Vertex function of an electron in a constant electromagnetic field

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The vertex function for an electron in a constant crossed field of arbitrary intensity is derived to the third order in the radiation field. It is shown that the radiation interaction smears out the Airy function which, in the external field, describes the intensity of the interaction of the electron with the photon as a function of the nonconserved momentum component. A qualitative relation $V^{(3)} \sim \alpha\chi^{2/3}V^{(1)}$ is found between the first-order and third-order vertex functions for large values of the dynamic parameter $\chi = [(eFp)^2]^{1/2}m^{-3}$. It is also shown that the radiation interaction does not change the order of magnitude of the squared mass of the system transferred at the vertex. The vertex function satisfies the Ward identity as modified by the external field.

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1. INTRODUCTION

At present there is much interest in research on the radiative corrections to electromagnetic processes in strong external fields, i.e., fields whose intensity is close to the characteristic quantum-electrodynamic value $F_0 = m^2c^3/e\hbar = 4.4 \cdot 10^{13}$ Oe. The mass and polarization operators in a constant field have been calculated to second order in perturbation theory,[1-10] as well as the fourth-order corrections to these quantities,[11-13] and also the radiative corrections to certain processes,[11-13] while the asymptotic behavior of the mass and polarization operators has also been investigated in higher orders of perturbation theory.[14-16] A more detailed bibliography and commentary on radiation corrections in the electrodynamics of an intense field is given in Ref. 17, page 272.

In all of these papers, however, the diagrams studied did not include the vertex function, which has not been found, up to this time, in electrodynamics in an external field. In the present paper we fill this gap and calculate the vertex to third order in the radiation field for an electron in a constant crossed field $\mathbf{E} \perp \mathbf{H}$, $E = H$, or arbitrary strength. This field is described by the four-potential[1)]

$$A_\mu = a_\mu(kx), \quad k^2 = ka = 0.$$

It is the low-frequency approximation of a plane wave field, and for ultrarelativistic particles it is a good approximation of an arbitrary constant field.

In fact, for an electron in an arbitrary constant field the vertex function depends on invariants composed of the momenta, the invariants χ, χ', and $\varkappa$ composed of the momenta and the field [Eq. (56)], and pure field invariants; for these last it is convenient to take the values ε and η of the electric and magnetic fields in a system in which they are parallel, as measured in units of F_0, i.e., the invariants $e\varepsilon m^{-2}$ and $e\eta m^{-2}$. For a crossed field these last invariants are equal to zero. If an arbitrary constant field is such that the pure field invariants are small both in comparison with unity and in comparison with χ, χ', and $\varkappa$:

$$e\varepsilon m^{-2}, \; e\eta m^{-2} \ll 1, \; \chi, \chi', \varkappa,$$

the vertex function for the field can be expanded in powers of the pure field invariants, and as the first approximation one gets the vertex function in a crossed field. The correction terms brought in by the deviation of the field from a crossed field are small owing to the stated conditions. Physically these conditions mean that the field is weak compared with F_0 and that the particles are ultrarelativistic.

2. THE MOMENTUM REPRESENTATION FOR THE VERTEX FUNCTION

In third-order perturbation theory the vertex function of an electron in a constant crossed field can be written in the form

$$V_\mu^{(3)}(x'', y, x') = e^2\gamma_\nu S^c(x'', y)\gamma_\mu S^c(y, x')\gamma_\nu D^c(x''-x'), \tag{1}$$

where $S^c(x,y)$ is the propagation function of an electron in the crossed field, as found by Schwinger[18]:

$$S^c(x, y) = e^{i\eta}S(x-y), \quad \eta = {}^1/_2 e(a, x-y)(k, x+y), \tag{2}$$

$$S(z) = \frac{1}{i(4\pi)^2}\int_0^\infty \frac{ds}{s^2}\exp\left\{i\frac{z^2}{4s} - is\left(m^2 + \frac{(eFz)^2}{12}\right)\right\}[m - i\gamma\pi z]e^{ie\sigma Fs/2}, \tag{3}$$

$$\pi_{\alpha\beta}(s) = \frac{1}{2s}\delta_{\alpha\beta} + \frac{1}{2}eF_{\alpha\beta} + \frac{s}{6}e^2(FF)_{\alpha\beta}, \tag{4}$$

and $D^c(z)$ is the propagation function of the photon, for which we shall use the proper-time representation:

$$D^c(z) = \frac{1}{i(4\pi)^2}\int_0^\infty \frac{dt}{t^2}\exp\left(i\frac{z^2}{4t} - i\mu^2 t\right). \tag{5}$$

Here μ is a small photon mass introduced to remove the infrared diffraction divergence of the vertex function.

We shall look for the vertex function in the E_p representation, which was introduced by one of us[1] and is in quantum electrodynamics in an external field the analog of the ordinary Fourier transformation. We find that the basis functions of the E_p representation in the crossed field case are of the form

$$E_p(x) = \exp\left\{ipx + i\frac{epa}{2kp}(kx)^2 - i\frac{e^2a^2}{6kp}(kx)^3 + i\frac{e\sigma F}{4kp}(kx)\right\}. \tag{6}$$

The matrices (6) are eigenfunctions of the operators[2)] $-i\partial_1, -i\partial_2, -i(\partial_0 + \partial_3), (\gamma\pi)^2$ with the eigenvalues p_1, p_2, p_-,

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p^2 and satisfy the equation

$$\gamma\Pi E_p(x)=E_p(x)\gamma p, \quad (7)$$

where $\Pi_\mu=-i\partial_\mu-eA_\mu$ is the kinetic momentum operator and γp is the γ-matrix eigenvalue of the operator $\gamma\Pi$.

The matrices E_p have the orthogonality and completeness properties

$$\int d^4x\,\bar{E}_q(x)E_p(x)=(2\pi)^4\delta(q-p),$$
$$\int\frac{d^4p}{(2\pi)^4}E_p(x)\bar{E}_p(y)=\delta(x-y), \quad (8)$$
$$\bar{E}_p(x)=\gamma_4E_p{}^+(x)\gamma_4;$$

We write the E_p transform of the function (1) in the form

$$V_\mu^{(3)}(q,p,l)=\int d^4x'\,d^4x''\,d^4y\,\bar{E}_q(x'')V_\mu^{(3)}(x'',y,x')E_p(x')e^{ily}. \quad (9)$$

Changing the variables of integration in Eq. (9) to

$$z''=x''-y,\quad z'=y-x',\quad y \quad (10)$$

and using a relation which follows directly from the explicit form of the E_p functions, Eq. (6),

$$E_p(y+z)=E_p(y)E_{\tilde{p}(y)}(z)e^{ie(az)(ky)}, \quad (11)$$

where

$$\tilde{p}_\mu(y)=p_\mu-ea_\mu(ky)+\frac{e(pa)}{kp}(ky)k_\mu-\frac{e^2a^2}{2kp}(ky)^2k_\mu \quad (12)$$

is the classical kinetic momentum of a charge at the point y_- with the initial value p_μ at the point $y_-=0$, we rewrite Eq. (9) in the form

$$V_\mu^{(3)}(q,p,l)=\int d^4y\,\bar{E}_q(y)\Lambda_\mu^{(3)}(\tilde{q}(y),\tilde{p}(y))E_p(y)e^{ily}, \quad (13)$$

where

$$\Lambda_\mu^{(3)}(\tilde{q},\tilde{p})=e^2\int d^4z'd^4z''\bar{E}_{\tilde{q}}(z'')\gamma_\nu S(z'')\gamma_\mu S(z')\gamma_\nu E_{\tilde{p}}(-z')D^c(z''+z')$$
$$\times\exp\{{}^1/_2\,ie\,[(az'')(kz'')-(az')(kz')]\}, \quad (14)$$

and $S(z)$ is the diagonal part of the electron propagation function, Eq. (3).

The resulting representation (13), (14) for $V^{(3)}(p,q,l)$ is a Fourier integral with respect to the argument l. Shifting the variable of integration y_- and using the relation (11), we can get for $V^{(3)}(q,p,l)$ a representation (13) with a different function $\Lambda_\mu^{(3)}$, which, unlike Eq. (14), depends explicitly on l. This representation is not a Fourier integral with respect to the variable l, but on the other hand allows us to simplify the dependence of the function $\Lambda_\mu^{(3)}$ on $\tilde{q},\tilde{p}_s$, i.e., on the variables of integration in Eq. (13).

Equation (13) differs from the E_p representation of the point vertex $V_\mu^{(1)}$ by the replacement $\gamma_\mu\to\Lambda_\mu$. Therefore the function $\Lambda_\mu(\tilde{q},\tilde{p})$ determines the correction to the vertex function in the E_p representation:

$$\Gamma_\mu(\tilde{q},\tilde{p})=\gamma_\mu+\Lambda_\mu(\tilde{q},\tilde{p}). \quad (15)$$

As can be seen from Eqs. (14) and (12), the vertex function in the E_p representation depends on the coordinate y_- of the absorption of the photon through the classical kinetic momenta $\tilde{p}_\mu(y)$ and $\tilde{q}_\mu(y)$ of the electron before and after the absorption, or more exactly through $\gamma\tilde{p}$ and $\gamma\tilde{q}$. In a certain sense these γ-matrices can be regarded as eigenvalues of the operator $\gamma\Pi$. This statement follows immediately from Eq. (7), which can be written in the form (see Ref. 17, page 9)

$$\gamma\Pi E_p(x)=\gamma\tilde{p}(x)E_p(x), \quad (16)$$

so that

$$\gamma\tilde{p}(x)E_p(x)=E_p(x)\gamma p. \quad (17)$$

Equation (17) can also be written in the form

$$\bar{E}_p(x)\gamma\tilde{p}(x)E_p(x)=\exp\left\{-i\frac{e\sigma F}{4kp}(kx)\right\}\gamma\tilde{p}(x)\exp\left\{i\frac{e\sigma F}{4kp}(kx)\right\}=\gamma p, \quad (18)$$

and accordingly the transition from the kinetic momentum $\gamma\tilde{p}$ to the quantum numbers γp or conversely is nothing but a Lorentz transformation.

The function $\Lambda_\mu(\tilde{q},\tilde{p})$, considered as a function of the quantum numbers $\tilde{q}$ and $\tilde{p}$, has independent meaning apart from the dependence of $\tilde{q}$ and $\tilde{p}$ on the coordinates. We shall show this with the example of the fourth-order vertex correction to the mass operator, which in the E_p representation is given by

$$M_V^{(4)}(q,p)=\int d^4x\,d^4x'\,\bar{E}_q(x)M_V^{(4)}(x,x')E_p(x'), \quad (19)$$

where

$$M_V^{(4)}(x,x')=ie^4\int d^4y\,d^4y'\,\gamma_\mu S^c(x,y)\gamma_\nu S^c(y,y')\gamma_\mu S^c(y',x')$$
$$\times\gamma_\nu D^c(x-y')D^c(y-x'). \quad (20)$$

Using the obvious equation $E_p(x)\bar{E}_p(x)=1$, we rewrite Eq. (9) in the form

$$M_V^{(4)}(q,p)=\int d^4x\,\bar{E}_q(x)E_p(x)\int d^4x'\,\bar{E}_p(x)M_V^{(4)}(x,x')E_p(x') \quad (21)$$

and show that the integral

$$M_V^{(4)}(p)=\int d^4x'\,\bar{E}_p(x)M_V^{(4)}(x,x')E_p(x') \quad (22)$$

does not depend on x.

Using the E_p representation of the electron propagator and the representation (13) for the function $V_\mu^{(3)}(q,p,l)$, we get

$$M_V^{(4)}(p)=-\frac{ie^2}{(2\pi)^8}\int d^4l\,d^4f\,d^4y\frac{e^{-il(x-y)}}{l^2-i\varepsilon}\bar{E}_p(x)\gamma_\mu E_f(x)$$
$$\times\frac{1}{m+i\gamma f-i\varepsilon}\bar{E}_f(y)\Lambda_\mu^{(3)}(\tilde{f}(y),\tilde{p}(y))E_p(y). \quad (23)$$

From the relations (11) and (18) we have

$$\bar{E}_p(x)\gamma_\mu E_f(x)\frac{1}{m+i\gamma f}\bar{E}_f(y)=\bar{E}_p(y)\bar{E}_{\tilde{p}(y)}(x-y)\gamma_\mu E_{\tilde{f}(y)}(x-y)\frac{1}{m+i\gamma\tilde{f}(y)}. \quad (24)$$

Therefore the expression (23) can be put in the form

$$M_V^{(4)}(p)=-\frac{ie^2}{(2\pi)^8}\int d^4y\bar{E}_p(y)\left\{\int d^4l\,d^4\tilde{f}\,\frac{e^{-ilz}}{l^2-i\varepsilon}\bar{E}_{\tilde{p}(y)}(z)\gamma_\mu E_{\tilde{f}}(z)\right.$$
$$\left.\times\frac{1}{m+i\gamma\tilde{f}-i\varepsilon}\Lambda_\mu^{(3)}(\tilde{f},\tilde{p}(y))\right\}E_p(y), \quad (25)$$

if in the curly brackets we change from the variable of integration f to $\tilde{f}=\tilde{f}(y)$, noting that the Jacobian for the change is equal to unity, and then set $x-y=z$.

After integration over the virtual momenta the expression in curly brackets in Eq. (25) is a γ-matrix invariant which depends only on $\gamma\tilde{p}(y)$, γz, and σF. The Lorentz transformation (18) takes $\gamma\tilde{p}(y)$ into γp and γz into $\gamma\tilde{z}$, where

$$\tilde{z}_\mu=z_\mu-\frac{ky}{kp}eF_{\mu\lambda}z_\lambda+\frac{(ky)^2}{2(kp)^2}e^2F_{\mu\lambda}F_{\lambda\sigma}z_\sigma, \quad (26)$$

and does not affect the invariant σF. Therefore

$$E_p(y)\{\ \}E_p(y)=\int d^4l d^4\tilde{f}\,\frac{e^{-il\tilde{z}}}{l^2-i\varepsilon}E_p(\tilde{z})\gamma_\mu E_{\tilde{f}}(\tilde{z})\frac{1}{m+i\gamma\tilde{f}-i\varepsilon}\Lambda_\mu^{(3)}(\tilde{f},p). \quad (27)$$

If now in Eq. (25) we change from the variable of integration y to $\tilde{z}$ (the Jacobian is unity) and leave off the unnecessary tildés from f and $\tilde{z}$, we finally get

$$M_V^{(4)}(p)=-\frac{ie^2}{(2\pi)^8}\int\frac{d^4f\,d^4l}{l^2-i\varepsilon}V_\mu^{(1)}(p,f,-l)\frac{1}{m+i\gamma f-i\varepsilon}\Lambda_\mu^{(3)}(f,p). \quad (28)$$

Accordingly, $M_V^{(4)}(p)$ does not depend on the coordinate x, and using Eq. (8) we get for the mass operator (21) the diagonal expression

$$M_V^{(4)}(q,p)=(2\pi)^4\delta(q-p)M_V^{(4)}(p), \quad (29)$$

which was also to be expected from general considerations.[1]

An analogous representation can be obtained also for the second-order mass operator:

$$M^{(2)}(p)=-\frac{ie^2}{(2\pi)^8}\int\frac{d^4f\,d^4l}{l^2-i\varepsilon}V_\mu^{(1)}(p,f,-l)\frac{1}{m+i\gamma f-i\varepsilon}\gamma_\mu. \quad (30)$$

Comparing formulas (28) and (30), we see that the change from the second-order mass operator to the vertex correction (28), (30) are remarkably similar to the vacuum expressions; in them only one vertex is "dressed" with E_p functions. This similarity, achieved with the technique of the E_p representation, decidedly facilitates the interpretation and the calculations.

3. CALCULATION OF THE VERTEX FUNCTION

Before proceeding to the direct calculation of the vertex function Λ_μ in third-order perturbation theory, we recall that the representation (14) is not unique. For example, by choosing in Eq. (9) integration variables different from those of Eq. (10) one can get for the vertex function a representation that depends explicitly on l. In the absence of an external field this arbitrariness of choice corresponds to the use of conservation laws. In our case, however the situation is less trivial, since in a constant field we have only three conserved quantum numbers and different representations of Λ_μ differ from one another in their γ-matrix structures. The passage from one representation to the other requires integration by parts with respect to the variable y_- in Eq. (13), so that in calculating $\Lambda_\mu^{(3)}$ it is convenient to start directly from Eq. (13). We shall try to express the vertex function in the form of an integral over the proper times of the electron and photon, in which the phase in the parts that do not depend on the field is expressed in terms of $\tilde{q}^2=q^2$, $\tilde{p}^2=p^2$ and l^2, and in the γ-matrix structure the matrices $\gamma\tilde{q}$ are on the left and the $\gamma\tilde{p}$ are on the right.

Choosing the coordinate system and the gauge so that the vector a is along the axis 1 and k is along axis 3, and integrating over the coordinates y_1, y_2, and $y_+=\frac{1}{2}(y_0+y_3)$, we put Eq. (13) in the form

$$V_\mu^{(3)}(q,p,l)=(2\pi)^3\delta(q_1-p_1-l_1)\delta(q_2-p_2-l_2)\delta(q_--p_--l_-)$$
$$\times\int_{-\infty}^{\infty}dy_-\,e^{if(\varphi)}\exp\left(-i\frac{e\sigma F}{4kq}\varphi\right)\Lambda_\mu^{(3)}(\tilde{q}(y),\tilde{p}(y))\exp\left(i\frac{e\sigma F}{4kp}\varphi\right), \quad (31)$$

where

$$f(\varphi)=-r\varphi+\frac{1}{2}\alpha\varphi^2-\frac{4\beta}{3}\varphi^3,\quad \varphi=ky=-k_+y_-,$$
$$r=\frac{q_+-p_+-l_+}{k_+},\quad \alpha=\frac{eqFp}{(kq)(kp)},\quad 8\beta=\frac{(eFl)^2}{(kq)(kp)(kl)}. \quad (32)$$

We call attention to the fact that when the external field is turned off the integral over y_- immediately gives

$$2\pi\delta(q_+-p_+-l_+)\Lambda_\mu^{(3)\,\mathrm{vac}}(q,p)$$

and leads to the conservation law $p_\alpha+l_\alpha=q_\alpha$ for all four momentum components. If we turn off the radiation interaction in Eq. (31), i.e., if we set $\Lambda_\mu^{(3)}\to\gamma_\mu$, then $V^{(3)}$ goes over into the vertex function $V^{(1)}$ given by Eqs. (22) and (23) of Ref. 19. In the presence of the field the first, second, and minus components of the momentum are conserved, and the plus component, conjugate to the minus coordinate, is not conserved; the kinetic momentum of the electron depends on the coordinate y_- of the absorption of the photon. Therefore after the integration over y_- the final momentum is not completely determined by the initial momenta p and l, the conservation law holding only for three of the four components, see Eq. (31). Moreover, if we regard Λ_α as a slowly varying function of y_- and abstract from spin effects, then because of the oscillation of the function $e^{if(\varphi)}$ the largest contribution to the integral (31) comes from the neighborhoods of points $\varphi=\varphi_c$, where $f'(\varphi_c)=0$. Since

$$-f'(\varphi)=r-\alpha\varphi+4\beta\varphi^2=k_+^{-1}(\tilde{q}_+(y)-\tilde{p}_+(y)-l_+), \quad (33)$$

this means the neighborhoods of points at which the conservation law holds also for the plus components of the momenta.

For the radiationless vertex function $V_\mu^{(1)}$, for which $\Lambda_\mu^{(1)}=\gamma_\mu$, the integral over y_- in Eq. (31) reduces essentially to an Airy function $\Phi(z)$ with the argument

$$z=(4\beta)^{-2/3}\left[\frac{r}{4\beta}-\left(\frac{\alpha}{8\beta}\right)^2\right], \quad (34)$$

which replaces the function $2\pi\delta(q_+-p_+-l_+)$. We omit here a common factor and the spin structure that leads to terms in the first and second derivatives of the Airy function.

We note that the argument z is gauge invariant, and determines the amount of detuning $q_+-p_+-l_+$ in terms of the field strength $F_{\mu\nu}$ and the values of the conserved components of the momenta p, q, and l. On the other hand, by means of the conservation laws we can give to the quantity $(q-p-l)_+$ the meaning of the transferred squared mass of the system in the charged or neutral channel:

$$(q-p-l)_+=\frac{(p+l)^2-q^2}{2q_-}=\frac{l^2-(q-p)^2}{2l_-}.$$

Accordingly, the interaction of the electron with the photon is intense only if z is not too large in absolute value, i.e., if the law of the conservation of the plus components of the momenta is not too strongly violated. When the radiation interaction is taken into account Λ_μ becomes a complicated function of y_- and the Airy function is considerably modified, cf. Eq. (54).

Using the representations (2) and (5) of the Green's functions and the explicit form (6) of the E_p functions, we carry through the integrations over z', z'' in Eq. (14) by means of the formula

$$J=\int\prod_{i=1}^{n} d^4x^{(i)} \exp\left\{-iq_\alpha^{(i)} x_\alpha^{(i)}+\frac{i}{4} x_\alpha^{(i)} w_{\alpha\beta}^{ik} x_\beta^{(k)}\right\} g(kx^{(i)})$$

$$=\frac{i^n(4\pi)^{2n}}{\sqrt{\det w}} \exp\{-iq_\alpha^{(i)} (w^{-1})_{\alpha\beta}^{ik} q_\beta^{(k)}\} g[2k_\alpha(w^{-1})_{\alpha\beta}^{im} q_\beta^{(m)}], \quad (35)$$

where w is a 4×4 matrix in the Lorentz indices α, β and an $n\times n$ matrix in the indices i, k that number the 4-coordinates of the integration and the prescribed four-momenta, w^{-1} is the matrix inverse to w, and g is an arbitrary function of the variables $x_-^{(k)}$, $k=1, 2, \ldots, n$.

In our case

$$w_{\alpha\beta}^{ik}=w_0^{ik}\delta_{\alpha\beta}+w_1^{ik}\frac{e}{kp}\tilde{p}_\lambda(F_{\lambda\alpha}k_\beta+F_{\lambda\beta}k_\alpha)$$

$$+w_2^{ik}\frac{e}{kq}\tilde{q}_\lambda(F_{\lambda\alpha}k_\beta+F_{\lambda\beta}k_\alpha)+w_3^{ik}e^2(FF)_{\alpha\beta}, \quad (36)$$

where the two-rowed matrices w_n are given by

$$w_0=\begin{pmatrix}\omega_1^{-1} & t^{-1}\\ t^{-1} & \omega_2^{-1}\end{pmatrix}, w_1=\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}, w_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, w_3=\frac{1}{3}\begin{pmatrix}s_1 & 0\\ 0 & s_2\end{pmatrix}, \quad (37)$$

s_1, s_2, and t are the respective proper times of the electrons with the momenta p, q and of the photon, and $\omega_i^{-1}=s_i^{-1}+t_i^{-1}$.

The inverse matrix w^{-1} also has the structure (36) with the matrix coefficients u_n instead of w_n:

$$u_0=w_0^{-1}=\frac{1}{\det w_0}\begin{pmatrix}\omega_2^{-1} & -t^{-1}\\ -t^{-1} & \omega_1^{-1}\end{pmatrix}$$

$$u_{1,2}=-w_0^{-1}w_{1,2}w_0^{-1}, \quad u_3=-w_0^{-1}w_3w_0^{-1}-w_0^{-1}\sigma_3w_0^{-1}\sigma_3w_0^{-1}, \quad (38)$$

where σ_3 is the Pauli matrix, and

$$\det w_0=\frac{1}{s_1s_2}+\frac{1}{t}\left(\frac{1}{s_1}+\frac{1}{s_2}\right). \quad (39)$$

After the integration Eq. (14) can be written in the form

$$\Lambda_\mu^{(3)}(\tilde{q}(y), \tilde{p}(y))=\frac{\alpha}{4\pi i}\iiint_0^\infty \frac{ds_1\, ds_2\, dt}{s_1^2s_2^2t^2}\exp[-i(m^2s_1+m^2s_2+\mu^2t)]Q_\mu$$

$$\times\exp\left\{-iqw^{-1}q+i\frac{4e^2a^2}{3kq}[k(w^{-1}q)^{(2)}]^3+i\frac{4e^2a^2}{3kp}[k(w^{-1}q)^{(1)}]^3\right\}, \quad (40)$$

where

$$Q_\mu=\exp\left[-i\frac{e\sigma F}{2kq}k(w^{-1}q)^{(2)}\right]\gamma_\nu\left[m+i\gamma\pi(s_2)\frac{\overleftrightarrow{\partial}}{\partial q^{(2)}}\right]$$

$$\times\exp\left(i\frac{e\sigma F}{2}s_2\right)\gamma_\mu\left[m+i\gamma\pi(s_1)\frac{\overleftrightarrow{\partial}}{\partial q^{(1)}}\right]$$

$$\times\exp\left(i\frac{e\sigma F}{2}s_1\right)\gamma_\nu\exp\left[-i\frac{e\sigma F}{2kp}k(w^{-1}q)^{(1)}\right], \quad (41)$$

and the double arrow over differential operators means that they act both to the right and to the left. After carrying out the differentiations we are to set

$$q^{(1)}=\tilde{p}(y), \quad q^{(2)}=\tilde{q}(y).$$

The result (40) has a very complicated dependence on $\tilde{p}$ and $\tilde{q}$, and consequently on y_-. This dependence can be considerably simplified by the following device. Substituting the expression (40) in Eq. (31) and making the changes of variables

$$z_-=y_-+\frac{2s_1s_2}{s_1+s_2+t}l_-, \quad (42)$$

$$s_1={}^1/_2\omega(1+v)(1+\eta), \quad s_2={}^1/_2\omega(1+v)(1-\eta),$$
$$t=\omega(1+v)/v, \quad (43)$$

we can rewrite Eq. (31) in the new form

$$V_\mu^{(3)}(q, p, l)=\int d^4z\, \bar{E}_q(z)\Lambda_\mu^{(3)}(\tilde{q}(z), \tilde{p}(z), l)E_p(z)e^{ilz}, \quad (44)$$

where the transformed vertex function $\Lambda_\mu^{(3)}(\tilde{q}, \tilde{p}, l)$ now depends on l, but in return has a very simple dependence on z:

$$\Lambda_\mu^{(3)}(\tilde{q}, \tilde{p}, l)=\frac{\alpha}{4\pi i}\int_0^\infty d\omega\int_0^\infty\frac{dv}{1+v}\int_{-1}^{1}d\eta$$

$$\times\exp\left\{-i\omega S-i\frac{\omega^3}{3}vR-\frac{i}{2}\omega^2v(1-\eta^2)e\tilde{q}F\tilde{p}\right\}Q_\mu(\tilde{q}, \tilde{p}, l), \quad (45)$$

$$S=m^2(1+v_\lambda)+{}^1/_2q^2(1-\eta)+{}^1/_2p^2(1+\eta)+{}^1/_4l^2v(1-\eta^2), \quad (46)$$

$$R={}^1/_2(eFq)^2(1-\eta)[1+{}^1/_2\eta(1+\eta)(1-v)]+{}^1/_2(eFp)^2(1+\eta)$$
$$\times[1-{}^1/_2\eta(1-\eta)(1-v)]+{}^1/_4(eFl)^2(1-\eta^2)[v-2+{}^1/_4(1-\eta^2)(1+v)^2]. \quad (47)$$

The variable $v_\lambda=v+\lambda(1+v)/v$, $\lambda=(\mu/m)^2$ contains the photon mass. The phase of the integrand depends only linearly on z_- through $e\tilde{q}F\tilde{p}=eqFp+e^2lFFz$. The quantity Q_μ is given by

$$Q_\mu(\tilde{q}, \tilde{p}, l)=-\left(m^2-i\frac{v}{\omega(1+v)^2}\right)\tilde{\gamma}_\mu-\frac{2+2v+v^2}{4(1+v)^2}[e\sigma F, \tilde{\gamma}_\mu]_+$$

$$-i\frac{\omega v}{8(1+v)^2}\left(2+v+v\frac{1+v}{3}\right)e^2\sigma F\gamma_\mu\sigma F+[\tilde{q}A\gamma-lB(\eta)\gamma]\tilde{\gamma}_\mu[\gamma A\tilde{p}+\gamma B(-\eta)l]$$
$$-im[\tilde{q}A\gamma-lB(\eta)\gamma]\tilde{\gamma}_\mu'-im[\tilde{q}A'\gamma+lB'(-\eta)\gamma]\tilde{\gamma}_\mu$$
$$-im\tilde{\gamma}_\mu'[\gamma A\tilde{p}+\gamma B(-\eta)l]-im\tilde{\gamma}_\mu[\gamma A'\tilde{p}-\gamma B'(\eta)l], \quad (48)$$

where

$$\tilde{\gamma}_\mu=\exp\left\{-i\frac{e\sigma F}{4}\omega[2+v(1+\eta)]\right\}\gamma_\mu\exp\left\{-i\frac{e\sigma F}{4}\omega[2+v(1-\eta)]\right\},$$
$$\tilde{\gamma}_\mu'=\exp\left\{i\frac{e\sigma F}{4}\omega v(1-\eta)\right\}\gamma_\mu\exp\left\{i\frac{e\sigma F}{4}\omega v(1+\eta)\right\}, \quad (49)$$

and the matrices $A, A', B(\eta), B'(\eta)$ are defined by the equations

$$A=\frac{1}{1+v}\left[1+\omega(2+v)eF+\frac{\omega^2}{3}(6+4v+v^2)e^2FF\right], \quad (50)$$

$$A'=\frac{1}{1+v}\left[1-\omega veF+\frac{\omega^2}{3}v(v-2)e^2FF\right], \quad (51)$$

$$B(\eta)=\frac{2+v(1-\eta)}{2(1+v)}+\frac{\omega}{4(1+v)}[v^2(1-\eta^2)+v(1-\eta)^2+4(2+v)]eF$$
$$+\frac{\omega^2}{4(1+v)}\left[8+4v\eta+3v(1-\eta)^2+\frac{2}{3}v(1+v)(1+\eta)\right.$$
$$\left.+v(1-v+v^2)\frac{\eta(1-\eta^2)}{3}\right]e^2FF, \quad (52)$$

$$B'(\eta)=\frac{v(1-\eta)}{2(1+v)}\left\{1+\frac{\omega}{2}[2+(1-v)(1+\eta)]eF+\frac{\omega^2}{2}\left[(2-v)(1+\eta)\right.\right.$$
$$\left.\left.-\frac{2v}{3}-v(1-\eta^2)+\frac{1}{3}(1-\eta)^2+v(v-4)\frac{\eta(1+\eta)}{3}\right]e^2FF\right\}. \quad (53)$$

In the transformation of Q_μ we have used the equality (33), which we already used in the integration by parts in Eq. (31) over the variable y_-.

We note that the matrices A and A' have a simple physical meaning, namely

$$A_{\alpha\beta}q_\beta=\pi_{\alpha\beta}z_\beta^{eff}, \quad A_{\alpha\beta}'q_\beta=\pi_{\alpha\beta}^*z_\beta^{eff},$$

where the matrix $\pi_{\alpha\beta}$ is defined in Eq. (4), $\pi_{\alpha\beta}^*$ differs from it in the sign of the charge or the field, and z^{eff} is the effective value of the relative coordinate of the classical electron with the momentum q (Ref. 5).

When the field is turned off the vertex function (45) goes over into the vacuum form, which was first obtained by Karplus and Kroll[20] (see also Ref. 21).

As already been mentioned, for the radiationless vertex function $V_\mu^{(1)}$ the integral over y_- in the representation (31) reduces essentially to the Airy function $\Phi(z)$ with the argument (34). It follows from Eq. (45) that the radiation interaction smears out the $\Phi(z)$, replacing it with the function

$$\frac{\alpha m^2}{4\pi i}\int_0^\infty d\omega\int_0^\infty \frac{dv}{1+v}\int_{-1}^{1} d\eta \exp\left(-i\omega S-\frac{i}{3}\omega^3 vR\right)\Phi(z-\zeta), \quad (54)$$

where z is the same argument (34) and ζ is a correction which depends on ω, v, and η:

$$\zeta=\omega^2 v(1-\eta^2)(4\beta)^{1/3}(kq)(kp). \quad (55)$$

4. PROPERTIES OF THE VERTEX FUNCTION IN STRONG FIELDS

We shall discuss two properties of the vertex function we have derived, which make it decidedly different from the corresponding function for an electron in vacuum. The interaction of particles with the external field is characterized by the relativistically and gauge invariant parameters

$$\chi=[(eFp)^2]^{1/2}/m^3,\quad \chi'=[(eFq)^2]^{1/2}/m^3,\quad \varkappa=[(eFl)^2]^{1/2}/m^3, \quad (56)$$

which, depending on the channel, satisfy the conservation laws

$$\chi+\varkappa=\chi',\quad \varkappa=\chi+\chi'. \quad (57)$$

It follows even from the radiationless vertex function $V_\mu^{(1)}\sim\Phi(z)$ and from the structure of the argument z that the squared mass transferred in the vertex is of order of

$$q^2-(p+l)^2\sim m^2\chi'^{2/3}(\varkappa/\chi)^{1/3},\quad (q+p)^2-l^2\sim m^2\varkappa^{2/3}(\varkappa^2/\chi\chi')^{1/3}, \quad (58)$$

is independent of m, and vanishes when the field is turned off

For large fields or momenta at least two of the parameters χ, χ', $\varkappa$ are large, and the transferred mass-squared is much larger than the square of the electron mass and increases with the field. For example, if $\chi\sim\chi'\sim\varkappa\gg 1$, then in any channel the transferred mass-squared is $\sim(eFp)^{2/3}$.

The radiation interaction does not change the validity of Eq. (58), since the important values of the variables of integration in Eq. (54) are such that ζ is always smaller than or of the order of unity.

Of still greater interest is the qualitative behavior of the vertex function $V_\mu^{(3)}$ at large values of the parameters (56). In this case the important values in the expression (54) are $\omega\sim\chi^{-2/3}$ and $\Phi(z-\zeta)\sim 1$, and we have the relation

$$V_\mu^{(3)}\sim\alpha\chi^{2/3}V_\mu^{(1)}. \quad (59)$$

Furthermore the γ-matrix terms[3)] Q_μ, which effectively depend on the field, are important here. The relation (59) indicates that $\alpha\chi^{2/3}$ is a universal parameter for the applicability of perturbation theory for large energies or fields. Arguments in favor of this sort of dependence (power-law, not logarithmic) of this parameter on χ were first given by one of the present writers[1] and were confirmed in Ref. 14.

5. THE WARD IDENTITY

The vertex function must satisfy a generalized Ward identity, which was first found by Fradkin.[22] It is convenient for us to use this relation in the E_p representation (see the paper by Mitter[23])

$$l_\mu V_\mu^{(3)}(q,p,l)=i[I(q,p,l)M^{(2)}(p)-M^{(2)}(q)I(q,p,l)], \quad (60)$$

where $M^{(2)}(q)$ is the second-order mass operator and the matrix $I(q,p,l)$ has the representation

$$I(q,p,l)=\int d^4x\bar{E}_q(x)E_p(x)e^{ilx} \quad (61)$$

and is a generalization of the four-dimensional function $(2\pi)^4\delta(q-p-l)$, which describes conservation of momentum at the vertex, to the case of a nonzero external field. Therefore $I(q,p,l)$ differs from $V_\mu(q,p,l)$ by replacement of Λ_μ with unity, see Eqs. (13) and (31).

By taking the limit $l\to 0$ in the identity (60) we can get the differential Ward identity

$$\left[\exp\left(-i\frac{eF}{kp}\frac{\partial}{\partial s}\right)\right]_{\mu\nu}\Lambda_\nu(p+sk,p,0)|_{s=0}=-i\frac{\partial M(p)}{\partial p_\mu}-i\frac{k_\mu k_\nu}{4(kp)^2}\frac{\partial M}{\partial p_\nu}e\sigma F$$
$$-\frac{e(Fp)_\mu k_\nu k_\rho}{2(kp)^2}\frac{\partial^2 M}{\partial p_\nu\partial p_\rho}+i\frac{e^2(FF)_{\mu\nu}k_\rho k_\lambda}{6(kp)^2}\frac{\partial^3 M}{\partial p_\nu\partial p_\rho\partial p_\lambda}. \quad (62)$$

By direct calculation it can be verified that our vertex function (45) and the previously found[12] mass operator of the electron in a crossed field satisfy the relation (62). This is a good check on these calculations. The choice of the integration variables in (13) for the vertex function was determined precisely by the requirement that they should be identical at $l=0$ with the integration variables in the expression for the mass operator.[12] It is not hard to see that for $l=0$ the integral over y_- in Eq. (45) gives a δ function for the plus component of the momentum, and the phase of the integrand in Eq. (45) no longer depends on the variable η, which is the fractional difference of the electron proper times, $\eta=(s_2-s_1)/(s_2+s_1)$. The phase now agrees with that of the second-order mass operator,[12] except for the natural change in the variables of the vertex

$$\omega^{-1}=(s_1+s_2)^{-1}+t^{-1},\quad v=(s_1+s_2)/t$$

the sum s_2+s_1 replaces the proper time s in the analogous variables of the mass operator. The variable ω can be called the proper time of the vertex function.

We note that for the representation (45) of the vertex function we have besides Eq. (62) the relation

$$k_\mu\frac{\partial^2\Lambda_\mu(p+sk,p,0)}{\partial s^2}\Big|_{s=0}=-\frac{i}{3}k_\mu\frac{\partial^2}{\partial s^2}\left(\frac{\partial M(p+sk)}{\partial p_\mu}\right)\Big|_{s=0}. \quad (63)$$

Our result for the vertex function contains of course a logarithmic divergence with respect to the proper time and requires regularization. Since the presence of the external field does not lead to any ultraviolet divergences beyond those of the vacuum case, to regularize the vertex function (45) it sufficies to subtract from it its value at $l=0$, $\gamma q=\gamma p=im$ and $F=0$;

$$\Lambda_\mu^{(3)}(\tilde{q},\tilde{q},0)|_{\gamma\tilde{q}=im,\,F=0}=L^{(3)}\gamma_\mu, \quad (64)$$

$$L^{(3)}=\frac{\alpha}{4\pi i}\int_0^\infty d\omega\int_0^\infty\frac{dv}{1+v}\int_{-1}^{1}d\eta\exp(-im^2\omega v_\lambda)\left[\frac{iv}{\omega(1+v)^2}+m^2\frac{2+2v-v^2}{(1+v)^2}\right]. \quad (65)$$

Then for the regularized vertex function we have

$$\Lambda_{R\mu}^{(3)}(\tilde{q}, \tilde{p}, l) = \Lambda_{\mu}^{(3)}(\tilde{q}, \tilde{p}, l) - L^{(3)}\gamma_{\mu}. \tag{66}$$

Introducing in Eq. (65) as the lower limit on the variable of integration ω the value $\omega_0 \to 0$ and performing the integrations, we get for $L^{(3)}$ the expression

$$L^{(3)} = \frac{\alpha}{2\pi}\left(\frac{1}{2}\ln\frac{1}{i\gamma m^2\omega_0} + \ln\lambda + 2\right), \quad \gamma = 1.781\ldots \tag{67}$$

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1) We use a system of units in which $\hbar = c = 1$, and the notations $p_\alpha = (\mathbf{p}, ip_0)$, $pq = \mathbf{p}\cdot\mathbf{q} - p_0 q_0$ $p_-, = p_0 - p_3$, $p_+ = 1/2(p_0 + p_3)$, $\alpha = e^2/4\pi\hbar c = (137.03\ldots)^{-1}$.

2) Expressions not written in invariant form are always given in a coordinate system with the axis 3 along the vector $\mathbf{k}$: $k_\mu = (0, 0, k_0, ik_0)$, $k_+ = k_0$.

3) A detailed derivation and analysis of the relation (59) is presented in our preprint (Fiz. Inst. Akad. Nauk, No. 84 1981), under the same title.

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