Spin-wave propagation in spatially disordered media

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(Submitted 18 August 1979)
Zh. Eksp. Teor. Fiz. **78**, 1867–1878 (May 1980)

A connection is established between the hydrodynamic approach and the phenomenological Lagrangian of spin waves in spatially disordered media. The hydrodynamic approach is generalized with allowance for the nonlinear interactions of the magnons. Account is taken of the relativistic corrections for the new possible phases due to total spontaneous breaking of the spin-rotation group.

PACS numbers: 75.30.Ds, 47.35. + i

The study of spin waves in spatially disordered magnetic media has received much attention of late.[1,2] Since spin waves are Goldstone excitations due to spontaneous breaking of symmetry with respect to spin rotations, it is possible to apply to them the general group-theoretical method of describing the dynamics of Goldstone excitations, the so called method of phenomenological Lagrangians (Refs. 3–5),[1)] which has been extensively used recently in elementary-particle physics.

The method of phenomenological Lagrangians is based on the remark that spontaneous symmetry breaking is in fact not a breaking of the symmetry, but its restructuring in such a way that the dynamic variables that transform in accordance with the linear representations of the corresponding group are replaced by some new dynamic variables with a nonlinear transformation law (the so called nonlinear realizations). The dynamic variables of the latter type are defined by a Lagrange function that is invariant to the transformations of the considered symmetry group. In the long-wave limit, the invariance requirement determines the Lagrange function of Goldstone fields accurate to a certain number of phenomenological constants. Depending on the subgroup of the invariance of the ground state, the excitation spectrum has one, two, or three Goldstone spin-wave branches. For the first two cases the subgroup of the invariance of the ground state is the group $O(2)$ and these cases correspond to ferromagnetic and antiferromagnetic media; in the case of antiferromagnetic media, the ground state is invariant also with respect to time reflection. In both considered cases small deviations of the system from the ground state are described by two angles [the coordinates of the homogeneous factor space $O(3)/O(2)$], which correspond to one or two branches of spin excitations, depending on whether the equations of motion contain derivatives of first or second order with respect to time.

The three spin-wave branches appear in the case of maximum breaking of the symmetry of the ground state, when the subgroup of its invariance is only the identity transformation. In this case deviations from the ground state are described by three angles [the parameters of the group $O(3)$], the dependence of which on the spatial coordinates and on the time determines in fact the character of the spin excitations.

The dispersion spectra of spin waves and of their interactions were considered in detail in Ref. 7 for the three indicated cases, for both spatially homogeneous and spatially inhomogeneous media.

The conclusion that total spontaneous symmetry breaking of spin rotations is possible and leads to the presence of three branches of spin waves, first formulated in Ref. 7 for the case of spatially disordered systems, was recently also proposed in an interesting paper by Halperin and Saslow[1] on the basis of the hydrodynamic approach used by them. They, however, took no account of the nonlinear effects, and the analysis pertained only to the particular case of a degenerate spin-wave spectrum. An analysis of spin waves for the case of a spatially disordered medium, similar to the analysis of Ref. 7, was recently presented also by Andreev,[2] who made, in particular, an interesting attempt to establish the form of the relativistic interactions.

The purpose of the present paper is, first, to establish a connection between the phenomenological method of Ref. 7, on the one hand, and the hydrodynamic approach of Ref. 1, on the other, and to generalize the latter to take into account all the possible nonlinear interactions and forms of the magnon spectrum. The second purpose is further development of the approach proposed in Ref. 7 to include allowance for relativistic interactions and for interaction with an external magnetic field.

We propose that, in contrast to the case of spatially ordered media, for which relativistic interactions contain terms both with and without gradients of the magnon fields, in the case of a spatially disordered medium there are possible phases due to the total spontaneous breaking of the group of spin rotations, for which allowance for the relativistic effects involves only the field gradients. As a result, the relativistic corrections for the considered phases do not change the Goldstone character of the spectrum and lead to a correlation between the polarization properties and the propagation direction of the spin waves, and to the appearance of spatial anisotropy.

1. The interaction between electron spins in magnetic media, neglecting relativistic effects, is invariant to simultaneous rotation of all the spins of the system—the transformations that make up the $O(3)$ symmetry group. In the case when some of the operators with spin indices averaged over the ground state of the system differ from zero and transform in accordance with a non-identical representation under transformations of

 0038-5646/80/050937-06$02.40

the spin-rotation group, spontaneous symmetry breaking takes place in the system. In states close to the ground state, local mean values in different spatial and temporal points differ in this case from one another by a certain rotation that belongs to the considered group of transformations. The wave processes that result from the indicated local differences manifest themselves as certain elementary excitations—spin waves having, depending on the structure of the medium, definite dispersion properties and a definite character of the interaction.[8,9]

In the long-wave limit, the properties of the spin waves, for the case of arbitrary magnetic media, can be described phenomenologically using only the symmetry requirement.[7] To clarify the relations that arise when the method of phenomenological Lagrangians is applied to magnetic media, we consider first the classical example of spontaneous breaking of $O(3)$ symmetry, namely mechanical rotation of an arbitrary solid. It is known that all the interactions between the individual components of the solid as a mechanical system remain invariant to spatial rotation. The formation, in the general case, of an asymmetrical solid, as a result of such interactions is spontaneous symmetry breaking. Just as in the general case of spontaneous symmetry breaking, in this case the spontaneous symmetry breaking does not mean vanishing of the symmetry. The symmetry with respect to spatial rotation manifests itself in the equivalence of different positions of the solid at rest, and also in the invariance of the equations of motion in the case of rotation of the solid. The requirement that the equations of motion be invariant to rotation and time reversal, together with the additional restriction that the degree of the derivatives of the Euler angles in the Langrangian not exceed two, leads to the well known Lagrange function[2)]

$$\mathscr{L}={}^1/_2 a_{\alpha\beta}\omega^\alpha(\theta_\lambda,\dot\theta_\lambda)\omega^\beta(\theta_\lambda,\dot\theta_\lambda), \tag{1}$$

where $a_{\alpha\beta}$ is the moment-of-inertia tensor, and $\omega^\alpha(\theta_\lambda,\dot\theta_\lambda)$ are the components of the vector of the angular velocity in the coordinate system tied to the rotating body; these components are represented in the form of functions of the Euler angles θ_λ and their derivatives with respect to time:

$$\begin{aligned}\omega^1(\theta_\lambda,\dot\theta_\lambda)&=\dot\theta_3\sin\theta_1\sin\theta_2+\dot\theta_1\cos\theta_2,\\ \omega^2(\theta_\lambda,\dot\theta_\lambda)&=\dot\theta_3\sin\theta_1\cos\theta_2-\dot\theta_1\sin\theta_2,\\ \omega^3(\theta_\lambda,\dot\theta_\lambda)&=\dot\theta_3\cos\theta_1+\dot\theta_2.\end{aligned} \tag{2}$$

The Euler angles $\theta_\lambda(t)$ $(\lambda=1,2,3)$ fix the position of the solid in space at a given instant of time. When the solid rotates, the Euler angles change. However, since the components $\omega^\alpha(\theta,\dot\theta)$ are considered in a rotating coordinate system, and consequently are invariants, the Lagrange function (1) is also invariant with respect to the rotations. As a result of the fact that the invariants (2) form a complete set of invariants, the form of the Lagrange function (1) follows uniquely from the symmetry requirements proposed above.

In the absence of invariance to time reversal, it is possible to add to the Lagrange function (1) terms that are linear in $\omega^\alpha(\theta_\lambda,\dot\theta_\lambda)$:

$$\mathscr{L}={}^1/_2 a_{\alpha\beta}\omega^\alpha(\theta_\lambda,\dot\theta_\lambda)\omega^\beta(\theta_\lambda,\dot\theta_\lambda)+b_\alpha\omega^\alpha(\theta_\lambda,\dot\theta_\lambda). \tag{3}$$

The Lagrange function (3) describes the so called magnetic top—a charged solid with an intrinsic magnetic moment that is rigidly fixed in the body.[12] Thus, the invariance requirements lead to a definition of the Lagrange function of the rotating body, accurate to phenomenological constants that have the meaning of the tensor of the moment of inertia and of the intrinsic magnetic moment.

To proceed to the consideration of spin waves, we generalize the Lagrangian (3) to the case of interacting tops that are uniformly distributed in space. To construct the invariant Lagrange in this case it suffices to add to (3) invariant terms that depend on the gradients of the Euler angles. The simplest invariants containing gradients are of the form

$$\omega^\alpha(\theta_\lambda,\nabla_i\theta_\lambda)D_{i\beta}(\theta_\lambda), \tag{4}$$

were $\omega^\alpha(\theta_\lambda,\nabla_i\theta_\lambda)$ are the invariant forms (2) in which the derivatives of the Euler angles with respect to time are replaced by their gradients, and $D_{i\beta}(\theta_\lambda)$ are known Euler functions, which transform the components of a vector from an immobile basis to a mobile one. Expression (4) has the meaning of relative rotation of the tops in neighboring points of space. The invariance of expression (4) relative to rotations is ensured by the fact that both the relative rotation and the relative displacement of the tops are expressed in a moving basis. Since rotation is an axial vector and the displacement is a vector, expression (4) reverses sign when the spatial coordinates are reversed. Taking the foregoing into account, we obtain in the long-wave approximation the following expression for the Lagrangian of interacting tops, which is at the same time the phenomenological Lagrangian of the Goldstone excitations that appear upon spontaneous breaking of the group of spatial rotations (without upsetting the translational invariance):

$$\begin{aligned}\mathscr{L}=&{}^1/_2 a_{\alpha\beta}\omega^\alpha(\theta_\lambda,\dot\theta_\lambda)\omega^\beta(\theta_\lambda,\dot\theta_\lambda)\\&+b_\alpha\omega^\alpha(\theta_\lambda,\dot\theta_\lambda)-{}^1/_2 d_{\alpha\beta;\gamma\delta}\omega^\alpha(\theta_\lambda,\nabla_i\theta_\lambda)\\&\times\omega^\beta(\theta_\lambda,\nabla_k\theta_\lambda)D_{i\gamma}(\theta_\lambda)D_{k\delta}(\theta_\lambda).\end{aligned} \tag{5}$$

The Lagrangian (5) can be used, in particular, to describe spin waves. In this case, when account is taken of only the exchange interactions, there is additional symmetry with respect to rotations only in spin space.

In the exchange-interaction approximation we have

$$d_{\alpha\beta;\gamma\delta}=c_{\alpha\beta}\delta_{\gamma\delta}, \tag{6}$$

which leads to the expression[7]

$$\begin{aligned}\mathscr{L}=&{}^1/_2 a_{\alpha\beta}\omega^\alpha(\theta,\dot\theta)\omega^\beta(\theta,\dot\theta)+b_\alpha\omega^\alpha(\theta,\dot\theta)\\&-{}^1/_2 c_{\alpha\beta}\omega^\alpha(\theta,\nabla_i\theta)\omega^\beta(\theta,\nabla_i\theta),\end{aligned} \tag{7}$$

in which the Euler angles determine the orientation of the basis in spin space, and the gradients form a scalar product and are not affected by the spin-rotation transformations. The terms of (5) which are not included in (7) are in this case small and describe the relativistic spin-orbit interaction. Some possible restrictions on the choice of the constants $d_{\alpha\beta;\gamma\delta}$ will be discussed later on. In the Lagrangian (7), or in the Lagrangian (5) if relativistic effects are taken into account, the Euler angles have a continuous dependence on the coordinates and the time, and are semiclassical magnon fields. When the Lagrangian is expanded in powers of these fields[3], kinetic terms (quadratic in the fields) and interaction terms are separated.

The kinetic terms determine the spectrum of the magnons. The interaction between an arbitrary number of magnons is determined by the terms of the interaction (7) via allowance for all the possible tree diagrams. The magnon spectrum and the matrix elements of the interaction on the mass shell determined on the basis of the Lagrangian (7) [or (5)] (i.e., under the condition that the energy and momentum of the magnons satisfy the spectral relations), do not depend in this case on the concrete parametrization of the rotation group.[3-5] The parametrization used above is convenient for consideration of the analogy with the rotation of a solid. When spin waves are considered, it is convenient to use other parametrizations, in which the group parameters enter in a more symmetrical fashion.

2. To change over to other possible parametrizations of the spin-rotation group, we introduce, for the description of the spin excitations, a local orthonormal frame $\mathbf{e}_\alpha(x,t)$ ($\alpha=1,2,3$ is the number of the unit vector):

$$\mathbf{e}_\alpha \mathbf{e}_\beta = \delta_{\alpha\beta}. \tag{8}$$

The orthonormality conditions (8) are preserved for different types of transformations:

$$\mathbf{e}_\alpha'(x,t) = \mathbf{e}_\alpha(x,t) + [\varepsilon_L \mathbf{e}_\alpha(x,t)], \tag{9}$$

$$e_{a\alpha}''(x,t) = e_{a\alpha}(x,t) - \varepsilon_{\alpha\beta\gamma}\varepsilon_R^\beta e_{a\gamma}(x,t) \tag{10}$$

with infinitesimally small transformation parameters ε_L^α and ε_R^α that are independent of the coordinates x_i and of the time t. The transformation (9) effects simultaneous rotation of all the reference frames and consequently of the entire system as a whole in immobile spin space. Neglecting relativistic interactions, the description of the spin waves should be invariant to such transformations. The transformation (10) means a transition to another orthonormalized frame. Since the orthonormalized frame is assumed to be rigidly connected with the parameters of the spontaneous symmetry breaking, the initial and transformed frames in (10) are never equivalent for the case of complete spontaneous symmetry breaking.[4] Relations (8) are the constraints on the components of the reference frame $\mathbf{e}_\alpha(x,t)$. These constraints can be lifted by substituting the unit vectors of the reference frame in the form of functions of three independent parameters

$$e_{a\alpha}(x,t) = D_{a\alpha}(\varphi_\lambda(x,t)). \tag{11}$$

This relation defines a transition to new variables, the transformation law for which, as a result of the nonlinearity of the constraints, is nonlinear and is determined from (9) and (11):

$$\mathbf{D}(\varphi(x,t)) + [\varepsilon_L \mathbf{D}_\alpha(\varphi(x,t))] = \mathbf{D}_\alpha(\varphi(x,t) + \delta_{\varepsilon_L}\varphi(x,t)). \tag{12}$$

To determine the law governing the transformation of the variables $\varphi_\lambda(x,t)$ we use, for example, the so called linear rational parametrization of the orthogonal matrices $D_{a\alpha}$(11), i.e.,

$$D = (1+T)/(1-T), \tag{13}$$

where T is a real antisymmetrical matrix $T_{a\alpha} = \frac{1}{2}\varepsilon_{a\lambda\alpha}\varphi_\lambda$. Expanding $(1-T)^{-1}$ in powers of T and recognizing that $T^3 = \frac{1}{2} T \,\mathrm{Sp}\, T^2 = -\frac{1}{4} T\varphi^2$, we obtain

$$D_{a\alpha}(\varphi_\lambda) = [1 + 2(T+T^2)/(1 - {}^1/_2 \,\mathrm{Sp}\, T^2)]_{a\alpha} = \delta_{a\alpha} + [\varphi_a\varphi_\alpha - \delta_{a\alpha}\varphi^2 + 2\varepsilon_{a\lambda\alpha}\varphi_\lambda]/2(1+\varphi^2/4). \tag{14}$$

Equations (12) and (14) lead to the following transformation law for the variables:

$$\delta_{\varepsilon_L}\varphi_\lambda = \varepsilon_L^\lambda + {}^1/_2 \varepsilon_{\lambda\beta\gamma}\varepsilon_L^\beta\varphi_\gamma + {}^1/_4(\varepsilon_L\varphi)\varphi_\lambda. \tag{15}$$

We note that Eq. (15) corresponds to a definite parametrization of a vector representation ($S=1$) of the rotation group. The transformation law (15) derived from it determines uniquely the parametrization of the other representations. In particular, the spinor representation ($S=1/2$) takes the form

$$g(\varphi) = [(1 - i\varphi_\lambda\sigma_\lambda/2)/(1 + i\varphi_\lambda\sigma_\lambda/2)]^{1/2}, \tag{16}$$

where σ_λ are Pauli matrices. Equation (12) with the generator $\varepsilon_{a\alpha}^\lambda$ replaced by the matrix $-iS_\lambda$, as well as the succeeding formulas (19)–(20), does not depend on the representation employed. To calculate the explicit form of the functions (15), the spinor representation (16) turns out to be more convenient. It can similarly be shown that the transformation (10) corresponds to the following law of parameter transformation:

$$\delta_{\varepsilon_R}\varphi_\lambda = \varepsilon_R^\lambda - {}^1/_2 \varepsilon_{\lambda\beta\gamma}\varepsilon_R^\beta\varphi_\gamma + {}^1/_4(\varepsilon_R\varphi)\varphi_\lambda. \tag{17}$$

Because of the connection of the transformation (15) with the transformation (12), the spin-wave equations of motion expressed in terms of the variables $\varphi_\lambda(x,t)$ should be invariant to the transformations (15). To construct expressions that are invariant with respect to (15) it is convenient, however, to use not the variables φ_λ directly, but the vectors $\mathbf{D}_\alpha(\varphi)$ (14). As a result of relations (8) and

$$\varepsilon_{abc} D_{a\alpha} D_{b\beta} D_{c\gamma} = \varepsilon_{\alpha\beta\gamma}, \tag{18}$$

all the possible invariant combinations of $\mathbf{D}_\alpha$ which do not contain derivatives are constants. The simplest invariant combinations that are linear in the derivatives with respect to time or with respect to the spatial coordinates x_i are the products

$$\begin{aligned} \mathbf{D}_\alpha \dot{\mathbf{D}}_\beta &= (D^{-1})_{\alpha a}\partial_t D_{a\beta} = \varepsilon_{\alpha\gamma\beta}\omega^\gamma(\varphi_\lambda, \dot{\varphi}_\lambda), \\ \mathbf{D}_\alpha \nabla_i \mathbf{D}_\beta &= (D^{-1})_{\alpha a}\partial_{x_i} D_{a\beta} = \varepsilon_{\alpha\beta\gamma}\omega^\gamma(\varphi_\lambda, \nabla_i\varphi_\lambda). \end{aligned} \tag{19}$$

As a consequence of (8), the products (19) are antisymmetrical with respect to permutation of the indices α and β, a fact explicitly taken into account the right-hand side of (19). Using the explicit form (14) of the function $\mathbf{D}_\alpha(\varphi)$, we obtain from (19) the following expressions for the left-hand invariant Cartan forms:

$$\omega^\alpha(\varphi_\lambda, \dot{\varphi}_\lambda) = (\delta_{\alpha\beta} - {}^1/_2\varepsilon_{\alpha\gamma\beta}\varphi_\gamma)\dot{\varphi}_\beta/(1+\varphi^2/4), \tag{20a}$$

$$\omega^\alpha(\varphi_\lambda, \nabla_i\varphi_\lambda) = (\delta_{\alpha\beta} - {}^1/_2\varepsilon_{\alpha\gamma\beta}\varphi_\gamma)\nabla_i\varphi_\beta/(1+\varphi^2/4). \tag{20b}$$

The expression for the phenomenological densities of the Lagrangians in the considered parametrization is obtained by substituting Eqs. (20) in (5) and (7).

Products of the type

$$\begin{aligned} \partial_t D_{a\alpha} D_{b\alpha} &= -\varepsilon_{acb}\tilde{\omega}^c(\varphi_\lambda, \dot{\varphi}_\lambda), \\ \nabla_i D_{a\alpha} D_{b\alpha} &= -\varepsilon_{acb}\tilde{\omega}^c(\varphi_\lambda, \nabla_i\varphi_\lambda), \end{aligned} \tag{21}$$

define the so-called right-hand differential Cartan forms

$$\tilde{\omega}^c(\varphi_\lambda, \dot{\varphi}_\lambda) = (\delta_{c\beta} + {}^1/_2\varepsilon_{c\gamma\beta}\varphi_\gamma)\dot{\varphi}_\beta/(1+\varphi^2/4), \tag{22a}$$

$$\tilde{\omega}^c(\varphi_\lambda, \nabla_i\varphi_\lambda) = (\delta_{c\beta} + {}^1/_2\varepsilon_{c\gamma\beta}\varphi_\gamma)\nabla_i\varphi_\beta/(1+\varphi^2/4), \tag{22b}$$

which are vectors with respect to the spinor rotations (9) and (15) and scalars with respect to the transformations (10) and (17).[5] The right and left Cartan forms are connected by the simple relation

$$\tilde{\boldsymbol{\omega}}(\varphi, \partial\varphi) = \mathbf{D}_\alpha\omega^\alpha(\varphi, \partial\varphi). \tag{23}$$

3. The recent[1] analysis[6] of spin waves in spatially disordered media was carried out with the aid of the so called hydrodynamic approach based on the representation of the Hamiltonian density in terms of conserved spin density and current. In Ref. 1, however, the transformation properties of the spin waves considered in Sec. 2 were not taken into account with sufficient consistency, and this led to incomplete allowance for the possible variants and to neglect of nonlinear effects.

We shall show that the Lagrangian density (6) admits of a transition to the hydrodynamic analysis with preservation of all its symmetry properties. To this end we change from the Lagrangian density (6) to the Hamiltonian density $\mathscr{H}$, defined as usual, as

$$\mathscr{H} = \pi_\lambda\dot{\varphi}_\lambda - \mathscr{L}, \tag{24}$$

where π_λ are the canonical momenta:

$$\pi_\lambda = \delta\mathscr{L}/\delta\dot{\varphi}_\lambda. \tag{25}$$

To verify that the Hamiltonian density (24) is a function of the spin density and current, we define the latter by means of the relations

$$-\delta\mathscr{L}/\delta\dot{\boldsymbol{\varepsilon}}(x,t) = \boldsymbol{\rho}(\varphi, \dot{\varphi}) = \mathbf{D}_\alpha(\varphi)\rho_\alpha(\varphi, \dot{\varphi}), \tag{26}$$

$$-\delta\mathscr{L}/\delta\nabla_i\boldsymbol{\varepsilon}(x,t) = \mathbf{j}(\varphi, \nabla_i\varphi) = \mathbf{D}_\alpha(\varphi)j_\alpha(\varphi, \nabla_i\varphi), \tag{27}$$

where $\varepsilon(x, t)$ is an infinitesimally small transformation of only the spin variables and depends on the coordinates and on the time. For the Lagrangian density (8), the spin density and current are of the form

$$\rho_\alpha(\varphi, \dot{\varphi}) = -a_{\alpha\beta}\omega^\beta(\varphi, \dot{\varphi}) - b_\alpha, \tag{28}$$

$$j_\alpha(\varphi, \nabla_i\varphi) = c_{\alpha\beta}\omega^\beta(\varphi, \nabla_i\varphi). \tag{29}$$

Taking (25) into account, as well as the relation

$$\pi_\lambda\dot{\varphi}_\lambda = -\rho_\beta(\varphi, \dot{\varphi})\,\omega_\beta(\varphi, \dot{\varphi}),$$

obtained by comparing (25) with (26), we obtain the following expression for the Hamiltonian density

$$\mathscr{H} = {}^1/_2(a^{-1})_{\alpha\beta}\rho_\alpha\rho_\beta + (a^{-1})_{\alpha\beta}b_\alpha\rho_\beta + {}^1/_2(c^{-1})_{\alpha\beta}j_\alpha(\nabla_i\varphi)j_\beta(\nabla_i\varphi), \tag{30}$$

which depends only on the density and current of the spin, expressed in the moving coordinate system.[7] For the Poisson brackets of φ_β with the spin densities ρ_α and $\boldsymbol{\rho}_\alpha$ (28) and for the brackets between the latter we obtain

$$\begin{aligned} \{\rho_\alpha(\mathbf{x}), \varphi_\beta(\mathbf{x}')\} &= -(\delta_{\alpha\beta} + {}^1/_2\varepsilon_{\beta\gamma\alpha}\varphi_\gamma + {}^1/_4\varphi_\alpha\varphi_\beta)\,\delta(\mathbf{x}-\mathbf{x}'), \\ \{\rho_\alpha(\mathbf{x}), \rho_\beta(\mathbf{x}')\} &= -\varepsilon_{\alpha\beta\gamma}\rho_\gamma(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}'), \\ \{\boldsymbol{\rho}_\alpha(\mathbf{x}), \varphi_\beta(\mathbf{x}')\} &= -(\delta_{\alpha\beta} - {}^1/_2\varepsilon_{\beta\gamma\alpha}\varphi_\gamma + {}^1/_4\varphi_\alpha\varphi_\beta)\,\delta(\mathbf{x}-\mathbf{x}'), \\ \{\boldsymbol{\rho}_a(\mathbf{x}), \boldsymbol{\rho}_b(\mathbf{x}')\} &= \varepsilon_{abc}\boldsymbol{\rho}_c(x)\,\delta(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{31}$$

The Poisson brackets for ρ_α and j_β are the consequence of (3) and (27), and duplicate the standard Poisson brackets of current algebra.[8]

The equations of motion corresponding to the Hamiltonian density $\mathscr{H}$ (30) take for the quantities ρ_α and φ_β the form of generalized Euler equations for the magnetic top:

$$\begin{aligned} \dot{\rho}_\alpha &= \varepsilon_{\alpha\beta\gamma}\rho_\beta(a^{-1})_{\gamma\sigma}(\rho_\sigma + b_\sigma) - \nabla_\lambda j_\alpha(\nabla_\lambda\varphi) + \varepsilon_{\alpha\beta\gamma}j_\beta(\nabla_\lambda)(c^{-1})_{\gamma\sigma}j_\sigma(\nabla_\lambda\varphi), \\ \dot{\varphi}_\alpha &= -(\delta_{\alpha\beta} + {}^1/_2\varepsilon_{\alpha\gamma\beta}\varphi_\gamma + {}^1/_4\varphi_\alpha\varphi_\beta)(a^{-1})_{\beta\gamma}(\rho_\gamma + b_\gamma). \end{aligned} \tag{32}$$

From (32) we obtain for the conserved density $\boldsymbol{\rho}_\alpha$ and the current $j_\alpha(\nabla_i\varphi)$ the continuity equation

$$\dot{\boldsymbol{\rho}}(\varphi, \dot{\varphi}) + \nabla_\lambda\mathbf{j}(\varphi, \nabla_\lambda\varphi) = 0. \tag{33}$$

The Hamiltonian density $\mathscr{H}$ (30) takes into account all the nonlinear exchange interactions of the magnons, and depends explicitly on the spontaneous-breaking parameters. Such a dependence arises in natural fashion when the microscopic volume integral or its hydrodynamic approximation is averaged. We note that in the latter case, when the Hamiltonian is a function of only the current variables, allowance for the spontaneous symmetry breaking calls for first replacing some of the products of the densities and currents by their mean values of the form $\langle \rho_a \ldots \rho_c \rangle = D_{a\alpha}(\varphi) \ldots D_{c\gamma}(\varphi) f_{\alpha\ldots\gamma}$, where $f_{\alpha\ldots\gamma}$ are the spontaneous symmetry-breaking parameters, followed by a transition to the long-wave approximation (30). When the indicated sequence is violated,[9] i.e., when the exchange integral is represented in the long-wave approximation by the lower powers of the density and current of the spin, followed by allowance for the spontaneous symmetry breaking, additional conditions are imposed on the coefficients in the Hamiltonian (30), namely $a_{\alpha\beta} \sim c_{\alpha\beta} \sim \delta_{\alpha\beta}$, and this raises the symmetry of the Hamiltonian at $b_a = 0$ to the symmetry of the $O(4)$ group.[10] For the Lagrangian density (6), which takes relativistic corrections into account, the spin current (27) takes the form

$$j_\alpha(\varphi, \nabla\varphi) = d_{\alpha\beta,\gamma\delta}\mathbf{D}_\gamma(\mathbf{D}_\delta\omega^\beta(\varphi, \nabla\varphi)), \tag{34}$$

and the Hamiltonian density $\mathscr{H}$ is correspondingly

$$\mathscr{H} = {}^1/_2(a^{-1})_{\alpha\beta}\rho_\alpha\rho_\beta + (a^{-1})_{\alpha\beta}b_\alpha\rho_\beta + {}^1/_2 f_{\alpha\beta,\gamma\delta}(j_\alpha(\nabla\varphi)\mathbf{D}_\gamma(\varphi))(j_\beta(\nabla\varphi)\mathbf{D}_\delta(\varphi)), \tag{35}$$

where the coefficients $f_{\alpha\beta,\gamma\delta}$ are defined by the relation

$$d_{\beta\rho,\alpha\rho'}f_{\rho\sigma,\rho'\sigma'}d_{\sigma\delta,\sigma'\gamma} = d_{\beta\delta,\alpha\gamma}.$$

For the spin current (34) and its density (28), the continuity equation does not hold, since the relativistic corrections disturb the invariance of the Hamiltonian (35) with respect to only spin rotations.

4. As noted above, in the Lagrangian (5), and accordingly in the Hamiltonian (35), there are terms that change the symmetry of the spin rotations to the symmetry of the group of total rotations. These terms arise as the result of relativistic interactions, and are therefore small. However, their presence leads to new effects which do not occur for the Lagrangian (8). These effects are the presence of spatial anisotropy in the propagation of spin waves, a correlation of the spin-wave polarization with their propagation direction, the onset of polarization in the scattering of spin waves, and others. As noted earlier, the Lagrangian density terms (5), which contain field gradients, take the following form when the relativistic interactions are taken into account:

$$-{}^1/_2 d_{\alpha\beta,\gamma\delta}(\mathbf{D}_\gamma\omega_\alpha(\nabla\varphi))(\mathbf{D}_\delta\omega_\beta(\nabla\varphi)). \tag{36}$$

This expression is a convolution with respect to the indices of quantities of three types: the gradients in the combinations $D_\alpha\nabla$, which represent the orbital motion, the currents $j_\alpha(\nabla_i\varphi)$, which represent the spin degrees of freedom, and the spontaneous symmetry breaking parameters $d_{\alpha\beta,\gamma\delta}$, which are also connected with the spin degrees of freedom. In expression (36) the components of all the foregoing quantities are represented in a moving frame.

Let us determine now which changes of the angular momentum for the oribtal and spin degrees of freedom, as functions of the coefficients $d_{\alpha\beta,\gamma\delta}$, are contained in (36). The coefficients $d_{\alpha\beta,\gamma\delta}$ can be either symmetrical to permutation of the indices α and β, in which case they are also symmetrical with respect to permutation of γ and δ, or else antisymmetrical with respect to the indicated pair of indices. In the former case the change of the orbital angular momentum in (36) is equal to zero or to 2, the change of the spin angular momentum, which is connected with the current, is also equal to 0 or 2, and the change of the spin angular momentum connected with the parameters of the spontaneous symmetry breaking can take on values from 0 to 4, in accordance with the conservation of the total angular momentum.

In expression (36) with antisymmetrical d with respect to α and β (and γ, δ), the spin momentum change connected with the currents is also equal to unity. The change of the spin angular momentum connected with the spontaneous breaking of the symmetry is equal to zero, unity, or two. It can be shown, however, that in the case when the change of the spin angular momentum as the result of symmetry breaking is equal to zero or two, expression (36) reduces to a total divergence. To prove this statement it suffices to rewrite (36) in the case of antisymmetrical coefficients $d_{\alpha\beta,\gamma\delta}$ in the form

$$-{}^1/_2\varepsilon_{\alpha\beta\sigma}f_{\sigma\rho}\varepsilon_{\rho\gamma\delta}(\mathbf{D}_\gamma\omega_\alpha(\nabla\varphi))(\mathbf{D}_\delta\omega_\beta(\nabla\varphi))$$

and to use the following identity

$$\varepsilon_{ikl}\omega_\alpha(\nabla_i\varphi)\omega_\beta(\nabla_k\varphi)D_{l\rho}\varepsilon_{\alpha\beta\mu}(f_{\rho\mu}+f_{\mu\rho}-\delta_{\mu\rho}\,\mathrm{Sp}\,f)$$
$$=-2\nabla_i\{\omega_\gamma(\nabla_k\varphi)\varepsilon_{ikl}D_{l\rho}f_{\rho\gamma}\},$$

which is the consequence of the Maurer–Cartan equations

$$\nabla_i\omega_\alpha(\nabla_k\varphi)-\nabla_k\omega_\alpha(\nabla_i\varphi)=-\varepsilon_{\alpha\beta\gamma}\omega_\beta(\nabla_i\varphi)\omega_\gamma(\nabla_k\varphi),$$
$$\nabla_i\{\omega_\gamma(\nabla_k\varphi)\varepsilon_{ikl}D_{l\rho}\varepsilon_{\rho\gamma\delta}f_\delta\}=0.$$

Thus, the number of independent variants defined by Eq. (36) is equal to 39. Six of these variants with orbital angular momentum equal to zero correspond to the exchange interaction.

We shall show now that by using the known properties of the microscopic relativistic interactions it is possible to decrease substantially the number of possible variants. To this end we note that the maximum number of particle-spin operators in the Hamiltonian of the microscopic relativistic interactions is equal to two, and on going over to the hydrodynamic description each expression represented in the form of a spin current corresponds to at least one spin operator of the microscopic Hamiltonian. Taking the foregoing into account, we obtain the following permissible invariants for the description of the relativistic interactions[11)]

$$\nabla_i(j_\alpha(\nabla_k\varphi))\varepsilon_{ikl}D_{\sigma\beta}\varepsilon_{\alpha\beta\gamma}f_\gamma, \tag{37a}$$

$$-{}^1/_2 d\{j_\alpha(\nabla_i\varphi)j_\beta(\nabla_k\varphi)+j_\beta(\nabla_i\varphi)j_\alpha(\nabla_k\varphi)$$
$$-{}^2/_3\delta_{ik}j_\alpha(\nabla\varphi)j_\beta(\nabla\varphi)\}D_{i\alpha}(\varphi)D_{k\beta}(\varphi). \tag{37b}$$

We note that the invariant $j_\alpha(\nabla\varphi)\mathbf{D}_\alpha(\varphi)$, which is compatible with the requirements considered above, violates spatial parity and therefore will be disregarded from now on.

From (37) and (36) we obtain the following expressions for the antisymmetrical and symmetrical components of the tensor $d_{\alpha\beta,\gamma\delta}$:

$$\overset{a}{d}_{\alpha\beta,\gamma\delta}=\varepsilon_{\alpha\beta\sigma}c_{\sigma\rho}\varepsilon_{\rho\mu\nu}f_\mu\varepsilon_{\nu\gamma\delta},$$
$$\overset{s}{d}_{\alpha\beta,\gamma\delta}=c_{\alpha\beta}\delta_{\gamma\delta}+d\{c_{\alpha\gamma}c_{\beta\delta}+c_{\beta\gamma}c_{\alpha\delta}-{}^2/_3\delta_{\gamma\delta}c_{\alpha\lambda}c_{\lambda\beta}\}. \tag{38}$$

5. The interaction with the external magnetic field h_i is introduced in standard fashion in the form $\gamma\boldsymbol{\rho}\cdot\mathbf{h}$. Allowance for the external magnetic field changes the magnetic-moment current, so that we can interpret $\gamma a_{\alpha\beta}$ as the components of the magnetic-susceptibility tensor. To determine the dispersion spectrum of the spin waves with allowance for the relativistic effects and for interactions with the external magnetic field, we obtain from the Lagrangian density (5), with account taken of the restrictions (38), the following free equations of motion for the spin waves relative to the equilibrium state with $\mathbf{b}\|\mathbf{h}$:

$$\omega^2 a_{\alpha\beta}\varphi^\beta+i\omega\tilde{b}_\gamma\varepsilon_{\alpha\beta\gamma}\varphi^\beta-\tilde{c}_{\alpha\beta}(k_i)\varphi^\beta=0, \tag{39}$$

where

$$\tilde{b}_\alpha=b_\alpha-\gamma(2a_{i\alpha}-\delta_{i\alpha}\,\mathrm{Sp}\,a)h_i,$$
$$\tilde{c}_{\alpha\beta}(k_i)=\tilde{c}_{\alpha\beta}(n_i)k^2-\gamma(\delta_{\alpha\beta}\mathbf{bh}-b_\alpha h_\beta),$$
$$\tilde{c}_{\alpha\beta}(n_i)=c_{\alpha\gamma}(\delta_{\beta\gamma}-{}^2/_3 dc_{\gamma\beta})+2dc_{i\alpha}c_{j\beta}n_in_j,$$
$$n_i=k_i/|\mathbf{k}|.$$

From (39) follows the dispersion equation[7]

$$\alpha\omega^6+3\beta\omega^4+3\tilde{\gamma}\omega^2+\delta=0, \tag{40}$$

where

$$\alpha=|a_{\alpha\beta}|,\quad \delta=-|\tilde{c}_{\alpha\beta}(k_i)|,$$
$$\beta=-({}^1/_3 b_\alpha a_{\alpha\beta}b_\beta+(1/3!)\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha'\beta'\gamma'}a_{\alpha\alpha'}\tilde{c}_{\gamma\gamma'}(k_i)),$$
$$\tilde{\gamma}={}^1/_2\tilde{b}_\alpha\tilde{c}_{\alpha\beta}(k_i)\tilde{b}_\beta+(1/3!)\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha'\beta'\gamma'}a_{\alpha\alpha'}\tilde{c}_{\beta\beta'}(k_i)\tilde{c}_{\gamma\gamma'}(k_i).$$

The dispersion equation (40) is the most general dispersion equation and it leads to the classification of all the possible magnetic structures by the type of their spectra and by the number of elementary branches, depending the number of nonzero eigenvalues of the matrices $a_{\alpha\beta}$, $\tilde{c}_{\alpha\beta}(k_i)$ and on the orientation of their eigenvectors relative to one another and to the vector $\mathbf{b}$.

We consider now a general case, when all the eigenvalues of the matrices $a_{\alpha\beta}$ and $\tilde{c}_{\alpha\beta}(k_i)$ differ from zero. In this case, for spin glass or for more complicated structures of the antiferromagnetic type, characterized by $\mathbf{b}=0$, Eq. (40) can be solved exactly in the absence of an external magnetic field ($\mathbf{h}=0$), and the spectrum is characterized by three branches of the Goldstone form[12)]

$$\omega_r^2=\varkappa_r(\mathbf{n})k^2 \quad (r=1,2,3). \tag{41}$$

For structures of the ferromagnetic type, characterized by $\mathbf{b}\neq 0$, an exact solution of (40) is obtained by using the well known Cardano formulas for cubic equations, and is quite complicated in form. We can, however, confine ourselves to the approximation of small $\mathbf{k}$ and, considering the dependence of the coefficients of (40) on $\mathbf{k}$, determine the character of the spectrum.[7] In the absence of an external magnetic field ($\mathbf{h}=0$) Eq. (40) has at $k=0$ only one nonzero root, inasmuch as in this case $\tilde{\gamma}=\delta=0$. The product of the roots that vanish at $k=0$ is proportional to $\delta\sim k^6$, and the sum of their pairwise products is proportional to $\tilde{\gamma}\sim k^2$; therefore at small k we have the following dependence of the roots of Eq. (40) on $\mathbf{k}$ and $\mathbf{n}$:

$$\omega_{\rm I}^2=A+D(\mathbf{n})k^2,\quad \omega_{\rm II}^2=F(\mathbf{n})k^2,\quad \omega_{\rm III}^2=G(\mathbf{n})k^4, \tag{42}$$

where the coefficients A, D, F, and G are determined by the following relations:

$$A=\frac{b_\alpha a_{\alpha\beta}b_\beta}{|a_{\alpha\beta}|},$$

$$D=\frac{\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha'\beta'\gamma'}a_{\alpha\alpha'}a_{\beta\beta'}\tilde{c}_{\gamma\gamma'}(\mathbf{n})}{2|a_{\alpha\beta}|}-\frac{b_\alpha a_{\alpha\beta}b_\beta}{b_\alpha\tilde{c}_{\alpha\beta}(\mathbf{n})b_\beta}$$

$$F=b_\alpha a_{\alpha\beta}b_\beta/b_\alpha\tilde{c}_{\alpha\beta}(\mathbf{n})b_\beta,\quad G=|\tilde{c}_{\alpha\beta}(\mathbf{n})|/b_\alpha\tilde{c}_{\alpha\beta}(\mathbf{n})b_\beta.$$

We note that in contrast to antiferromagnetic media, in ferromagnetic media there appears one non-Goldstone branch because $\mathbf{b}\neq 0$ in such media. An important general property of the dispersion relations (41) and (42) is that the relativistic corrections do not change the Goldstone character of the spectrum, but lead to a correlation of the polarization properties with the direction of propagation of the spin waves, to the appearance of spatial anisotropy, and to onset of polarization in spin-wave scattering.

In the presence of $O(2)$ symmetry in the ground state of the system, the behavior of the spin waves is determined by two angles φ_λ (see, e.g., Ref. 7). A description of such systems is obtained by making the substitutions $a_{11}=a_{22}\neq 0$, (with the remaining coefficients equal to zero) in the Lagrangian density (7) and in the subsequent formulas.

1)See also Chap. VII of the monograph[6].

2)See, e.g., Ref. 10, Sec. 35.

3)At any point except $\theta_1=0$ or $\theta_1=\pi$.

4)In the case when the parameters of the spontaneous symmetry breaking are tensors of third and higher order, a symmetry with respect to the transformations (10) can appear in the long-wave limit. However, since there are no grounds whatever for the vanishing of the lower-order tensors, this case is not very likely.

5)We note that the vector $\tilde{\omega}(\varphi_\lambda, \partial\varphi_\lambda)$ is not the only possible vector that is linear in the derivatives. Such vectors are also the derivatives $\partial D_\alpha(\varphi)$. Using the latter, we can write down terms quadratic in the derivatives in the Lagrangian density (7) in the form $\tilde{a}_{\alpha\beta}\partial_t D_\alpha(\varphi)\partial_t D_\beta(\varphi)$, and the terms with the gradients in the Lagrangian density (5) in the form $\tilde{d}_{\alpha\beta,\nu\delta}(D_\alpha\nabla)D_\beta(D_\nu\nabla)D_\delta$.

6)See also Ref. 11.

7)We note the analogy between the considered procedure and the change from a Lagrangian and angular velocities to a Hamiltonian and the angular momentum for the case of a rotating solid.

8)See, e.g., Ref. 11.

9)See, e.g., formula (7) and its subsequent proof in Ref. 11.

10)See, e.g., Ref. 2. The general mechanism of the possible raising of the symmetry in the phenomenological description of Goldstone excitations was discussed in Ref. 5.

11)We note that this assumption may not hold if averaging the microscopic Hamiltonian leads to derivatives of local spin operators. Thus, for example, $\nabla_i S_i(x)$ can go over, when spontaneous symmetry breaking is taken into account, into $\nabla_i D_{i\alpha}b_\alpha = {}_{\alpha\beta\nu}\omega_\alpha(\nabla_i\varphi)b_\beta D_{i\nu}$.

12)This can be easily verified by using the diagonalizing t—transformation, as a result of which Eq. (40) is represented in the form

$$\omega^6-\Big[\sum_r \varkappa_r+\sum_r B_r^2\Big]\omega^4+\Big[\sum_{r>s}\varkappa_r\varkappa_s+\sum_r \varkappa_r B_r^2\Big]\omega^2-\varkappa_1\varkappa_2\varkappa_3=0,$$

where

$$\varkappa_r(\mathbf{k})\delta_{rs}=(ta^{-1/2}\tilde{c}(\mathbf{n})a^{-1/2}t^{-1})_{rs},$$
$$B_r=(ta^{1/2})_{rs}b_s.$$

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Translated by J. G. Adashko