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Low-frequency asymptotic form of the self-energy parts of a superfluid Bose system at $T=0$

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A functional integration method is used to obtain the first two terms of the asymptotic form of the Green's functions at $(\omega,\mathbf{k})=p\to 0$ and the principal asymptotic terms of the self-energy parts of three-dimensional and two-dimensional superfluid Bose systems at $T=0$. It is shown that the anomalous self-energy part tends to zero like $(\ln Rp)^{-1}$ for three-dimensional system and like p for the two-dimensional system.

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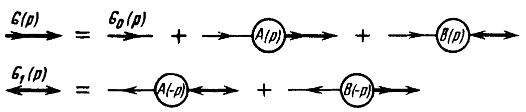
It was shown by A. A. and Yu. A. Nepomnyashchikh[1,2] that the anomalous self-energy part of a three-dimensional superfluid Bose system at $T=0$ is exactly equal to zero at $p=(\omega,\ \mathbf{k})=0$. This result is somewhat unexpected from the point of view of perturbation theory, where (for the Bose-gas model) the first-order approximation for the anomalous self-energy part is constant and differs from zero at small p.[3] The result of Refs. 1 and 2 indicates that the approach of Gavoret and Nozieres,[4] who assume a nonzero anomalous self-energy part at $p=0$, is incorrect.

In this paper we calculate the asymptotic forms of the self-energy parts of three-dimensional and two-dimensional Bose systems at $T=0$ with the aid of functional methods.[5] The obtained formulas (28) and (30) yield anomalous self-energy parts that vanish in the limit as $p\to 0$ in accord with Refs. 1 and 2.

We calculate first the asymptotic Green's functions, and obtain the self-energy parts from the Dyson–Belyaev equations:



(1)

Here $G(p)$ and $G_1(p)$ are the total normal and anomalous Green's functions, $A(p)$ and $B(p)$ are the normal and anomalous self-energy parts,

$$G_0(p)=(i\omega-k^2/2m+\mu)^{-1} \tag{2}$$

is the perturbed Green's function, and μ is the chemical potential.

The low-frequency asymptotic form of the Green's functions at $T=0$

$$G(p)\approx -G_1(p)\approx -m\rho_0/\rho p^2,\quad p^2=k^2+\omega^2c^{-2} \tag{3}$$

were first obtained by N. N. Bogolyubov.[6] We obtain here for the asymptotic Green's functions the terms of order higher than p^{-2}, which are needed to determine the asymptotic forms of the self-energy parts.

The normal and anomalous Green's functions of the Bose system are determined by the formulas

$$G(x,y)=-\langle\psi(x)\bar\psi(y)\rangle,\quad G_1(x,y)=-\langle\psi(x)\psi(y)\rangle, \tag{4}$$

where $x\equiv(\tau,\mathbf{x})$, $y\equiv(\tau',\mathbf{y})$, $\mathbf{x},\mathbf{y}\in V$ are the spatial variables, $\tau,\tau'\in[0,\beta]$, and $\beta^{-1}=T$ is the absolute temperature. The formulas for $T=0$ are obtained by taking the thermodynamic limit as $V\to\infty$ and $T\to 0$.

The averaging symbol $\langle\ldots\rangle$ in (4) can be understood as the quotient of the continual integrals

$$\langle A\rangle=\int Ae^S d\bar\psi d\psi \Big/ \int e^S d\bar\psi d\psi, \tag{5}$$

where S is the functional of the action:

$$S=\int d^4x\left(\bar\psi(x)\partial_\tau\psi(x)-\frac{1}{2m}\vec\partial\bar\psi(x)\vec\partial\psi(x)+\mu\bar\psi(x)\psi(x)\right)-\frac{1}{2}\int d\tau\, d^3x\, d^3y\, u(\mathbf{x}-\mathbf{y})\bar\psi(x)\bar\psi(y)\psi(y)\psi(x); \tag{6}$$

$u(\mathbf{x}-\mathbf{y})$ is the paired interaction potential of the Bose

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particles.

We use the method developed in Ref. 5, Secs. 18 and 19. Its idea consists of consecutively integrating first with respect to "fast" and then with respect to "slow" fields, using different perturbation-theory schemes during the different stages. This approach gets around the infrared divergences at $p\to 0$.

We define the "slow" field $\psi_0(x)$ as that part in the expansion

$$\psi(x)=\frac{1}{(\beta V)^{1/2}}\sum_{\omega,\mathbf{k}} e^{i(\omega\tau+\mathbf{k}\mathbf{x})}a(\omega,\mathbf{k}) \tag{7}$$

with k smaller than a certain k_0. The remaining part of the sum will be called the "fast" field $\psi_1(x)$. Thus,

$$\psi(x)=\psi_0(x)+\psi_1(x). \tag{8}$$

The order of magnitude of k_0 depends on the concrete Bose system. It was determined for a Bose gas in Ref. 5, Sec. 18.

After integration over the "fast" fields ψ_1 and $\bar\psi_1$ we arrive at the formulas

$$G(x,y)\approx-\langle\psi_0(x)\bar\psi_0(y)\rangle_0,\quad G_1(x,y)\approx-\langle\psi_0(x)\psi_0(y)\rangle_0, \tag{9}$$

where ψ_0 and $\bar\psi_0$ are the "slow" fields, $\langle\ldots\rangle_0$ denotes averaging over the "slow" fields with weight $\exp S_h$, where S_h is the functional of the "hydrodynamic" action, defined by

$$\exp S_h=\int\exp S\, d\bar\psi_1 d\psi_1 \tag{10}$$

and calculated in Ref. 5.

Since we are dealing with "slow" fields, it is convenient to change over to the variables density and phase (polar coordinates) in accordance with the formulas

$$\psi_0(x)=\rho^{1/2}(x)e^{i\varphi(x)},\quad \bar\psi_0(x)=\rho^{1/2}(x)e^{-i\varphi(x)}. \tag{11}$$

The expression for S_h will be taken in the quadratic form[5]

$$\int dx\left(-\frac{p_\mu}{2m}(\partial\varphi)^2-\frac{p_{\mu\mu}}{2}(\partial_\tau\varphi)^2+ip_{\mu\rho_0}\pi\partial_\tau\varphi+\frac{1}{2}p_{\rho_0\rho_0}\pi^2\right) \tag{12}$$

Here

$$\pi(x)=\rho(x)-\rho_0(k_0), \tag{13}$$

where $\rho_0(k_0)$ is determined from the condition $\partial p/\partial\rho_0=0$, with $p=S_h/\beta V$ calculated under the condition $\varphi(x)=0$, $\rho(x)=\rho_0=\text{const}$. The quantity $\rho_0(k_0)$ tends to the density ρ_0 of the condensate as $k_0\to 0$. The coefficients p_μ, $p_{\mu\mu}$, $p_{\mu\rho_0}$, $p_{\rho_0\rho_0}$ in (12) are the derivatives of $p=S_h/\beta V$ with respect to the variables μ and ρ_0.

We express the Green's functions in the form

$$\begin{aligned}G(x,y)&=-\langle[\rho(x)\rho(y)]^{1/2}\exp\{i(\varphi(x)-\varphi(y))\}\rangle_0,\\ G_1(x,y)&=-\langle[\rho(x)\rho(y)]^{1/2}\exp\{i(\varphi(x)+\varphi(y))\}\rangle_0.\end{aligned} \tag{14}$$

Putting $\rho(x)=\rho_0(k_0)+\pi(x)$, $\rho(y)=\rho_0(k_0)+\pi(y)$ and expanding in powers of $\pi(x)$, $\pi(y)$, $\varphi(x)\pm\varphi(y)$, we obtain the equations

$$\begin{aligned}G(x,y)&=-\rho_0-\rho_0\langle\varphi(x)\varphi(y)\rangle_0+\frac{i}{2}(\langle\pi(x)\varphi(y)\rangle_0\\&-\langle\pi(y)\varphi(x)\rangle_0)-\frac{1}{4\rho_0}\langle\pi(x)\pi(y)\rangle_0-\frac{\rho_0}{2}\langle\varphi(x)\varphi(y)\rangle_0^2+\ldots,\\ G_1(x,y)&=-\rho_0+\rho_0\langle\varphi(x)\varphi(y)\rangle_0-\frac{i}{2}(\langle\pi(x)\varphi(y)\rangle_0+\langle\pi(y)\varphi(x)\rangle_0)\\&-\frac{1}{4\rho_0}\langle\pi(x)\pi(y)\rangle_0-\frac{\rho_0}{2}\langle\varphi(x)\varphi(y)\rangle_0^2+\ldots,\end{aligned} \tag{15}$$

in which only the terms that are significant as $|x-y|\to\infty$ are indicated. Taking the Fourier transforms

$$\begin{aligned}G(p)&=\int e^{-ip(x-y)}G(x-y)d(x-y),\\ G_1(p)&=\int e^{-ip(x-y)}G_1(x-y)d(x-y),\end{aligned} \tag{16}$$

we obtain for small $p\neq 0$:

$$\begin{aligned}G(p)&=-\rho_0 g_{\varphi\varphi}(p)-ig_{\varphi\pi}(p)-\frac{1}{4\rho_0}g_{\pi\pi}(p)-\frac{\rho_0}{2}g_{\varphi\varphi}*g_{\varphi\varphi}(p)+\ldots,\\ G_1(p)&=\rho_0 g_{\varphi\varphi}(p)-\frac{1}{4\rho_0}g_{\pi\pi}(p)-\frac{\rho_0}{2}g_{\varphi\varphi}*g_{\varphi\varphi}(p)+\ldots\end{aligned} \tag{17}$$

The Green's functions $g_{\varphi\varphi}$, $g_{\varphi\pi}$, $g_{\pi\pi}$ of the slow fields were calculated in Ref. 5, Sec. 19. They are given by

$$\begin{aligned}&g_{\varphi\varphi}(p)=-p_{\rho_0\rho_0}Z^{-1},\quad g_{\varphi\pi}(p)=-g_{\pi\varphi}(p)=p_{\mu\rho_0}\omega Z^{-1},\\ &g_{\pi\pi}(p)=\left(\frac{p_\mu}{m}k^2+p_{\mu\mu}\omega^2\right)Z^{-1},\quad Z=p_{\mu\rho_0}^2\omega^2-p_{\rho_0\rho_0}\left(\frac{p_\mu}{m}k^2+p_{\mu\mu}\omega^2\right).\end{aligned} \tag{18}$$

In particular, the function $g_{\varphi\varphi}(p)$ can be written in the form

$$g_{\varphi\varphi}(p)=\frac{m}{\rho(k^2+\omega^2/c^2)}=\frac{m}{\rho p^2}, \tag{19}$$

if we use the formulas (Ref. 5, Sec. 19)

$$p_\mu=\rho,\quad p_{\mu\mu}-p_{\mu\rho_0}^2/p_{\rho_0\rho_0}=\rho/mc^2, \tag{20}$$

where ρ is the total density of the system and c^2 is the square of the speed of sound. The convolution

$$g_{\varphi\varphi}*g_{\varphi\varphi}(p)=\frac{1}{(2\pi)^4}\int_{|q|<k_0}d^4q\, g_{\varphi\varphi}(q)g_{\varphi\varphi}(p-q) \tag{21}$$

takes upon substitution of (19) the following asymptotic form as $p\to 0$:

$$g_{\varphi\varphi}*g_{\varphi\varphi}(p)=\frac{m^2c}{8\pi^2\rho^2}\ln\frac{2k_0}{p}. \tag{22}$$

From (17)–(19) and (22) we easily obtain the asymptotic forms of the self-energy parts. Solving the Dyson–Belyaev equations (1) with respect to the self-energy parts $A(p)$ and $B(p)$ we obtain

$$\begin{aligned}A(p)&=G_0^{-1}(p)-\frac{G(-p)}{G(p)G(-p)-G_1^2(p)},\\ B(p)&=\frac{G_1(p)}{G(p)G(-p)-G_1^2(p)}.\end{aligned} \tag{23}$$

The denominator in (23) can be simply expressed in terms of the functions g:

$$G(p)G(-p)-G_1^2(p)=g_{\varphi\pi}^2+g_{\varphi\varphi}g_{\pi\pi}+2\rho_0^2g_{\varphi\varphi}(g_{\varphi\varphi}*g_{\varphi\varphi}), \tag{24}$$

and for the numerators we can confine ourselves to the first terms in the right-hand side of (17) ($\pm\rho_0 g_{\varphi\varphi}$). The result is

$$\begin{aligned}B(p)&\approx A(p)-G_0^{-1}(p)\approx\\ &\approx\rho_0 g_{\varphi\varphi}[g_{\varphi\pi}^2+g_{\varphi\varphi}g_{\pi\pi}+2\rho_0^2g_{\varphi\varphi}(g_{\varphi\varphi}*g_{\varphi\varphi})]^{-1}\\ &=\rho_0[g_{\pi\pi}+g_{\varphi\pi}^2g_{\varphi\varphi}^{-1}+2\rho_0^2(g_{\varphi\varphi}*g_{\varphi\varphi})]^{-1}.\end{aligned} \tag{25}$$

By virtue of (18) we have

$$g_{\pi\pi}+g_{\varphi\pi}^2g_{\varphi\varphi}^{-1}=-1/p_{\rho_0\rho_0}. \tag{26}$$

Substituting (22) and (26) in (25) we get

$$B(p)\approx A(p)-G_0^{-1}(p)\approx\rho_0\left[\frac{m^2c\rho_0^2}{4\pi^2\rho^2}\ln\frac{2k_0}{p}-\frac{1}{p_{\rho_0\rho_0}}\right]^{-1}. \tag{27}$$

The dependence on the auxiliary parameter k_0 should drop out of the final answer, and therefore the quantity $p^{-1}_{\rho_0\rho_0}$ itself should depend logarithmically on k_0. This reasoning allows us to rewrite (27) in the form

$$B(p)\approx A(p)-G_0^{-1}(p)\approx b/\ln(1/Rp),\quad b=4\pi^2\rho^2/m^2c\rho_0, \tag{28}$$

where R is a constant with the dimension of the length. We have obtained the sought asymptotic formulas for the self-energy parts of a three-dimensional Bose system at $T=0$. The anomalous self-energy part $B(p)$ tends to zero as $p\to 0$ in accordance with Refs. 1 and 2.

Carrying out analogous calculations for a two-dimensional Bose system at $T=0$, we obtain asymptotic formulas for the Green's functions:

$$G(p)=-\frac{m\rho_0}{\rho p^2}-\frac{m^2c\rho_0}{32\rho^2 p}+o\left(\frac{1}{p}\right),$$
$$G_1(p)=\frac{m\rho_0}{\rho p^2}-\frac{m^2c\rho_0}{32\rho^2 p}+o\left(\frac{1}{p}\right) \tag{29}$$

and for the self-energy parts we get

$$B(p)\approx A(p)-G_0^{-1}(p)=8\rho^2 p/m^2c\rho_0+o(p). \tag{30}$$

The anomalous self-energy part tends here also to zero as $p\to 0$, and furthermore more rapidly ($\propto p$) than in the three-dimensional case, where it decreases in proportion to $\ln^{-1}(1/Rp)$.

APPENDIX

We trace the cancellation of the dependence on k_0 in (27) for a model of a low-density Bose gas. We use the formula

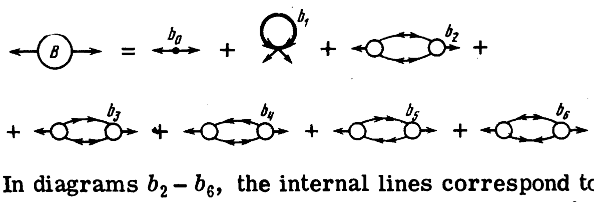
$$p_{\rho_0\rho_0}=-\frac{1}{\rho_0(k_0)}B_{k_0}(0). \tag{A.1}$$

It expresses $p_{\rho_0\rho_0}$ in terms of $\rho_0(k_0)$ and the self-energy part $B_{k0}(0)$ calculated during the course of integration with respect to the "fast" fields by perturbation theory, in which the integrals with respect to the momenta are cut off at the lower limit k_0. At $k_0=0$, Eq. (A.1) goes over into $p_{\rho_0\rho_0}=-\rho_0^{-1}B(0)$ and is given in this form in Ref. 5, [(16.23)]. The formula is valid, however, also at $k_0 \neq 0$.

We subject k_0 to the condition

$$mc\exp(-\theta^{-3/2})\ll k_0\ll mc, \tag{A.2}$$

where θ is the gas parameter ($\theta\sim a\rho^{1/3}$, a is the effective interaction radius of the Bose particles, and ρ is the density). In this case $\rho_0(k_0)$ differs little from the condensate density ρ_0, and the main contribution to $B_{k9}(0)$ is made by the sum of diagrams



(A.3)

In diagrams b_2-b_6, the internal lines correspond to the first-approximation normal and anomalous Green's functions

$$G(p)=-\frac{i\omega+k^2/2m+mc^2}{\omega^2+\varepsilon^2(\mathbf{k})},\quad G_1(p)=\frac{mc^2}{\omega^2+\varepsilon^2(\mathbf{k})},$$
$$\varepsilon^2(\mathbf{k})=k^4/4m^2+c^2k^2. \tag{A.4}$$

The circles with three arrows in (A.3) correspond to the constant factors $2\rho_0^{1/2}t_0$, where ρ_0 is the density of the condensate, t_0 is the value of the paired t matrix at zero energy and zero momenta. The thick line of diagram b_1 corresponds to the total anomalous Green's function so that (A.3) is an equation. The calculation of b_0+b_1 is explained in Sec. 17 of Ref. 5. At $T=0$, the analog of formula (17.33) of Ref. 5 can be written in the form

$$b_0+b_1=t_0\rho_0+\frac{t_0^2\rho_0}{2(2\pi)^3}\int d^3k\left(\left(\frac{k^2}{2m}\right)^{-1}-\frac{1}{\varepsilon(\mathbf{k})}\right)$$
$$=t_0\rho_0+\frac{t_0^2\rho_0 m^2c}{\pi^2}. \tag{A.5}$$

The integral in this formula does not diverge logarithmically at the lower limit, so that we can assume that it is taken over all $\mathbf{k}$, and not only over the region $|\mathbf{k}|>k_0$. The contribution of diagrams b_2-b_6 at $p=0$ is given by

$$\sum_{i=2}^{6}b_i=-\frac{2\rho_0t_0^2}{(2\pi)^4}\int_{|\mathbf{q}|>k_0}d^4q[3G_1^2(q)+2G^2(q)+4G(q)G_1(q)]$$
$$=\frac{\rho_0t_0^2mc}{4\pi^2}\left(9-\ln\frac{4mc}{k_0}\right), \tag{A.6}$$

which shows the logarithmic dependence on k_0. Adding with the formula (A.5) for b_0+b_1 and dividing by $(-\rho_0)$, we obtain

$$p_{\rho_0\rho_0}=-t_0\left(1-\frac{t_0m^2c}{4\pi^2}\left(\ln\frac{4mc}{k_0}-13\right)\right),$$
$$p^{-1}_{\rho_0\rho_0}=-t_0^{-1}-\frac{m^2c}{4\pi^2}\left(\ln\frac{4mc}{k_0}-13\right). \tag{A.7}$$

When the left-hand inequality of (A.2) is satisfied, the first term $(-t_0)^{-1}$ in (A.7) is the principal one, and the second gives a small correction. The coefficient preceding the logarithm in (A.7) was calculated in first-order approximation. In this approximation $\rho\approx\rho_0$, and we can rewrite (A.7) in the form

$$p^{-1}_{\rho_0\rho_0}=-t_0^{-1}-\frac{m^2c\rho_0^2}{4\pi^2\rho^2}\left(\ln\frac{4mc}{k_0}-13\right). \tag{A.8}$$

Substituting (A.8) in (27) we verify that k_0 cancels out. We obtain Eq. (28), where

$$\ln\frac{1}{Rp}=\ln\frac{8mc}{p}-13+\frac{4\pi^2\rho^2}{m^2c\rho_0^2t_0}, \tag{A.9}$$

so that

$$R=(8mc)^{-1}\exp(13-4\pi^2\rho^2/m^2c\rho_0^2t_0). \tag{A.10}$$

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