Momentum-transfer dependence of the cross section for e^+e^- pair production by muons on nuclei

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For the case of e^+e^- pair production by ultrarelativistic muons on nuclei, the dependence of the cross section and the energy spectrum of the pair on $t = k^2$, the square of the four-momentum transfer, is calculated. Asymptotic formulas for small and large values of the momentum transfer are examined. Qualitatively, we have $d\sigma \sim dt/t$ for $t \ll m^2$ and $d\sigma \sim dt/t^2$ for $t \gg m^2$. A decided asymmetry is found in the energy distribution between the components of the pair in the "deep-inelastic" region, where the energy k_0 of the pair is of the order of the square of the momentum transfer divided by the mass of the electron, $k_0 \sim k^2/m$; although the energy of the pair is large in this case, $k_0 \gg 2m$, there is considerable probability that almost all of the energy is carried away by one particle of the pair, the energy of the other particle being only of the order of its mass. In accordance with the uncertainty relation, the distribution in momentum transfer gives a (qualitative) distribution in impact parameter. By treating the process as pair production by an external field, one can estimate probabilities for multiple pair production in collisions of fast nuclei with charges eZ and eZ'. In this way the ratio of the cross sections for production of two pairs and for a single pair is found to be $\sigma_2/\sigma_1 \sim (\alpha^2 ZZ'/\pi)^2 \ln\gamma$. Similarly, $\sigma_{n+1}/\sigma_n \sim (\alpha^2 ZZ'/\pi)^2 \ln^2\gamma$, $n \geq 2$, where γ is the Lorentz factor of the incident nucleus.

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1. INTRODUCTION

The first theoretical papers on pair production by charged particles appeared more than 40 years ago. A paper[1] by Landau and Lifshitz gave the main term of the asymptotic formula for the total cross section at large energies. The total cross section and the distribution of electron and positron energies for production by a lepton on a Coulomb force center were found in the remarkable papers by Racah.[2,3] Let us also note some later papers. Kel'ner[4] took screening of the nucleus by electrons into account and found the energy distribution of the electrons and positrons in the case of complete screening, and later[5] found the energy loss of muons through e^+e^- pair production and gave numerical values of functions describing the energy spectrum of pairs for muons of several different energies passing through earth or lead. The total cross section for e^+e^- pair production in collision of a muon and an atom was found by Pichkurov and the present writer,[6] and the energy spectra of the pairs were found later.[7] The state of the problem up to 1970 is described in a book by Bugaev, Kotov, and Rozental'.[8]

In the present paper we examine the dependence of the cross section on $k^2 = t$, where $k_\mu = P_\mu - P'_\mu$ is the four-momentum transferred from the incident lepton; the target is a Coulomb center of force. We shall be interested in the case in which the initial lepton is ultrarelativistic: $P_0 \gg M$ and $t \ll M^2$, where M is the mass of the incident lepton (for definiteness a muon). For such distant collisions the state of motion of the colliding particles is almost unchanged in the process of production of an e^+e^- pair which carries away only a small fraction of the energy of the incident particle.

Our starting point is a formula of Racah[2] which describes the differential cross section as a function of the external parameters P_0 and M and the variables t, k_0, and w; w is the energy of the electron, and $k_0 = w + w'$ is that of the pair. The purpose of the work is to integrate the Racah formula first over w (which gives the distribution in k_0 for fixed t) and then to integrate over k_0, obtaining the distribution in t. For $t \ll M^2$ this last distribution can also be regarded as the momentum distribution of the recoil nuclei.

The integration of the exact formula turns out to be difficult, and therefore two important limiting cases are considered. In Sec. 2 the Racah formula is written in a form which is convenient for analytic integration over w for $t \ll m^2$ and $t \gg m^2$, where m is the mass of the electron.

In Sec. 3 the integration over w is carried out in the simplest case, when the energies of all the particles are ultrarelativistic. The next integration over k_0 (within the limits of applicability of the formula) gives an expression from which one can easily perceive the structure of the distribution over t for t in the range $\gamma^{-1} \ll t^{1/2} \ll \gamma$; namely, the cross section is proportional to a polynomial of second degree in $\ln(2\gamma)$. The coefficients χ_2, χ_1, χ of this polynomial are functions of t only if $t \ll M^2$, and χ_2 and χ_1 are already determined from this consideration of the case of ultrarelativistic energies. The function χ is derived for $t \ll m^2$ in Sec. 4, and for sufficiently large t/m^2 in Sec. 5. Qualitatively $d\sigma \sim dt/t$ for $t \ll m^2$ and $d\sigma \sim dt/t^2$ for $t \gg m^2$.

By means of the uncertainty relation the distribution in momentum transfer can be made to yield (qualitative) information about the distribution in impact parameter. By regarding the process as pair production by an external field[1] one can easily estimate the probabilities

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for production of several pairs in collisions of fast nuclei; see Eqs. (32) and (33).

In Sec. 4 the integration of $d\sigma$ over w is performed as in Refs. 2 and 9, to give the distribution in k_0 and t for $t \ll m^2$. Although the original expression here is somewhat more complicated than in Refs. 9 and 2, the answer can be expressed in terms of the same functions. Subsequent integration over k_0 gives the distribution in t.

Section 5 deals with distributions like those of Sec. 4, but for the case of sufficiently large t/m^2. Here there is a particular complication because at $k_0 \sim t/m \gg m$ small electron (or positron) energies, $w \sim m$, are in the effective range of integration over w. The fact that values $w \sim m$ are favored at $k_0 \sim t/m \gg m$ can also be seen from the electron propagator, in which the denominator is small under these conditions.

2. THE RACAH FORMULA

Let $w = |\mathbf{p}|$ and $p = |\mathbf{p}|$ be the energy and momentum of the electron (and w', p' those of the positron); P_0 and $P = |\mathbf{P}|$ are the energy and momentum of the incident (and P_0', P' of the scattered) muon; $k^2 = (\mathbf{P} - \mathbf{P}')^2 - (P_0 - P_0')^2 = \mathbf{k}^2 - k_0^2$ is the square of the transferred four-momentum; $w^2 - p^2 = m^2 = 1$, $P_0^2 - \mathbf{P}^2 = M^2$, $k_0 = w + w' = P_0 - P_0'$. The cross section for the process of interest here is given by a formula of Racah[2] (for remarks on the derivation of this formula by the Feynman technique see Ref. 6; $\hbar = c = m = 1$, $\alpha = 1/137$)

$$d\sigma = \frac{(4\alpha r_0 Z)^2}{\pi}\frac{P_0^2}{P^2}\left\{\frac{\Phi}{\eta^4}\left(\frac{P_0'}{P_0} - \frac{t}{4P_0^2}\right) + \left(\frac{\Phi}{2\eta^2} + \frac{\Gamma}{t}\right)\left[\frac{1}{\eta^2}\left(\frac{P_0'}{P_0} - \frac{t}{4P_0^2}\right) + \frac{1}{2P_0^2} - \frac{1}{\gamma^2 t}\right]\right\} dt\, dk_0\, dw, \qquad (1)$$

where $r_0 = \alpha/m$, $\gamma = P_0/M$, $t = k^2$, $\eta^2 = k_0^2 + t$, $w^2 = p^2 + 1$, and

$$\Phi = \Phi_0 + \Phi_\Lambda \Lambda + \Phi_L L + \Phi_{L'} L' + \Phi_{L\Lambda} L\Lambda + \Phi_{L'\Lambda} L'\Lambda, \qquad (2)$$

$$\Gamma = \Gamma_0 + \Gamma_\Lambda \Lambda + \Gamma_L L + \Gamma_{L'} L' + \Gamma_{L\Lambda} L\Lambda + \Gamma_{L'\Lambda} L'\Lambda + \Gamma_{LL'} LL', \qquad (3)$$

$$\Lambda = \Lambda(k_0, w) = \ln\frac{t/2+1+\rho+pp'}{A^{1/2}} = \frac{1}{2}\ln\frac{t/2+1+\rho+pp'}{t/2+1+\rho-pp'}, \qquad (4)$$

$$L = \ln\frac{t/2+k_0 w+p\eta}{A^{1/2}} = \frac{1}{2}\ln\frac{t/2+k_0 w+p\eta}{t/2+k_0 w-p\eta}, \quad \rho = ww', \qquad (5)$$

$$\begin{aligned} A &= t^2/4+t+k_0^2+\rho t = [t/2+1+\rho+pp'][t/2+1+\rho-pp'] \\ &= [t/2+k_0 w+p\eta][t/2+k_0 w-p\eta] \\ &= [t/2+k_0 w'+p'\eta][t/2+k_0 w'-p'\eta]; \end{aligned} \qquad (6)$$

in Eq. (2) we have

$$\begin{aligned} \Phi_0 &= -\frac{\eta^2(3t+12+4\rho)}{24A^2}pp' - \frac{t\eta^2(k_0^2-4\rho)}{6A^3}pp' + \frac{t(k_0^2-2)-k_0^2+\rho(3k_0^2-4-2t)-4\rho^2}{8App'}, \\ \Phi_\Lambda &= \frac{\eta^2\rho(t+4\rho)}{12A^2} + \frac{\eta^2}{4A} + \frac{2\rho+2-k_0^2}{8p^2p'^2} - \frac{(t+4\rho)[2\rho(k_0^2-t)-2k_0^2+t(k_0^2-2)]}{32Ap^2p'^2}, \\ \Phi_L &= -\frac{p'}{8\eta p^2} - \frac{(4\rho+t)(2k_0 w+t)}{32A\eta p^2}p', \\ \Phi_{L\Lambda} &= \frac{k_0 w+2\rho+t}{8\eta p^3} \end{aligned} \qquad (7)$$

and in Eq. (3)

$$\begin{aligned} \Gamma_0 &= \frac{8k_0^2+t(12\rho+2k_0^2+12+3t)}{48A^2}pp' + \frac{t^2(2-t)(k_0^2-4\rho)}{24A^3}pp' + \frac{\rho(2t-k_0^2-2)+2t-2+k_0^2(2-t)}{8App'}, \\ \Gamma_\Lambda &= \frac{3k_0^2-3t-4\rho}{12A} \quad \frac{t\rho(2k_0^2+3t+4\rho)}{24A^2} + \frac{k_0^2-2-2\rho}{8p^2p'^2} + \frac{(t-2)[t(k_0^2-2)-2k_0^2+2\rho(k_0^2-t)]}{32Ap^2p'^2}, \\ \Gamma_L &= \frac{p'}{8\eta p^2} + \frac{(t-2)(2k_0 w+t)}{32A\eta p^2}p', \\ \Gamma_{L\Lambda} &= \frac{1}{4\eta p} + \frac{1-t-k_0 w}{8\eta p^3}, \quad \Gamma_{LL'} = -\frac{1}{4\eta^2}. \end{aligned} \qquad (8)$$

The primed quantities (*e.g.*, L', $\Phi_{L'}$, etc.) are obtained from the corresponding unprimed quantities by the exchanges $w \leftrightarrow w'$, $p \leftrightarrow p'$. The important property of symmetry in w and w' is a consequence of the Born approximation. We note that in our notation, unlike Racah's, Φ and Γ include the factor pp'.

We also note that the square of the momentum transfer is related to the scattering angle ϑ of the muon by the equation

$$k^2 = t = 2P_0P_0' - 2PP'\cos\vartheta - 2M^2.$$

We shall here be interested only in the ultrarelativistic case $\gamma \gg 1$. Besides, we confine ourselves to the range $t \ll M^2$. The minimum value of t for given k_0 is $t_{\min} = (k_0/\gamma)^2$, and the maximum value of k_0 for given t is $k_{0\max} = \gamma t^{1/2}$. Equation (1) takes the form

$$d\sigma = \frac{(4\alpha r_0 Z)^2}{\pi}\left\{\frac{3\Phi}{2\eta^4} + \frac{\Gamma}{t\eta^2} - \frac{1}{\gamma^2 t}\left(\frac{\Phi}{2\eta^2} + \frac{\Gamma}{t}\right)\right\} dt\, dk_0\, dw. \qquad (9)$$

The dependence on P_0 and M enters here only through the Lorentz factor γ. Moreover, since Φ and Γ depend only on t, k_0, and w, and $1 \leqslant w \leqslant k_0 - 1$, $2 \leqslant k_0 \leqslant k_{0\max}$, after integration over w and k_0 we have $d\sigma/dt = f(t, \gamma t^{1/2})$, so that the only dependence on γ is that through $k_{0\max}$.

3. DISTRIBUTIONS IN PAIR ENERGY AND IN MOMENTUM TRANSFER IN THE ULTRARELATIVISTIC CASE

If all of the particles involved in the process are ultrarelativistic, the expression (9) can be simplified[2,4]:

$$d\sigma = \frac{(2\alpha r_0 Z)^2}{\pi}\left[\ln(2k_0\tau) - \frac{1}{2} - \frac{1}{2}\ln\gamma_1\right]\left\{a\left[\frac{1}{t} - \frac{\tau}{\gamma_1}\right] - \frac{bz}{t^2} - \frac{\tau c}{\gamma_1^2}\right\} dt\,\frac{dz}{2z}\,dx; \qquad (10)$$

here

$$\begin{gathered} z = (k_0/\gamma)^2,\ \gamma_1 = 1+t\tau,\ a = b + {}^4/_3(1-\tau)\tau z, \\ b = 1 - {}^4/_3\tau,\ c = {}^1/_3 z\tau + {}^1/_3 - {}^4/_3\tau,\ x = w/k_0,\ \tau = x(1-x), \\ (2 \ll k_0 \ll P_0;\ t \ll k_0;\ 1 \ll w, w'). \end{gathered} \qquad (11)$$

Unfortunately the region of applicability of this simple formula becomes narrower as t increases. Formally this is a result of the fact that its derivation from Eq. (9) is based on the requirement that $mk_0 \gg k^2$, i.e., $k_0 \gg t$. Physically it is due to the decidedly uneven distribution of the energy of the pair between its components for $k_0 \sim t \gg 1$; there is a considerable probability that one element of the pair will have a low energy. In fact, simply from the expression for γ_1 in (11) it can be seen that the lower limit on values of w that are important is given by the condition

$$t\tau \simeq tww'/k_0^2 \sim tw/k_0 \sim 1,$$

i.e., for $k_0 \sim t \gg 1$, w can be of the order of unity, whereas Eq. (10) holds for $w, w' \gg 1$.

It follows from the kinematics of the problem that for fixed k we have $t \geqslant z$, and for fixed t the maximum energy of the pair is $k_{0\max} = \gamma t^{1/2}$. It follows that Eq. (10) holds for values of t that are neither too large nor too small:

$$2\gamma^{-1} \ll t^{1/2} \ll \gamma. \tag{12}$$

The inequality on the left is necessary for $k_{0\max} \gg 2$, and that on the right is necessary for $k_{0\max} \gg t$.

Integrating Eq. (10) with respect to x over the range from 0 to 1 and with respect to k_0 over the range from $\bar{k}_0 \gg t$ to $k_{0\max} = \gamma t^{1/2}$, we get $(dz/z = 2dk_0/k_0)$

$$d\sigma = \frac{(2\alpha r_0 Z)^2}{\pi} F_{\rm I}\, dt, \tag{13}$$

where

$$F_{\rm I} = 2\int\limits_{\sqrt{k_0}}^{\gamma\sqrt{t}} \frac{dk_0}{k_0}\left\{\ln(2k_0)\chi_2 + \psi_1 + z\left[\left(\ln(2k_0) + \frac{1}{2}\right)\psi_2 + \psi_3\right]\right\}, \tag{14}$$

$$\chi_2 = \frac{1}{3t^2} + B(t)\left[-\frac{1}{3t^2} + \frac{5}{12t} - \frac{5}{12(t+4)}\right]\ln\frac{t_2}{t_1}, \tag{15}$$

$$\psi_1 = -\frac{1}{3t^2} + B(t)\left\{\left(-\frac{1}{3t^2} - \frac{1}{4t} + \frac{1}{4(t+4)}\right)\ln\frac{t_2}{t_1} + \left(\frac{1}{6t^2} - \frac{5}{24t} + \frac{5}{24(t+4)}\right)a_3\right\}, \tag{16}$$

$$\psi_2 = \frac{2}{3t^3} - \frac{1}{12t^2} + \frac{1}{48t} - \frac{1}{48(t+4)} + B(t)\left[-\frac{2}{3t^3} - \frac{1}{4t^2} + \frac{1}{12t} - \frac{1}{12(t+4)} - \frac{1}{12(t+4)^2}\right]\ln\frac{t_2}{t_1}, \tag{17}$$

$$\psi_3 = -\frac{4}{3t^3} + \frac{1}{8t^2} - \frac{1}{32t} + \frac{1}{32(t+4)} + B(t)\left\{\left(\frac{3}{8t^2} - \frac{5}{48t} + \frac{5}{48(t+4)} + \frac{1}{24(t+4)^2}\right)\ln\frac{t_2}{t_1} + \left(\frac{1}{3t^3} + \frac{1}{8t^2} - \frac{1}{24t} + \frac{1}{24(t+4)} + \frac{1}{24(t+4)^2}\right)a_3\right\}, \tag{18}$$

$$a_3 = \frac{3}{2}\ln^2\left(\frac{t_2}{t_1}\right) + 2\,\mathrm{Li}_2\left(\frac{1}{t_2}\right) + \mathrm{Li}_2\left(\frac{t_1+t_2}{t_2^2}\right), \tag{19}$$

$$B(t) = (1/4 + 1/t)^{1/2}, \quad t_{1,2} = B(t) \mp 1/2. \tag{20}$$

Here $\mathrm{Li}_2(y)$ is Euler's dilogarithm

$$\mathrm{Li}_2(y) = -\int\limits_0^y \frac{dx}{x}\ln(1-x). \tag{21}$$

The integrand in Eq. (14) gives the distribution in k_0. The functions χ_2, ψ_1, ψ_2, ψ_3 depend only on the variable t. For $t \ll 1$ we find

$$\chi_2 = \frac{7}{18t} - \frac{1}{15}, \quad \psi_1 = -\frac{109}{108t} + \frac{71}{600}, \quad \psi_2 = -\frac{7}{18t^2} + \frac{4}{45t}, \quad \psi_3 = \frac{65}{54t^2} - \frac{1127}{5400t}. \tag{22}$$

Accordingly, for $t \ll 1$ the cross section satisfies $d\sigma \sim dt/t$. Small t means a large effective distance: $t^{-1/2} \gg m^{-1}$.

For $t \gg 1$ we have

$$\chi_2 = \left(\frac{2}{3t^2} - \frac{2}{t^3}\right)\ln t + \frac{1}{3t^2} + \frac{4}{3t^3},$$
$$\psi_1 = -(3\ln^2 t + \pi^2)\left(\frac{1}{6t^2} - \frac{1}{2t^3}\right) - \frac{2}{3}\left(\frac{1}{t^2} + \frac{1}{t^3}\right)\ln t - \frac{1}{3t^2} - \frac{2}{3t^3},$$
$$\psi_2 = -\frac{2}{3}\left(\frac{1}{t^3} + \frac{1}{t^4}\right)\ln t + \frac{1}{3t^3}, \tag{23}$$
$$\psi_3 = \left(\frac{1}{2}\ln^2 t + \frac{\pi^2}{6}\right)\left(\frac{1}{t^3} + \frac{1}{t^4}\right) + \left(\frac{2}{3t^3} + \frac{1}{3t^4}\right)\ln t - \frac{5}{6t^3} - \frac{4}{3t^4}.$$

The terms with the factor z in the integrand in Eq. (14) are important only for $z \sim t$ (i.e., for $k_0 \sim \gamma t^{1/2}$) and are small for $z \ll t$. For fixed t we have qualitatively $d\sigma \sim dk_0/k_0$ right up to $k_0 = k_{0\max}$.

Performing the elementary integration over k_0 in Eq. (14), we get

$$F_{\rm I} = \ln^2(2\gamma)\chi_2 + \ln(2\gamma)\chi_1 - \ln^2(2\bar{k}_0)\chi_2 - 2\ln(2\bar{k}_0)\psi_1 - (\bar{k}_0/\gamma)^2[\ln(2\bar{k}_0)\psi_2 + \psi_3] + \psi_4, \tag{24}$$

where

$$\chi_1 = \chi_2 \ln t + 2\psi_1 + t\psi_2 = -\frac{1}{12t} + \frac{1}{12(t+4)} + \frac{\ln t}{3t^2} + B(t)a_3\left[\frac{1}{3t^2} - \frac{5}{12t} + \frac{5}{12(t+4)}\right] + B(t)\ln\left(\frac{t_2}{t_1}\right)\left\{\left(-\frac{1}{3t^2} + \frac{5}{12t} - \frac{5}{12(t+4)}\right)\ln t - \frac{4}{3t^2} - \frac{3}{4t} + \frac{3}{4(t+4)} + \frac{1}{3(t+4)^2}\right\}$$

$$\psi_4 = {}^1/_4 \ln^2(t)\chi_2 + \ln(t)\psi_1 + {}^1/_2 t\ln(t)\psi_2 + t\psi_3.$$

In order to get rid of the parameter $\bar{k}_0$ in Eq. (24) we must go back to the expression (9) and integrate it over w, and then over k_0 from 2 to $\bar{k}_0$. It is obvious that for $\bar{k}_0 \gg 2$ we get

$$F_{\rm II} = \ln^2(2\bar{k}_0)\chi_2 + 2\ln(2\bar{k}_0)\psi_1 + (\bar{k}_0/\gamma)^2[\ln(2\bar{k}_0)\psi_2 + \psi_3] + \psi_5. \tag{25}$$

Accordingly $F_{\rm II}$ is determined by means of $F_{\rm I}$ up to an additive constant ψ_5, which is subject to calculation.

Replacing the function $F_{\rm I}$ in Eq. (13) with $F = F_{\rm I} + F_{\rm II}$, where

$$F = \ln^2(2\gamma)\chi_2 + \ln(2\gamma)\chi_1 + \chi, \quad \chi = \psi_4 + \psi_5, \tag{26}$$

we get the distribution over the square of the four-momentum transfer in the range (12) (where the function χ must depend only on the variable t):

$$\frac{d\sigma}{dt} = \frac{(2\alpha r_0 Z)^2}{\pi}\{\ln^2(2\gamma)\chi_2 + \ln(2\gamma)\chi_1 + \chi\}, \quad (\gamma^{-2} \ll t \ll M^2, \gamma^2). \tag{27}$$

In the reversed coordinate system in which the incident particle and the target are interchanged, the expression (27) is of the same form, and describes the distribution over the variable t for the target. As noted in the Introduction, for $t \ll M^2$ the pair gets only a small fraction of the energy, so that the distribution over t does not depend on the nature of the colliding particles.

One can tabulate $\chi(t)$ by starting directly from Eq. (9). The behavior of $\chi(t)$ for $t \ll 1$ and for sufficiently large t will be studied in the following sections. According to the results found there we have for $t \ll 1$

$$\chi_1 = \left(\frac{7}{18t} - \frac{1}{15}\right)\ln t - \frac{65}{27t} + \frac{293}{900}$$

$$\chi = \left(\frac{7}{72t} - \frac{1}{60}\right)\ln^2 t - \left(\frac{65}{54t} - \frac{293}{1800}\right)\ln t - \pi^2\left(\frac{7}{108t} - \frac{1}{90}\right) + \frac{259}{54t} - \frac{129}{250}; \tag{28}$$

for χ_2 see Eq. (22). The corresponding results for sufficiently large t ($t \gg 1$) are

$$\chi_1 = -\frac{1}{3t^2}[\ln^2 t + 5\ln t + \pi^2 + 1] + \frac{1}{t^3}\left[\ln^2 t - \frac{2}{3}\ln t + \pi^2 - \frac{4}{3}\right],$$
$$\chi = \frac{1}{t^2}\left[\frac{\ln^3 t}{18} + \frac{5\ln^2 t}{12} + \left(\frac{11}{6} + \frac{\pi^2}{9}\right)\ln t - \frac{7}{9} + \frac{7}{18}\pi^2 + \zeta(3)\right], \tag{29}$$

where $\zeta(3) = 1.202\ldots$; for χ_2 see Eq. (23). The function χ in Eq. (29) is given in lowest approximation; the calculation of the next term is difficult.

We now estimate qualitatively the probability of production of several pairs in the collision of two fast nuclei with charges eZ and eZ'. We use for this the approach of Landau and Lifshitz.[1] Let $w(r)$ be the probability of producing one pair in a collision between the nuclei with impact parameter r; assume $w(r) \ll 1$. Then, neglecting effects of the identical nature of the particles, we have for the cross section for production of n pairs[1)]

$$\sigma_n = \int_0^\infty w^n(r) 2\pi r\, dr. \tag{30}$$

It is now natural to suppose that, in the present situation with logarithmic formulas, $w(r) \sim r^{-2}$ for r in the range $1 \leqslant r \leqslant \gamma$. Otherwise we can assume that in order of magntiude $t = r^{-2}$. Then from Eqs. (26), (27) and (30) it follows that

$$w(t) = (2\alpha^2 ZZ'/\pi)^2 t^2 F. \tag{31}$$

Setting for a qualitative estimate [see Eqs. (26) and (27)]

$$t^2 F \approx \begin{cases} {}^7/_{18}\, t \ln^2 2\gamma, & t \leqslant 1 \\ 0, & t > 1 \end{cases},$$

we find

$$\frac{\sigma_2}{\sigma_1} \sim \left(\frac{\alpha^2 ZZ'}{\pi}\right)^2 \ln \gamma. \tag{32}$$

Similarly,

$$\frac{\sigma_{n+1}}{\sigma_n} \sim \left(\frac{\alpha^2 ZZ'}{\pi}\right)^2 \ln^2 \gamma, \quad n \geqslant 2. \tag{33}$$

The difference between the ratios (32) and (33) is due to the fact that in Eq. (30) $t_{\rm eff}^{-1/2} = r_{\rm eff} \sim 1$ for $n \geqslant 2$, i.e., large impact parameters are no longer effective for production of two or more pairs.

4. THE CASE OF SMALL MOMENTUM TRANSFERS

In this section we assume that $t \ll 1$ (i.e., $t \ll m^2$), $\gamma \gg 1$, and $\gamma t^{1/2}$ is unrestricted. We are not bound here by the limitation expressed by the left-hand part of the inequality (12). In the lowest approximation the quantity Φ in Eq. (9) (which describes the contribution of "scalar" photons[6)]) is to be omitted, and Γ is to be taken at $t = 0$, i.e., on the mass shell of the photon. We then have

$$P(k) = \frac{16}{k_0}\int_1^{k_0-1} dw\,\Gamma(t=0) = \frac{692+468k+76k^2+108k^3}{27(1+k)^3}K(k)$$
$$-\frac{692+360k+692k^2}{27(1+k)^3}E(k) - 4\left(\frac{1-k}{1+k}\right)^2 J_- + 16\left(\frac{1-k}{1+k}\right)^2 J_{--}, \tag{34}$$

where

$$J_- = \int_0^k \frac{dk}{1-k}K(k), \quad J_{--} = \int_0^k \frac{dk}{1-k^2}J_-, \quad k = \frac{k_0-2}{k_0+2},$$

in agreement with the expression obtained by Racah [see Eq. (10) in Ref. 3]. Here $K(k)$ and $E(k)$ are the elliptic integrals of the first and second kinds.

In this approximation Eq. (9) with $\Phi = 0$ and Eq. (34) describe the distribution in k_0 and t. Integrating this distribution over k_0 from 2 to $k_{0\max}$, we get for the distribution in t an analog of Eq. (13), in which F_I must be replaced with

$$F = \frac{1}{2t}P_1(k) - \frac{1}{\gamma^2 t^2}\left\{\frac{4}{27}\left[\frac{-5+189k+65k^2-81k^3}{(1-k)^2(1+k)}K(k)\right.\right.$$
$$\left.\left.+\frac{5-270k+5k^2}{(1-k)^2(1+k)}E(k) + 81J_-\right] - 8J_{--} + 32\int_0^k \frac{dk}{1-k^2}J_{--}\right\}, \tag{35}$$
$$k = (k_{0\max}-2)/(k_{0\max}+2).$$

Here $P_1(k)$ is defined by the expression

$$P_1(k) = \int_0^k \frac{dk}{1-k^2}P(k) = \frac{28}{9}J_- - 4\left(\frac{1-k}{1+k}\right)^2 J_{--}$$
$$-\frac{432+772k+456k^2+84k^3}{27(1+k)^3}K(k) + \frac{432+688k+432k^2}{27(1+k)^3}E(k).$$

As was to be expected, the function $P_1(k)$ agrees with that calculated by Racah [see Eq. (16a) in Ref. 3].

Setting $k = (\bar{k}_0 - 2)/(\bar{k}_0 + 2)$ in Eq. (35), we get (in the approximation under consideration) an analog of Eq. (25) which holds for all $\bar{k}_0 > 2$.

When we keep also the next term in the expansion, we get from Eqs. (9) and (2):

$$\int_1^{k_0-1} dw\left[\frac{3}{2k_0^4}\Phi + \frac{1}{t\eta^2}\Gamma\right] = \frac{1}{32t}\frac{1-k}{1+k}P(k) - \frac{3}{8}\left(\frac{1-k}{1+k}\right)^3 J_{--}$$
$$+\frac{9+14k+9k^2}{2^6(1+k)^6}(1-k)^3 J_-$$
$$-\frac{37841-7525k-28194k^2+20650k^3+17233k^4+6075k^5}{2^6\cdot 3^3\cdot 5^2(1+k)^6}(1-k)K(k)$$
$$+\frac{37841-13600k-7586k^2-13600k^3+37841k^4}{2^6\cdot 3^3\cdot 5^2(1+k)^6}(1-k)E(k),$$
$$\int_1^{k_0-1} dw\left[\frac{\Phi}{2\eta^2} + \frac{\Gamma}{t}\right] = \frac{1}{8t}\frac{1+k}{1-k}P(k) - \frac{1}{2}\left(\frac{1-k}{1+k}\right)^3 J_{--} + \frac{9+14k+9k^2}{2^4(1+k)^3}(1-k)J_-$$
$$-\frac{24353+9675k-7602k^2+13050k^3+15889k^4+6075k^5}{2^4\cdot 3^3\cdot 5^2(1+k)^4(1-k)}K(k)$$
$$+\frac{24353+3600k+862k^2+3600k^3+24353k^4}{2^4\cdot 3^3\cdot 5^2(1+k)^4(1-k)}E(k). \tag{36}$$

Equations (9) and (36) describe the distribution in k_0. In particular, for $k_0 \gg 2$ they lead to Eqs. (13) and (22), and for $k_0 - 2 \ll 1$ we get

$$\frac{d\sigma}{dt\,dk_0} = \frac{(\alpha r_0 Z)^2}{24}(k_0-2)^3\left[\frac{k_{0\max}^2 - k_0^2}{tk_{0\max}^2} + \frac{24-7k_{0\max}^2}{8k_{0\max}^2}\right]. \tag{37}$$

By integration over k_0 we can get from Eq. (9) and (36) a correction term for Eq. (35). We consider here only the limiting cases. For $k_{0\max} - 2 \ll 1$ we get by integrating (37) over k_0:

$$\frac{d\sigma}{dt} = \frac{(\alpha r_0 Z)^2}{96}(\gamma\sqrt{t}-2)^4\left[\frac{\gamma\sqrt{t}-2}{5t} - \frac{1}{8}\right]. \tag{37'}$$

In Eqs. (37) and (37′) we are assuming that the energy of the pair is not too small (that $k_0 - 2 \gg \alpha^2 Z^2$), so that the Born approximation can be used. (For more details about the conditions for its validity in this case see also Sec. 92 in Ref. 11.)

For $\gamma t^{1/2} \gg 2$ we get from Eqs. (9) and (36) the results of Eqs. (26) and (28). The asymptotic forms of J_-, J_{--}, ... for $k_0 \gg 1$ are given in Refs. 2 and 12.

5. THE CASE OF LARGE MOMENTUM TRANSFER

In this section we consider sufficiently large values of t/m^2 and derive the distribution in k_0 for fixed t; we also find the function $\chi(t)$ [see Eq. (27)].

The distribution over k_0 is simpler in form if we deal with various ranges of k_0 separately.

1) $2 \leqslant k_0 \ll t^{1/2}$. Here $pp'/t, p\eta/t, \ldots \ll 1$. Expanding A^{-1}, L, Λ, etc., in powers of the small parameter, we reduce the integrals over w to elliptic integrals. In particular, it is easily verified that in the expressions for

$$\int_1^{k_0-1} \Phi\, dw, \quad \int_1^{k_0-1} \Gamma\, dw \tag{38}$$

there is complete cancellation not only of the terms independent of t, but also of those $\propto t^{-1}$. To the accuracy with which we are concerned this region contributes nothing in the subsequent integration over k_0.

2) $2 \ll k_0 \ll t$. From the condition $w + w' = k_0 \gg 2$ it follows that the only possibilities for small particle energy are $w \sim 1$, or else $w' \sim 1$. We note that because Φ and Γ are symmetric in w and w' we can integrate only over the half-interval $1 \leqslant w \leqslant k_0/2$. In such integrals we can of course set $p' \to w'$. Although $w \sim 1$ gives contributions to some integrals over w (and can easily be taken into account if t is large enough), only values $w \gg 1$ are important in the expressions (38). That the contributions from $w \sim 1$ cancel out in these expressions can be seen from Eqs. (7) and (8), which for $\rho = ww' \ll t$ and $k_0 \ll t$ take the form

$$\begin{gathered}\Phi_0 = -\Phi_\Lambda \Lambda = -\Gamma_0 = \Gamma_\Lambda \Lambda = (k_0^2 - 2\rho)/2tpp', \\ -\Phi_L = \Phi_{L\Lambda}\Lambda = \Gamma_L = -\Gamma_{L\Lambda}\Lambda = p'/4\eta p^2.\end{gathered} \tag{39}$$

In the integral $\int^{k_0-1} LL'\, dw$ values $w \gg 1$ are important; more precisely, $w_{eff} \sim k_0 \gg 2$. For $p \approx w$ and $p' \approx w'$ we get from (4) and (5)

$$\begin{gathered}\Lambda \approx \frac{1}{2}\ln\left(1 + \frac{4\rho}{t}\right) = \frac{1}{2}\ln\left[\left(1 - \frac{w}{w_3}\right)\left(1 - \frac{w}{w_4}\right)\right], \\ L \approx \frac{1}{2}\left[\ln\left(1 - \frac{w}{w_3}\right) - \ln\left(1 - \frac{w}{w_4}\right)\right], \\ L' \approx \ln\frac{w_4}{-w_3} - L, \quad w_{3,4} = \frac{k_0 \mp \eta}{2}, \quad \eta^2 = k_0^2 + t.\end{gathered} \tag{40}$$

From (6) we have

$$A \approx \frac{t^2}{4} + \rho t = t(w - w_3)(w_4 - w), \quad 2 \ll k_0 \ll t. \tag{41}$$

The integrals over w can be expressed in terms of the Euler dilogarithm, Eq. (21); in the intermediate steps trilogarithms also occur (see the analogous integrals in Ref. 13). The results are

$$\begin{gathered}\int_1^{k_0-1} dw\Phi = -\frac{k_0^3}{2t^2} - \frac{4k_0}{3t} + \left(\frac{\eta^3}{6t^2} + \frac{11\eta}{12t} - \frac{5}{12\eta}\right)\ln\frac{w_4}{-w_3} \\ - \frac{k_0}{4t}\ln^2\frac{w_4}{-w_3} - \frac{1}{24\eta}\ln^3\frac{w_4}{-w_3} + \frac{\eta}{6t}[\ldots],\end{gathered} \tag{42}$$

$$\begin{gathered}\int_1^{k_0-1} dw\Gamma = \frac{13k_0}{12t} - \left(\frac{5\eta}{12t} + \frac{1}{8\eta}\right)\ln\frac{w_4}{-w_3} + \frac{1}{24\eta}\ln^3\frac{w_4}{-w_3} \\ + \left(\frac{\eta}{6t} - \frac{1}{3\eta}\right)[\ldots].\end{gathered} \tag{43}$$

Here

$$\begin{gathered}[\ldots] = \frac{1}{2}\ln\left(\frac{w_4}{-w_3}\right)\ln\left(\frac{4\eta^2}{t}\right) + \mathrm{Li}_2\left(\frac{-w_3}{\eta}\right) - \mathrm{Li}_2\left(\frac{w_4}{\eta}\right), \\ \ln\frac{w_4}{-w_3} = 2\,\mathrm{Arsh}\frac{k_0}{\sqrt{t}}.\end{gathered} \tag{44}$$

3) $k_0^2 \gg t$. Only the region $k_0 \sim t$ needs to be considered, since the case $k_0 \gg t$ is contained in Eqs. (13), (14), and (23). From Eq. (6) we have

$$\begin{gathered}A = t(w - w_1)(w_2 - w) \approx t(w - w_1)(k_0 - w), \\ w_{1,2} = k_0/2 \mp (k_0^2/4 + t/4 + 1 + k_0^2/t)^{1/2}, \quad -w_1 \approx t/4k_0 + k_0/t.\end{gathered} \tag{45}$$

For $k_0 \sim t$ we get

$$\int_1^{k_0-1} dw\Phi = \frac{tu^3}{24}\left[\ln u + \frac{1}{2}\ln t - \frac{3}{2}\right], \quad u = \frac{2k_0}{t}, \tag{46}$$

$$\begin{gathered}\int_1^{k_0-1} dw\Gamma = \frac{u\ln u}{12} + u\left(\frac{7}{24} - \frac{5}{24}\ln t - \frac{\pi^2}{36}\right) + \frac{u}{24}\ln^2(tu^2) \\ + u^3\left[\frac{\ln u}{3(u^2-1)^3} + \frac{\ln u}{6(u^2-1)^2} - \frac{1}{6(u^2-1)^2}\right] - \frac{u^3}{4}\frac{\ln u}{u^2-1} \\ + \left(\frac{u}{12} - \frac{1}{2u}\right)\mathrm{Li}_2(1-u^2) + \frac{\pi^2}{12u} + \frac{1}{2u}\mathrm{Li}_2(1-u) \\ - \frac{1}{2u}\mathrm{Li}_2\left(\frac{1}{1+u}\right) - \frac{1}{4u}\ln^2(1+u).\end{gathered} \tag{47}$$

Substitution of Eqs. (42), (43), and (46), (47) into Eq. (9) gives the distributions in k_0 and t in the corresponding ranges of k_0.

To get the distributions in t we must integrate over k_0. Integrating the expression (42) over k_0 with the weight η^{-4} and the limits 2 and k_0, where $t^{1/2} \ll k_0 \ll t$, we get

$$\begin{gathered}\int_2^{\tilde{k}_0} \frac{dk_0}{\eta^4}\int_1^{k_0-1} dw\Phi = \frac{1}{t^2}\left\{\frac{1}{6}\ln^2(2\tilde{k}_0) - \left(\frac{1}{2} + \frac{1}{6}\ln t\right)\ln(2\tilde{k}_0)\right. \\ \left. + \frac{1}{24}\ln^2 t + \frac{1}{4}\ln t + \frac{16}{9}\ln 2 - \frac{5}{18} - \frac{\pi^2}{36} - \frac{1}{4}\zeta(3)\right\}.\end{gathered} \tag{48}$$

It is easy to verify that this expression holds also for $k_0 \geqslant t$; in it we can set $\tilde{k}_0 = \gamma t^{1/2}$.

Similarly, we find from Eq. (43)

$$\begin{gathered}\frac{1}{t}\int_2^{\tilde{k}_0} \frac{dk_0}{\eta^2}\int_1^{k_0-1} dw\Gamma = \frac{1}{t^2}\left\{\frac{1}{72}\ln^3(\tilde{z}t) - \frac{5}{48}\ln^2\tilde{z}\right. \\ + \left(\frac{13}{24} - \frac{\pi^2}{72} - \frac{5}{24}\ln t\right)\ln\tilde{z} + \left(\frac{13}{24} - \frac{\pi^2}{72}\right)\ln t - \frac{5}{48}\ln^2 t \\ \left. - \frac{8}{3}\ln 2 + \frac{\pi^2}{18} + \frac{5}{8}\zeta(3)\right\}, \quad \tilde{z} = \left(\frac{2\tilde{k}_0}{t}\right)^2.\end{gathered} \tag{49}$$

The corresponding result from Eq. (47), taking $\tilde{z} \ll 1$, is

$$\frac{1}{t}\int\limits_{\tilde{k}_0}^{k_0}\frac{dk_0}{k_0^2}\int\limits_1^{k_0-1}dw\Gamma=\frac{1}{t^2}\Big\{-\frac{1}{72}\ln^3(\tilde{z}t)+\frac{5}{48}\ln^2\tilde{z}$$
$$-\left(\frac{13}{24}-\frac{\pi^2}{72}-\frac{5}{24}\ln t\right)\ln\tilde{z}+\frac{1}{72}[\ln^3(tu^2)-8\ln^3(u)]$$
$$-\frac{5}{12}\ln^2 u+\left(\frac{7}{12}-\frac{5}{12}\ln t-\frac{\pi^2}{12}\right)\ln u+\frac{1}{3}+\frac{1}{12(u^2-1)}$$
$$-\frac{\ln u}{6(u^2-1)}-\frac{\ln u}{6(u^2-1)^2}+\pi^2\left(\frac{1}{16}-\frac{1}{12u^2}\right)$$
$$+\left(\frac{1}{2u^2}-\frac{3}{8}\right)\mathrm{Li}_2(1-u^2)-\frac{\ln u}{6}\mathrm{Li}_2\left(\frac{1}{u^2}\right)-\frac{1}{6}\mathrm{Li}_3\left(\frac{1}{u^2}\right)$$
$$+\left(\frac{1}{2u^2}-\frac{1}{2}\right)\Big[\mathrm{Li}_2\left(\frac{1}{1+u}\right)-\mathrm{Li}_2(1-u)+\frac{1}{2}\ln^2(1+u)\Big]\Big\},$$
$$u=\frac{2k_0}{t}\qquad \tilde{z}=\left(\frac{2\tilde{k}_0}{t}\right)^2,\quad \mathrm{Li}_3(z)=\int\limits_0^z\frac{\mathrm{Li}_2(z)}{z}dz. \tag{50}$$

As was to be expected, when Eqs. (49) and (50) are combined the dependence on $\tilde{z}$ drops out.

For $k_0 \ll \gamma t^{1/2}$ the terms in Eq. (9) with the factor γ^{-2} can be neglected. Then Eqs. (9) and (48)–(50) give the integral distribution in k_0 for fixed t. For $\gamma \gg t^{1/2}$ these equations give (in lowest approximation) the result stated in Eqs. (26) and (29). Here the relation

$$-\frac{4}{\gamma^2 t}\int\limits_2^{\gamma\sqrt{t}}dk_0\int\limits_1^{k_0-1}dw\left[\frac{\Phi}{2k_0^2}+\frac{\Gamma}{t}\right]\approx\frac{1}{t^2}\Big\{\left(\frac{1}{3}-\frac{2}{3}\ln t\right)\ln(2\gamma)$$
$$+\frac{1}{6}\ln^2 t+\frac{5}{6}\ln t-\frac{5}{6}+\frac{\pi^2}{6}\Big\},$$

has been used for the terms $\sim\gamma^{-2}$ in Eq. (9); it follows from Eqs. (46) and (47) for $u_{eff} \gg 1$. As can be seen from Eq. (50), for $t^{1/2}$ the expression for $d\sigma/dt$ is rather cumbersome, and we shall not consider it.

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APPENDIX A

Some integrals encountered in the asymptotic formulas for small t

A remarkable result of Racah's work[2,3,9,12] is that the functions he gave are convenient for the subsequent integrations; they have simple asymptotic forms, do not involve elliptic integrals of the third kind, and occur in a number of different processes and distributions. The integrals occurring in our case for small momentum transfer can also be expressed in terms of Racah's functions. We shall point out some important properties of these integrals. The most surprising is that in the first and second approximations, which we consider here, only Racah function are encountered in the expressions for $d\sigma$.

We define the notations

$$J_n=\int\limits_1^{k_0-1}\frac{dw}{pp'}w^n,\quad I_n=\int\limits_1^{k_0-1}\frac{dw}{pp'}\rho^n,\quad \rho=ww'. \tag{A.1}$$

By a standard procedure we obtain[14]

$$J_0=\frac{2}{k_0}K(k_1)=(1-k)K(k),\quad k=\frac{k_0-2}{k_0+2},$$
$$k_1^2=1-k_1'^2,\quad k_1'=2/k_0. \tag{A.2}$$

The calculation of J_1 by the same method leads to the elliptic integral Π_1 of the third kind[14] (with a special ratio of the arguments). We note, however, that the change of variable of integration $w \to w'$ causes the interchange $w \longleftrightarrow w'$ in the integrand, and the limits are not changed. Accordingly, $J_1=k_0 J_0-J_1$, so that $J_1=(k_0/2)J_0$. Consequently,

$$\Pi_1(k'-1,k)=\frac{1+k'}{2k'}K(k),\quad k^2=1-k'^2. \tag{A.3}$$

The calculations are much more compact in terms of I_n (than in terms of J_n). Owing to this it is helpful to have the appropriate recurrence relations. Integrating the expressions for the derivative of $(k_0-2w)pp'\rho^n$ with $n=0, 1$, we get

$$I_2=\left(\frac{k_0^2}{2}-\frac{1}{3}\right)J_0+\left(\frac{k_0^2}{6}-\frac{4}{3}\right)I_1,$$
$$I_3=\left(\frac{8}{15}-\frac{23}{30}k_0^2\right)J_0+\left(\frac{23}{15}+\frac{11}{30}k_0^2+\frac{k_0^4}{30}\right)I_1. \tag{A.4}$$

For a complete list of the integrals encountered in the asymptotic formulas for small t see Ref. 15. It is interesting to note that the integrals

$$\int\limits_1^{k_0-1}\frac{l}{p'}\frac{1}{w\pm 1}dw.\qquad l=\mathrm{Arch}\,w,$$

disappear in the final expressions for the quantities (38).

APPENDIX B

Calculation of asymptotic forms of integrals over w for $k_0 \sim t \gg 1$

The derivation of this asymptotic formalism is the most complicated and laborious part of this work. We shall illustrate the method of calculation with a single integral as an example. For $k_0 \sim t \gg 1$ we have

$$A\approx t(w-w_1)(k_0-w),\quad -w_1\approx t/4k_0+k_0/t. \tag{B.1}$$

Let us consider the integral

$$\int\limits_1^{k_0-1}\frac{dw}{A}\Lambda=2\int\limits_1^{k_0/2}\frac{dw}{A}\Lambda=2\int\limits_1^{\bar{w}}\frac{dw}{A}\Lambda+2\int\limits_{\bar{w}}^{k_0/2}\frac{dw}{A}\Lambda. \tag{B.2}$$

We choose $\bar{w}$ so that

$$-w_1\ll\bar{w}\ll k_0. \tag{B.3}$$

Then in the last integral in (B.2) we can use the approximation (40), and we then find [see Eqs. (2) and (3) on page 266 of Ref. 13]

$$\int\limits_{\bar{w}}^{k_0/2}\frac{dw}{A}\Lambda\approx\frac{1}{4tk_0}\left[\ln^2\frac{4k_0^2}{t}-\ln^2\frac{4k_0\bar{w}}{t}-\frac{\pi^2}{6}\right]. \tag{B.4}$$

For the integral from 1 to $\bar{w}$ we have

$$\int\limits_1^{\bar{w}}\frac{dw}{A}\Lambda\approx\frac{1}{tk}\int\limits_1^{\bar{w}}\frac{dw}{w-w_1}\Lambda. \tag{B.5}$$

Making the replacement $k_0 \to k_0'$ in Λ, we have for the derivative with respect to k_0':

$$\frac{\partial}{\partial k_0'}\int\limits_1^{\bar{w}}\frac{dw}{w-w_1}\Lambda(k_0',w)\approx\frac{1}{2(k_0'-k_0)}\ln\frac{k_0}{k_0'}$$
$$-\frac{2k_0}{4k_0k_0'-t^2}\ln\frac{4k_0k_0'}{t^2}+\frac{1}{2k_0'}\ln\frac{4k_0'\bar{w}}{t}. \tag{B.6}$$

Here we have used the approximation (B.1) for w_1.

Finding the indefinite integral (with respect to k_0') of the right-hand side of (B.6), replacing k_0' with k_0, and requiring that for k_0 satisfying the inequality $t^{1/2} \ll k_0 \ll t$ this gives the correct expression for region 2) in Sec. 5, we find

$$\int_1^{\bar{w}} \frac{dw}{w-w_1} \Lambda(k_0, w) \approx \frac{1}{2} \mathrm{Li}_2(1-u^2) + \frac{1}{4} \ln^2(2u\bar{w}) - \frac{\pi^2}{12}, \qquad u = 2k_0/t. \tag{B.7}$$

Finally from Eqs. (B.2), (B.4), (B.5), and (B.7) we get

$$\int_1^{k_0-1} \frac{dw}{A} \Lambda \approx \frac{2}{t^2 u} \left[\mathrm{Li}_2(1-u^2) + \frac{1}{2} \ln^2(tu^2) - \frac{\pi^2}{3} \right]. \tag{B.8}$$

For the calculation of the other integrals over w see Ref. 15.

Note added in proof (23 January 1979). It can be seen from Eq. (45) that for $k_0^2 \gg t \gg m^2$ the important values of w (or w') are $\sim -w_1 \ll k_0$. Moreover, an examination of the electron propagator shows that a strongly virtual spacelike photon gives almost all its energy to the member of the pair that becomes free after interacting with it. The other particle of the pair, which becomes free after being scattered by the Coulomb field, carries only a small fraction of the energy.

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