of a certain solution of the "complete" tetrahedra equations (3.8). If this is so, then there exists apparently a larger family of commuting operators $T(\theta, v)$ that depend, besides on the direction θ of the auxiliary line s, also on the rate v at which this line is shifter over the lattice $\mathscr{L}$. The family $T(\theta)$ determined by us is then the limiting case of $T(\theta) = T(\theta, v)|_{v\to 0}$.

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APPENDIX

In the cited equations we used the following abbreviations:

$$\theta_{12}=\theta_1+\theta_2, \quad \theta_{23}=\theta_2+\theta_3, \quad \theta_{123}=\theta_1+\theta_2+\theta_3;$$

$$\begin{aligned}&b(\pi-\theta_{12}, \theta_2)c(\pi-\theta_{123}, \theta_{23})l(\pi-\theta_{123}, \theta_3)c(\theta_3, \pi-\theta_{23})\\&+u(\pi-\theta_{12}, \theta_2)\sigma(\theta_1, \theta_{23})u(\theta_{12}, \theta_3)b(\theta_3, \pi-\theta_{23})\\&=b(\theta_2, \pi-\theta_{23})c(\theta_{12}, \pi-\theta_{123})l(\pi-\theta_{123}, \theta_1)c(\theta_1, \pi-\theta_{12})\\&+u(\theta_2, \pi-\theta_{23})\sigma(\theta_{12}, \theta_3)u(\theta_1, \theta_{23})b(\theta_1, \pi-\theta_{12});\end{aligned} \tag{A.1}$$

$$\begin{aligned}&c(\theta_2, \pi-\theta_{12})b(\theta_{23}, \pi-\theta_{123})b(\theta_{12}, \pi-\theta_{123})c(\pi-\theta_{23}, \theta_2)\\&=l(\theta_2, \theta_3)c(\theta_3, \theta_{12})c(\theta_{23}, \theta_1)l(\theta_1, \theta_2)\\&+h(\theta_2, \theta_3)l(\theta_3, \pi-\theta_{123})l(\pi-\theta_{123}, \theta_1)h(\theta_1, \theta_2);\end{aligned} \tag{A.2}$$

$$\begin{aligned}&b(\theta_1, \pi-\theta_{12})u(\theta_{23}, \pi-\theta_{123})u(\theta_{12}, \pi-\theta_{123})c(\theta_3, \pi-\theta_{23})\\&+u(\theta_1, \pi-\theta_{12})s(\theta_1, \theta_{23})s(\theta_{12}, \theta_3)v(\theta_3, \pi-\theta_{23})\\&=s(\theta_2, \theta_3)v(\theta_3, \pi-\theta_{123})l(\theta_{23}, \pi-\theta_{123})u(\theta_1; \pi-\theta_{12}).\end{aligned} \tag{A.3}$$

$$\begin{aligned}&g(\theta_2, \pi-\theta_{12})t(\theta_1, \pi-\theta_{123})h(\theta_3, \pi-\theta_{123})b(\theta_2, \theta_3)\\&+h(\theta_2, \pi-\theta_{12})h(\theta_{23}, \pi-\theta_{123})v(\theta_{12}, \theta_3)c(\theta_2, \theta_3)\\&=v(\theta_3, \pi-\theta_{23})c(\theta_3, \pi-\theta_{123})v(\theta_1, \theta_{23})v(\theta_1, \pi-\theta_{12}),\\&+u(\theta_3, \pi-\theta_{23})g(\theta_3, \theta_{12})c(\theta_1, \pi-\theta_{123})c(\theta_1, \pi-\theta_{12});\end{aligned} \tag{A.4}$$

$$\begin{aligned}&g(\theta_2, \pi-\theta_{12})t(\theta_1, \pi-\theta_{123})l(\theta_3, \pi-\theta_{123})h(\theta_3, \pi-\theta_{23})\\&+h(\theta_2, \pi-\theta_{12})h(\theta_{23}, \pi-\theta_{123})u(\theta_{12}, \theta_3)g(\theta_3, \pi-\theta_{23})\\&=v(\theta_3, \pi-\theta_{23})c(\theta_3, \pi-\theta_{123})c(\theta_1, \theta_{23})c(\theta_1, \theta_2)\\&+u(\theta_3, \pi-\theta_{23})g(\theta_3, \theta_{12})v(\theta_1, \pi-\theta_{123})v(\theta_1, \theta_2);\end{aligned} \tag{A.5}$$

$$\begin{aligned}&h(\theta_2, \pi-\theta_{12})l(\theta_{23}, \pi-\theta_{123})s(\theta_3, \theta_{12})t(\theta_2, \pi-\theta_{23})\\&=v(\theta_3, \pi-\theta_{23})v(\theta_3, \pi-\theta_{123})t(\theta_1, \theta_{23})h(\theta_2, \pi-\theta_{12})\\&+u(\theta_3, \pi-\theta_{23})h(\theta_3, \theta_{12})h(\theta_{23}, \pi-\theta_{123})l(\theta_2, \pi-\theta_{12}).\end{aligned} \tag{A.6}$$

1) This method was first proposed by Karowski, Thun, Truong, and Weisz.[3]

2) The triangles equations are in fact a component part of the quantum inverse-problem problem, since this equation is satisfied by the R matrix that defines the commutation relations between the elements of the global monodromy matrix (see Ref. 13).

3) Of course, the lattice $\mathscr{L}_{NM}(\alpha)$ does not differ in its coordinate structure from a rectangular lattice, and we speak of a lattice of parallelograms only to maintain the geometric meaning of the parameter α.

4) The idea of the derivation presented below stems from the papers of Baxter[8] and of Faddeev, Sklyanin, and Takhtadzhyan.[12]

5) In the "lattice" interpretation (see Sec. 4) of this model, the condition for allowed states corresponds to the fact that in the three-dimensional lattice $\mathscr{L}(\{n_a\}\{\xi_a\})$ it is permissible to color the faces black and white only in a way that the black faces form closed surfaces without edges.

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Influence of spatial dispersion on the image forces and electron energy spectrum above the surface of liquid helium

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An analytic expression is obtained for the potential of the electrostatic image forces above the surface of liquid helium, with account taken of the spatial dispersion of its dielectric constant. The calculated frequencies of the transition between the surface electron levels agree well with the experimental data for He^3 and He^4.

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1. INTRODUCTION

The first to point out the possibility of the onset of localized electronic states over the surface of liquid helium under the influence of electrostatic image forces were Cole and Cohen[1,2] and Shikin.[3] The existence of such surface (two-dimensional) states was experimentally confirmed by Brown and Grimes[4,5] for He^4, and

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then by Édel'man and Volodin[6,7] for He^3.

In first approximation, the problem of the energy spectrum of the electrons localized near the surface of liquid helium can be considered on the basis of a simple model potential,[8] which corresponds to infinite repulsion ($V_0 \to \infty$) on the liquid–vapor interface ($x=0$), and coincides in the gas phase ($x>0$) with the potential of the classical image forces

$$W_0(x) = -\frac{e^2}{4x}\frac{\varepsilon-1}{\varepsilon+1}, \quad (1)$$

where ε is the dielectric constant of liquid helium, equal to $\varepsilon_3 = 1.0572$ for He^4 and $\varepsilon_3 = 1.0428$ for He^3.[9,10] In this case the problem reduces to the one-dimensional Schrödinger equation, which is identical with the equation for the radial wave function of the hydrogen atom in the s state, and the electron spectrum takes the form

$$E_n^0(k_\parallel) = \frac{\hbar^2 k_\parallel^2}{2m} - \frac{(Z^*e)^2}{2a_0 n^2}; \quad n=1,2,3,\ldots \quad (2)$$

where $\hbar k_\parallel$ is the two-dimensional momentum of the electrons along the surface, $a_0 = \hbar^2/me^2$ is the Bohr radius, e and m are the charge and mass of the electron, and $Z^*e \equiv e(\varepsilon-1)/4(\varepsilon+1)$ is the effective image charge.

This model leads to a prefectly satisfactory (within 5%) quantitative agreement between the theoretical and experimental frequencies f_{1n} of the $1 \to n$ transition frequencies for He^4 and He^3 (see the table). The reason is that, by virtue of the smallness of Z^* for helium, the binding energy

$$|E_n^0| \approx \frac{6.6}{n^2} \cdot 10^{-4} \text{ eV}$$

is small at $k_\parallel = 0$, compared with the potential of the volume repulsion of the electrons from the filled shells of the atoms, $V_0 \approx 1$ eV, and the maximum of the electron wave function $u_n(x)$ is located at the large distance $a_0/Z^* \approx 100$ Å from the surface of the liquid, where the classical expression (1) is valid with high accuracy. Contributing to the good agreement between theory and experiment is also the partial cancellation of two effects: on the one hand, the lowering of the electron levels E_n on account of the finite height of the potential barrier V_0, and on the other, the expulsion of the levels from the potential well of finite depth, inasmuch as the real image forces do not diverge at the point $x=0$, in contrast to (1).

To describe the finite (nonsingular) image forces it is customary to use various model potentials.[2,11-14] In particular, to eliminate the divergence of $W_0(x)$ on the surface, Grimes, Brown, Burns, and Zipfel[13] have used a device known in the theory of metal surfaces,[15] that of shifting the origin (the image plane) into the interior of the liquid helium by a certain distance x_0, so that the effective potential energy of the electron takes the form

$$W_0(x) = \begin{cases} -Z^*e^2/(x+x_0), & x \geq 0 \\ V_0 > 0, & x < 0 \end{cases}. \quad (3)$$

With the aid of the method developed by Sanders and Weinreich,[12] in first-order perturbation theory, the following expression was obtained for the self-energy of the electron in the n-th state[13]:

$$E_n = E_n^0 + \frac{\hbar^2}{2m}\left\{x_0 - \left(\frac{\hbar^2}{2mV_0}\right)^{1/2}\right\}\left[\frac{du_n(x)}{dx}\right]^2_{x=0}. \quad (4)$$

Comparison with experiment yielded for He^4 at $V_0 = 1$ eV the value $x_0 = 1.04$ Å,[13] and for He^3 at $V_0 = 0.9$ eV the value $x_0 = 1.25$ Å.[6] In a sufficiently strong clamping electric field $\mathscr{E}_\perp$, however, when the electrons are localized much closer to the surface of the liquid phase than at $\mathscr{E}_\perp = 0$, a noticeable (albeit small) discrepancy is observed between the experimental and theoretical $\mathscr{E}_\perp$ dependences of the transition frequencies f_{12} and f_{13}, calculated on the basis of the phenomenological model (3), both for He^4 (Ref. 13) and for He^3.[6,7]

Hipolito, de Felicio, and Farias[14] obtained for a potential in the form (3) an exact solution of the Schrödinger equation in terms of confluent hypergeometric functions, and found the eigenvalues of the energy for the surface electronic states with $n = 1$, 2, and 3 as functions of the parameter x_0, which was chosen from the condition of equality of the frequencies f_{12} and f_{13} to their experimental values for He^4 (in the absence of the clamping field), and turned out to be equal to $x_0 = 1.01$ Å. Calculations of the dependences of f_{12} and f_{13} on $\mathscr{E}_\perp$, carried out with this value of x_0,[14] lead to a splendid agreement with experiment[13] in a wide clamping-field interval.

In the present paper, on the basis of the Green's function of the longitudinal self-consistent field, which describes the screening of the Coulomb interaction near the interface between media with spatial dispersion,[16] we calculate the potential of the image forces of a point charge located over the surface of liquid helium, and show that for a correct choice of the asymptotic form of the dielectric constant $\varepsilon(k) = 1 + \text{const}/k^4$ as $k \to \infty$ the effects of spatial dispersion ensure continuity (finiteness) of both the potential $W(x)$ and the electrostatic attraction force $F_x = -\partial W/\partial x$ on the liquid–vapor interface. This potential is approximated with high accuracy by the model potential (3) and it is this which explains the success of the phenomenological theories[13,14] in the description of the surface electronic states.

2. POTENTIAL OF IMAGE FORCES NEAR THE SURFACE OF LIQUID HELIUM

As shown by us earlier,[16] the potential electrostatic energy of a point charge located in vacuum at a distance x from the surface of a semi-infinite medium with dielectric constant $\varepsilon(k)$, is given by

$$W(x) = -e^2 \int_0^\infty q\,dq\,D(q; x, x), \quad (5)$$

where $D(q; x, x')$ is the Green's function of the Poisson

TABLE I.

		Experimental values of f_{1n}, GHz	Hydrogenlike spectrum of $f_{1n}^{(0)}$, GHz	Theoretical values Δf_{1n}, GHz	Theoretical values f_{1n}, GHz
He^4	f_{12}	125.9±0.2 [13]	119.2	6.75	125.95
	f_{13}	148.6±0.3 [13]	141.3	7.45	148.75
He^3	f_{12}	69.8±0.15 [6,7]	67.6	3.3±0.4	70.9±0.4
	f_{13}	83.15±0.25 [6,7]	80.1	3.6±0.4	83.7±0.4

equation for the longitudinal self-consistent field, and equals in this case

$$D(q;x,x)=\frac{e^{-2qx}}{2q}\frac{1-qa(q)}{1+qa(q)}\quad(x>0),\tag{6}$$

$$a(q)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dk_\perp}{(k_\perp^2+q^2)\varepsilon((k_\perp^2+q^2)^{1/2})},\quad q=(q_y^2+q_z^2)^{1/2}.\tag{7}$$

In contrast to metals, for which the dielectric constant is relatively well known,[16,17] in the case of dielectrics and semiconductors with covalent bonds the question of the calculation of $\varepsilon(k)$ from first principles, with allowance for the forbidden band in the electron spectrum, remains open at the present time. Various phenomenological models[18–20] were propsed in this connection for $\varepsilon(k)$. The simplest of these is the Inkson model,[19] according to which the dielectric constant of a semiconductor (dielectric) is given by

$$\varepsilon(k)=1+\frac{\varepsilon-1}{1+k^2(\varepsilon-1)/\lambda^2};\quad k=(k_\perp^2+q^2)^{1/2},\tag{8}$$

where ε is the static permittivity in a homogeneous electric field (as $k\to 0$), and the parameter λ^{-1} plays a role of the length of screening by the bound (valence) electrons, and is of the same order of magnitude as the atomic radius r_0. At large momentum transfers we have $\varepsilon(k)\approx 1+\lambda^2/k^2$, which agrees formally with the Thomas-Fermi approximation (TFA) for the conduction electrons in a metal.

In the case of liquid helium, when $\varepsilon-1\ll 1$, we can assume with good accuracy that

$$\frac{1}{\varepsilon(k)}=1-\frac{\varepsilon-1}{1+k^2(\varepsilon-1)/\lambda^2}.\tag{9}$$

Substituting (9) in (7), and then in (6) and (5), and integrating with respect to $k_\perp$ and q, we obtain

$$W(x)\approx-\frac{Z^*e^2}{x}+\frac{2Z^*e^2\lambda}{(\varepsilon-1)^{1/2}}\left\{\frac{\pi}{2}\left[\mathbf{H}_1\left(\frac{2\lambda x}{(\varepsilon-1)^{1/2}}\right)-N_1\left(\frac{2\lambda x}{(\varepsilon-1)^{1/2}}\right)\right]-1\right\},\tag{10}$$

where $Z^*\approx(\varepsilon-1)/8$, and $\mathbf{H}_1(z)$ and $N_1(z)$ are Struve and Neumann functions. It follows therefore that at $x\gg(\varepsilon-1)^{1/2}/2\lambda$ we have

$$W(x)\approx-\frac{Z^*e^2}{x}\left(1-\frac{(\varepsilon-1)^{1/2}}{2\lambda x}\right)\tag{11}$$

and at $x\ll(\varepsilon-1)^{1/2}/2\lambda$

$$W(x)\approx-\frac{2Z^*e^2\lambda}{(\varepsilon-1)^{1/2}}\left\{1-\frac{2\lambda x}{(\varepsilon-1)^{1/2}}\left[\ln\left(\frac{(\varepsilon-1)^{1/2}}{\gamma\lambda x}\right)+\frac{1}{2}\right]\right\}\tag{12}$$

where $\gamma=1.78\ldots$ is the Euler constant.

We see that as $x\to 0$ the potential $W(0)$ of the image forces is finite, but its derivative $\partial W(x)/\partial x|_{x=0}$ diverges logarithmically. The reason is that the Inkson model (as well as the TFA) does not take into account the quantum effects that lead, in particular, to the following asymptotic form of the dielectric constant of a gas of free electrons in the random-phase approximation[17]:

$$\varepsilon(k)=1+\frac{\alpha}{3}\left(\frac{2k_F}{k}\right)^4\quad(k\to\infty),\tag{13}$$

where $\alpha=1/\pi a_0k_F$, and k_F is the Fermi momentum.

Schulze and Unger[20] have proposed an interpolation formula for (k) of a dielectric (semiconductor):

$$\varepsilon(k)=1+\frac{\varepsilon-1}{1+k^2(\varepsilon-1)/\lambda^2}\frac{1}{1+3k^2/4k_F^2},\tag{14}$$

which takes into account the quantum corrections phenomenologically and leads to a correct asymptotic behavior of the form (13) as $k\to\infty$, when all the electrons act as free ones.

Calculating the electrostatic energy (5) on the basis of (14) accurate to terms $\sim(\varepsilon-1)^2$, we obtain

$$W(x)=-\frac{Z^*e^2}{x}-2Z^*e^2\left(\frac{3}{4k_F^2}-\frac{\varepsilon-1}{\lambda^2}\right)^{-1}\left\{\frac{\sqrt{3}}{2k_F}-\frac{(\varepsilon-1)^{1/2}}{\lambda}-\frac{\pi\sqrt{3}}{4k_F}\left[\mathbf{H}_1\left(\frac{4k_Fx}{3^{1/2}}\right)-N_1\left(\frac{4k_Fx}{3^{1/2}}\right)\right]+\frac{\pi(\varepsilon-1)^{1/2}}{2\lambda}\left[\mathbf{H}_1\left(\frac{2\lambda x}{(\varepsilon-1)^{1/2}}\right)-N_1\left(\frac{2\lambda x}{(\varepsilon-1)^{1/2}}\right)\right]\right\}.\tag{15}$$

Expression (15) goes over into (10) as $k_F\to\infty$.

As $x\to 0$ it follows from (15) that

$$W(x)\approx-\frac{2Z^*e^2}{3^{1/2}/2k_F+(\varepsilon-1)^{1/2}/\lambda}\left[1-\frac{x}{3^{1/2}/2k_F-(\varepsilon-1)^{1/2}/\lambda}\ln\frac{3^{1/2}\lambda}{2k_F(\varepsilon-1)^{1/2}}\right].\tag{16}$$

We see that the asymptotic form $\varepsilon(k)=1+\text{const}/k^4$ as $k\to\infty$ ensures continuity, at the point $x=0$, of both the potential $W(x)$ and the image forces $F_x=-\partial W/\partial x$.

3. SHIFT OF LEVELS OF SURFACE ELECTRON STATES BY SPATIAL DISPERSION EFFECTS

As noted above, the shift of the levels of the surface electronic states is due to the fact that the potential barrier V_0 on the liquid-vapor boundary is finite, and to the difference between the real potential of the image forces $W(x)$ from the classical potential $W_0(x)$. Therefore for nonsingular potentials in the first-order perturbation theory these effects can be considered independently,[12] and the total shift of the self-energy of the electron is equal to

$$\Delta E_n=\Delta E_n'+\Delta E_n'';\tag{17}$$

$$\Delta E_n'=-\left(\frac{\hbar^2}{2m}\right)^{3/2}V_0^{-1/2}\left[\frac{du_n(x)}{dx}\right]^2_{x=0};\tag{18}$$

$$\Delta E_n''=\int_0^\infty dx\,u_n^2(x)\left[W(x)+\frac{Z^*e^2}{x}\right],\tag{19}$$

where $u_n(x)$ are hydrogenlike wave functions of the one-dimensional Coulomb problem, which takes for the first three levels the form

$$\begin{aligned}u_1(x)&=2x\beta^{3/2}e^{-\beta x};\quad \beta=Z^*/a_0;\\ u_2(x)&=2^{-1/2}x\beta^{3/2}(1-{}^1/_2\beta x)e^{-\beta x/2};\\ u_3(x)&=2\cdot3^{-3/2}x\beta^{3/2}[1-{}^2/_3\beta x+{}^2/_{27}(\beta x)^2]e^{-\beta x/3},\end{aligned}\tag{20}$$

from which it follows, in particular, that

$$\left[\frac{du_n(x)}{dx}\right]^2_{x=0}=4\left(\frac{\beta}{n}\right)^3.\tag{21}$$

Substituting (15) and (20) in (19) we see that $\Delta E_n''$ is expressed in terms of the generalized hypergeometric series ${}_3F_2$ and in terms of spherical Legendre functions of the second kind Q_μ^ν, with the aid of the formulas[21]

$$\int_0^\infty z^\nu e^{-z}\mathbf{H}_1(bz)\,dz=\frac{b^2\Gamma(\nu+3)}{2\pi^{1/2}\Gamma(5/2)}\,{}_3F_2\left(1,\frac{\nu+3}{2},\frac{\nu}{2}+2;\frac{3}{2},\frac{5}{2};-b^2\right);\tag{22}$$

$$\int_0^\infty z^{\mu-1}e^{-z}N_1(bz)\,dz=-\frac{2}{\pi}\Gamma(\mu+1)\left(\frac{b^2}{1+b^2}\right)^{\mu/2}Q_{\mu-1}^1\left[\left(\frac{b^2}{1+b^2}\right)^{1/2}\right],\tag{23}$$

where $\Gamma(\nu)$ is the gamma function and the parameter takes on the value $2\lambda/\beta(\varepsilon-1)^{1/2}$ or $4k_F/3^{1/2}\beta$, i.e., $b\gg 1$.

We express (22) with the aid of the MacRobert transformation in terms of hypergeometric functions of the small argument $1/b^2$, and as a result of simple but cumbersome calculations, retaining the first nonvanishing terms, we obtain

$$\Delta E_n'' = \frac{3^{1/3} e^2}{2k_F a_0^2}\left(\frac{Z^*}{n}\right)^3 [1+O(Z^*)]. \qquad (24)$$

Thus, according to (21) and (24), the shift of the n-th level is equal to

$$\Delta E_n = \frac{1}{n^3}(\Delta E_1' + \Delta E_1''), \qquad (25)$$

and the corresponding corrections to the frequencies of the transitions between the levels f_{1n} are determined by the expression

$$\Delta f_{1n} = \frac{|\Delta E_1|}{2\pi\hbar}\left(1 - \frac{1}{n^2}\right). \qquad (26)$$

Substituting in (18) and (21) the values $V_0 = 1.3$ eV and $Z^* = 6.95 \times 10^{-3}$ for He^4 and $V_0 = 0.9$ eV and $Z^* = 5.24 \times 10^{-3}$ for He^3 (see Ref. 2), we get

$$\Delta E_1'(He^4) = -5.94 \cdot 10^{-5}\ \text{eV}; \quad \Delta E_1'(He^3) = -3.06 \cdot 10^{-5}\ \text{eV}. \qquad (27)$$

If we substitute in (24) the value of the Fermi momentum k_F calculated for the total electron density (assuming two electrons per atom), so that $k_F = 1.09$ and $k_F = 0.99$ Å^{-1} for He^4 and He^3, respectively, then we obtain for Δf_{12} and Δf_{13} values that are somewhat too high. A much better agreement with experiment is obtained for He^4 (see the table) by putting $k_F = \lambda = \lambda_0$ and regarding λ_0 as the parameter of the model (14), a parameter chosen from the condition that the potential (15) coincide with (3) at the point $x = 0$ for the optimal choice of $x_0 = 1.01$ Å.[13] In this case $\lambda_0 = 0.545$ Å^{-1} and $\Delta E_1'' = 2.74 \times 10^{-5}$ eV.

For these values of the parameter x_0 and λ_0, the dependence of the dimensionless potential $w(\xi) \equiv W(x)/e^2\lambda_0$ on $\xi = \lambda_0 x$, calculated in accordance with (15) for He^4, is shown in the figure by the solid curve. The dash-dot curve shows in the same figure the model potential (3), while the dashed curve shows the potential of the classical image forces (1). We see that the analytic expression (15) obtained in the present paper with allowance for the spatial-dispersion effects is approximated with high accuracy (within 3%) by the model potential (3), and this explains the splendid agreement between the calculations of Ref. 4 and the experiment of Ref. 13.

For He^3, taking into account the approximate estimate[6] $x_0 = (1.25 \pm 0.15)$ Å, we get $\lambda_0 = (0.428 \pm 0.051)$ Å^{-1} and $\Delta E_1'' = (1.50 \pm 0.18) \times 10^{-5}$. In this case the agreement between the theoretical values of f_{1n} and experiment is somewhat worse than for He^4 (see the table).

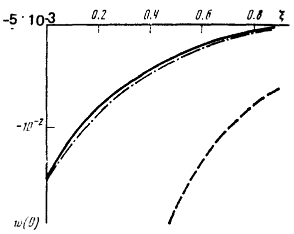

FIG. 1. Dependence of the dimensionless potential $w(\xi) = W(x)/e^2\lambda_0$ on $\xi = \lambda_0 x$ for He^4 at $\lambda_0 = \lambda = k_F = 0.545$ Å^{-1} and $Z^* = 6.95 \times 10^{-3}$ (solid curve), of the model potential $\tilde{w}_0(\xi) = -Z^*/(\xi + \lambda_0 x)$ at $x = 1.01$ Å (dash-dot) and of the potential $w_0(\xi) = -Z^*/\xi$ of the classical image forces (dashed).

It should be noted in conclusion that there exists an additional contribution to the shift of the transition frequencies f_{1n}, due to the renormalization of the self-energy of the electrons on account of their interaction with the zero-point surface oscillations (ripplons). For the nonsingular potential (15), the constant of the electron-ripplon interaction is equal to

$$g_{nm}(q) = \xi_q \int_0^\infty dx \left\{ u_n(x) u_m(x) \frac{dW(x)}{dx} + \left[U_n(x)\frac{dU_m}{dx} + U_m(x)\frac{dU_n}{dx}\right] W(x) \right\} \qquad (28)$$

and

$$\xi_q = \left(\frac{\hbar q}{2\rho\omega_q S}\right)^{1/2}; \quad \omega_q = \left(q\left(g + q^2\frac{\sigma}{\rho}\right)\right)^{1/2}; \qquad (29)$$

where ρ is the density of the liquid phase, S is the area of the surface, σ is the surface-tension coefficient, and g is the acceleration due to gravity (see Ref. 8).

Calculations using integrals of the type (22) and (23) show that $g_{nm}(q) \sim (Z^*)^4$. In second-order perturbation theory the shift of the n-th level is equal to

$$\delta E_n = -\sum_q \frac{2g_{nn}^2(q)}{\hbar\omega_q}. \qquad (30)$$

Thus, $\delta E_n \sim (Z^*)^8$ for He^4, and is negligibly small for He^3. For other dielectrics[1,2], however, such a "polaron" contribution can be quite substantial.[23]

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Contribution to the theory of the two-dimensional mixed state in type-*I* superconductors

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We consider the nature of the two-dimensional mixed state produced on the inner surface of a hollow cylinder when the superconductivity is destroyed by current. The two-dimensional mixed-state layer constitutes a structure periodic along the cylinder axis, consisting of alternating annular superconducting regions and regions in which the macroscopic phase coherence is disturbed and the order-parameter phase undergoes at certain instants of time 2π jumps at a frequency satisfying the Josephson condition, while the order parameter oscillates between zero and a certain finite value. This picture is analogous to the phase slippage centers in the resistive state of a narrow superconducting channel. The current-voltage characteristic of the sample is calculated, and one of its peculiarities is the presence of an excess current that depends little on the sample voltage.

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1. INTRODUCTION

In the study of the properties of current-carrying super-conductors a situation frequently arises wherein, despite of the presence of a constant electric field in the sample, purely thermodynamic factors favor the formation of a superconducting state either in the entire sample or in definite sections of the sample (if the temperature of the superconductor and the magnetic field in these sections are lower than the critical values). Thus, the coexistence of a constant electric field and superconductivity is observed in narrow (quasi-one-dimensional) superconducting channels in a certain range of current (the so-called resistive state; see, e.g., Refs. 1 and 2). One more example is connected with the destruction of the superconductivity by current in solid type-I superconductors, when the sample becomes stratified into alternating normal and superconducting domains (the intermediate state). The superconducting domains cannot be in touch with one another on the macroscopic sections, for otherwise the sample becomes short-circuited. It is clear nevertheless that near the cylinder axis, where the magnetic field is weak, the formation of the superconducting state should be favored.

A peculiar situation takes place when the superconductivity is destroyed by current in hollow type-I cylindrical samples. As noted by L. Landau,[3] when the current through the sample exceeds $\mathscr{T}_c (r_1^2+r_2^2)/2r_1r_2$ (where $\mathscr{T}_c = cH_c r_2/2$ is the critical current, and r_1 and r_2 are the radii of the inner and outer surfaces of the cylinder), the intermediate state in the interior of the sample vanishes and goes over into the normal state. At the same time, on the inner surface the field is weak, therefore the normal state is unstable there. The state produced near the inner surface of the cylinder is, however, not purely superconducting, since a constant electric field is present in the sample. Such a state is called two-dimensional mixed (TM), and was experimentally observed by I. Landau and Sharvin.[4] A qualitatively similar picture appears on the surface of a bulky superconducting sample when an external magnetic field exceeding the critical value is turned off. When turned off, the magnetic field in space vanished rapidly, whereas in the sample volume, on account of the induced eddy currents, it retains a large value for a rather long time. As a result, the formation of the TM state turns out to be convenient on the surface. This situation was investigated experimentally in detail by Dorozhkin and Dolgopolov.[5]

In all the listed examples, in some sections of the superconductor there exists simultaneously a constant electric field and superconductivity. The primary reason is that the constant electric field penetrates into the superconductor to a finite depth l_E. It is known that this depth as a rule greatly exceeds the coherence length $\xi(T)$ as well as the penetration depth $\lambda(T)$ of a constant magnetic field (for alloys without paramagnetic impurities we have near the critical temperature $l_E = l_\varepsilon(4T/\pi\Delta)^{1/2}$,[6] where l_E is the diffusion length of the quasiparticles $l_\varepsilon = (\mathscr{D}\tau_{ph})^{1/2}$).

We are dealing thus with a situation in which the established superconductivity exists against the background of a constant electric field. If the conditions of the problem are such that the field differs from zero in macroscopic sections of the sample, then the scalar potential φ can assume large values. It is clear that when the latter increases the superconductivity should become destroyed in the entire volume. This, however,

