field on the crystal electrons reduces to application of a moment of forces in an angular direction without production of a resultant rotation of the crystal as a whole.

By adding to Eqs. (35) and (35a) an external force having the indicated properties, we can obtain resonant excitation of transverse waves in a Wigner crystal at the frequencies ω_n from (38). In this case the Q of the exciting LC circuit should decrease resonantly.

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[1)] The electric field (2), taken on the surface of the solenoid, i.e., at $x^2+y^2=r^2$, coincides with the field $E=ZI(t)$, where $Z=-i\omega c^{-2}L$ is the impedance of the solenoid, L is the inductance of the round solenoid per unit solenoid length, $L=2\pi RN$, n is the number of turns per unit length, and ω is the frequency of the alternating current $I(t)$.

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Recursion equation for the percolation problem

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A recursion equation allowing for an arbitrary change in scale is obtained for the percolation problem. The percolation threshold for two-dimensional space coincides with the exact value $P_c = 0.5$, and the corresponding threshold for three-dimensional space is $P_c = 0.16$, which agrees with the available data. A calculation is given of the critical index which governs the power-law behavior of the conductivity near the percolation threshold. The recursion approach confirms the hypothesis of scaling invariance.

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1. The percolation theory methods are being used widely to solve various problems in the physics of disordered systems.[1,2] Numerical calculations in the lattice problems and in the continuum analog of the percolation problem have yielded the most important characteristics of disordered systems and the hypothesis of scaling invariance has been put forward[3-6] by analogy with the theory of phase transitions. As a result of the scaling invariance and universality, percolation along channels of dimensions of the order of the lattice constant has little effect on the large-scale properties of such systems and this justifies the application of the renormalization group methods to the calculation of the percolation threshold and critical indices. The first attempts have been made[7,8] to use the recursion equation in solving the two-dimensional problem of bonds[7] and the problem of sites.[8] The doubling procedure is used in Refs. 7 and 8 and this suffers from low precision.[9]

We shall propose an improved recursion equation which can be applied to systems of any dimensions in space and which allows for an arbitrary change in scale which increases considerably the precision of the results. Our method is readily seen to be similar to the recursion approach of A. A. Migdal, who first applied this to the theory of phase transitions and gauge fields.[9]

2. We shall derive the recursion equation by selecting in a d-dimensional system a region in the form of a d-dimensional cube with the characteristic size L, which is much less than the correlation radius[2] but much greater than the lattice constant. We shall assume that this region is described approximately by a single parameter $P(L)$, which is the probability that the region is conducting. Then, the probability that the region is not conducting is $1-P(L)$. We shall combine N^d cubes into one with a characteristic size NL and calculate the probability $P(NL)$ that the cube of dimensions NL is conducting. If we connect current-carrying conductors to two opposite $(d-1)$-dimensional faces of the cube NL, we find that the probability that the current passes through the cube can be calculated as follows. We considered a "column" of N d-dimensional cubes joining these faces. The probability that the column is conducting is $[P(L)]^N$. The current can pass between the faces if at least one of the N^{d-1} columns is conducting. The probability of this happening is $1-(1-P^N)^{N^{d-1}}$. This is the probability that the cube NL is conducting:

$$P(NL)=1-[1-P^N(L)]^{N^{d-1}}. \tag{1}$$

Equation (1) allows only for the shortest leakage paths between the opposite faces of the cube NL. The most accurate results may be expected in the limit $N\to 1$ (Ref. 9) with Eq. (1) continued analytically to nonintegral values of N. If we assume that $NL=L+dL$, where dL is an infinitesimally small change in scale, and expand both sides of Eq. (1) as series up to the first order in dL/L, we obtain the following equation which describes the dependence of P on L for an arbitrary change in scale: