Conformal anomaly and the production of massless particles by a conformally flat metric

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(Submitted 13 February 1981)
Zh. Eksp. Teor. Fiz. **81**, 417–428 (August 1981)

It is shown that a conformally flat gravitational field can produce gauge bosons through the anomaly in the trace of the energy–momentum tensor. An equation is obtained for the single-particle wave function of a photon with allowance for the electromagnetic corrections in an arbitrary conformally flat gravitational field. The solution of this equation determines the amplitude for photon production by the field. The production of scalar particles is discussed. It is shown that the theory can be formulated in such a way that the production of massless scalar particles, in contrast to photons, is forbidden.

PACS numbers: 04.40. + c, 14.80.Kx, 12.25. + e

1. INTRODUCTION

It is well known that a variable gravitational field (like, for that matter, any other force field) can produce elementary particles from the vacuum.[1] However, it was already noted in the first papers[2–5] devoted to this question that in the physically interesting case of a conformally flat metric, which includes, for example, the Friedmann and de Sitter metrics, the production of particles is strongly suppressed, and in the case of zero mass is forbidden altogether. Massless particles can be produced in a nonisotropic space,[5] leading to rapid isotropization[6] of the space. With regard to the assertion that there is no production of massless particles in conformally flat space, it is valid only for noninteracting particles and is based on the conformal invariance of the theory. It is however known that interaction breaks conformal invariance despite the fact that the original Lagrangian can be formally conformally invariant. A well-known example of the breaking of conformal invariance is the anomaly in the trace of the energy–momentum tensor in the theory of gauge fields, which in flat space has the form

$$T_{\mu\mu}^{\text{anom}}=\frac{\alpha\beta}{8\pi}G_{\mu\nu}^{a}G_{\mu\nu}^{a}, \tag{1.1}$$

where $\alpha=g^2/4\pi$ is the gauge coupling constant, $G_{\mu\nu}^a$ is the field tensor of the vector field A_μ^a, and β is a numerical coefficient and depends on the form of the theory. If SU(N) is a gauge group with N_f families of fundamental fermions, then in the lowest order in α

$$\beta=11N/3-2N_f/3.$$

We recall that in the tree approximation $T_{\mu\mu}=0$ for massless particles, but this equation is violated in loop diagrams because of the divergences. The point is that the gauge-invariant regularization of the divergent diagrams is not conformally invariant. In particular, in the case of Pauli–Villars regularization one introduces into the theory unphysical massive fields which are chosen to make the result finite, and in the final result the masses are taken equal to infinity. The violation of conformal invariance (due to the fact that the regular fields are massive) in the intermediate stage of the calculations is reflected in the final result in, for example, the form of the anomaly (1.1).

Gauge-invariant regularization by means of analytic continuation with respect to the dimension of space (by transition to dimension $d=4-\varepsilon$) is also not conformally invariant, since a Lagrangian which is conformally invariant in a space of dimension $d=4$ is no longer such for $d\neq4$. In the limit $\varepsilon\to0$, the deviations from conformal invariance in the Lagrangian are formally small ($\sim\varepsilon$), but the loop graphs can contain poles with respect to ε, so that finite noninvariant corrections arise in some amplitudes.

It is easy to see that in the limit of a weak conformally flat gravitational field the particle production amplitude is proportional to the trace of the operator of the energy-momentum tensor of these particles. Indeed, for an appropriate choice of the coordinates, a conformally flat metric can be written in the form

$$g_{\mu\nu}=a^2(x_\alpha)\eta_{\mu\nu}, \tag{1.2}$$

where $\eta_{\mu\nu}=\text{diag}(1,-1,-1,-1)$ is the metric tensor in Minkowski space, and $a(x_\alpha)$ is a function of the coordinates.

The Lagrangian of the interaction of particles with a weak gravitational field has the form

$$L_{int}=T_{\mu\nu}h_{\mu\nu}, \tag{1.3}$$

where $T_{\mu\nu}$ is the energy–momentum tensor of the quantized fields, and $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$ is the deviation of the metric from flatness. In (1.3), summation with the metric tensor $\eta_{\mu\nu}$ is understood.

It can be seen from (1.2) and (1.3) that the amplitude for the production of particles by a weak conformally flat gravitational field is proportional to $T_{\mu\mu}$. Because of the anomaly (1.1), $T_{\mu\mu}$ does not vanish even for massless particles and, thus, the production of massless gauge bosons (and, in higher orders in the coupling constant α, other particles as well) becomes possible.[7]

A similar circumstance has also been noted[8] for massless scalar particles in the $\lambda\varphi^4$ theory (see also Ref. 9). However, in this case the situation is not so unambiguous as in gauge theory. The point is that when the divergent parts are regularized in the energy momentum tensor of the scalar field an anomaly can also arise in $T_{\mu\mu}$

but, in contrast to gauge theory, in which the requirement of gauge invariance fixes the magnitude of the anomalous term, the corresponding term in the $\lambda\varphi^4$ theory is not determined and may be arbitrary, including zero. The last requirement does not seem natural, since the logarithmic renormalizations clearly introduce into the theory a dimensional parameter, breaking thereby the conformal invariance, but formally it cannot be eliminated.

It is natural to consider whether it might not be possible to construct a theory in which $\beta \equiv 0$ [see (1.1)], so that an anomaly does not arise in $T_{\mu\mu}$. It is possible that supersymmetric theories provide an example of this. It was shown in Ref. 10 in a definite supersymmetric model that $\beta = 0$ up to the three-loop graphs. The future will show if this property is preserved in all perturbation orders, and also whether supersymmetric theories bear a relationship to the real world. If the answers to these questions are in the affirmative, we may conclude that massless elementary particles are not produced by a conformally flat metric. But if $\beta \neq 0$, then particles will be produced, in disagreement with the current point of view.

Below, the results of Ref. 7 on the production of massless vector particles obtained in the lowest order in the gravitational field are extended to the case of an arbitrary gravitational field. In Sec. 2, we derive a wave equation with radiative corrections for photons in the metric (1.2). Pauli–Villars regularization is used in the calculation. Although the gauge invariance makes it possible to establish the form of the equation from the known charge renormalization in flat space, an explicit calculation of the diagrams in the metric (1.2) is given in the Appendix. In Sec. 3, we discuss the rate of photon production near the singularity in a Friedmann cosmology, and in Sec. 4 we briefly consider the production of particles with spin 0.

2. WAVE EQUATION WITH RADIATIVE CORRECTIONS FOR PHOTONS IN CONFORMALLY FLAT SPACE

It is well known (see, for example, the reviews of Refs. 11 and 12) that the amplitude for the production of particles in an external field is determined by the solutions of the single-particle wave equation in this field. Suppose that in the limits $t \to \pm\infty$ the field is switched off and the asymptotic behavior of the solution of the wave equation has the form

$$\varphi_{\mathbf{k}} = e^{i\omega t}, \quad t \to -\infty, \qquad \varphi_{\mathbf{k}} = \alpha_{\mathbf{k}} e^{i\omega t} + \beta_{\mathbf{k}} e^{-i\omega t}, \quad t \to +\infty, \tag{2.1}$$

where $\varphi_{\mathbf{k}}$ is the particle wave function in the momentum representation, $\omega = |\mathbf{k}|$, and $|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1$. Then the amplitude for the production of a pair of particles with momenta $\mathbf{k}$ and $-\mathbf{k}$ is $\beta_{\mathbf{k}}/\alpha_{\mathbf{k}}^2$.

Let us consider interacting electron–positron and electromagnetic fields in the metric (1.2). The Lagrangian of of the system together with the massive regulating fermions fields has the form[1)]

$$L = a^4 \{ -\eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta} F_{\mu\nu}/4 + \sum_{j=0}^{2} \bar{\psi}_j [C_j (i\underset{\rightarrow}{\hat{D}} - m_j)/2 - C_j (i\underset{\leftarrow}{\hat{D}} + m_j)/2 + e a^{-1} \hat{A}] \psi_j \}, \tag{2.2}$$

where $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$; γ^μ are the ordinary γ matrices satisfying the condition $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$; $\bar{\psi} = \psi^+ \gamma^4$; and

$$\underset{\rightleftarrows}{\hat{D}} = a^{-1}\gamma^\mu (\underset{\rightleftarrows}{\partial_\mu} + {}^3/_2 \partial_\mu \ln a),$$

where the arrow under the derivative indicates the direction in which it acts, ψ_0 is the operator of the physical electron–positron field, and $\psi_{1,2}$ are the heavy regularizing fields. We set $m_0 = 0$ and $m_{1,2} \to \infty$. The constants C_j satisfy the conditions

$$C_0 = 1, \quad C_0 + C_1 + C_2 = 0, \quad C_1 m_1^2 + C_2 m_2^2 = 0. \tag{2.3}$$

In deriving the expression for the spinor part of the Lagrangian, we have used the tetrad formalism, the details of which can be found, for example, in Weinberg's book.[13]

Going over to new operators of the spinor fields in accordance with

$$\psi = a^{-3/2} \chi,$$

we rewrite the Lagrangian in the form

$$L = -(F_{\mu\nu})^2/4 + \sum_{j=0}^{2} \bar{\chi}_j [C_j (i\underset{\rightarrow}{\hat{\partial}} - m_j a)/2 - C_j (i\underset{\leftarrow}{\hat{\partial}} + m_j a)/2 + e\hat{A}] \chi_j. \tag{2.4}$$

Here and below, summation is performed with the Minkowski metric $\eta_{\mu\nu}$.

It can be seen that for massless fields L reduces to the ordinary Lagrangian of quantum electrodynamics in flat space-time. It follows from this in particular that in the tree approximation there is no production of massless particles in the metric (1.2).

The radiative corrections change the situation. The equation for the single-particle photon wave function with radiative corrections due to the interaction with the electron–positron field has the form

$$\partial_\nu F_{\mu\nu}(x_1) + ie^2 \int dx_2 \Pi_{\mu\nu}(x_1, x_2) A_\nu(x_2) = 0, \tag{2.5}$$

where $\Pi_{\mu\nu}(x_1, x_2)$ is the photon polarization operator. In the lowest order in the electromagnetic interaction, it is

$$\Pi_{\mu\nu}^{(1)}(x_1, x_2) = \sum_j C_j \operatorname{Sp} \gamma_\mu G_j(x_2, x_1) \gamma_\nu G_j(x_1, x_2), \tag{2.6}$$

where G_j is the propagator of the fermion with mass m_j in the metric (1.2); it satisfies the equations

$$[i\underset{\rightarrow}{\hat{\partial}_2} - m_j a(x_2)] G_j(x_1, x_2) = \delta(x_1 - x_2), \tag{2.7a}$$

$$G_j(x_1, x_2) [i\underset{\leftarrow}{\hat{\partial}_1} + m_j a(x_1)] = -\delta(x_1 - x_2). \tag{2.7b}$$

For $m = 0$, G_0 is equal to the propagator of a free electron in flat space. If $m_j \neq 0$, and explicit expression for G_j cannot in general be found. However, since m_j must be set equal to infinity in the final result, the answer will contain $G_j(x_1, x_2)$ in the limit $x = (x_1 - x_2) \to 0$.

We expand $a(x_2)$ in Eq. (2.7a) in powers of x:

$$a(x_2) = a(x_1) + x_\alpha \partial_\alpha a(x_1) + {}^1/_2 x_\alpha x_\beta \partial_\alpha \partial_\beta a(x_1) + \ldots \equiv a(x_1) + V(x, x_1). \tag{2.8}$$

It is clear from dimensional considerations that in this expansion it is sufficient to consider only the terms quadratic in x. The contribution to $\Pi_{\mu\nu}$ from terms of higher order vanishes as $m_j \to \infty$.

One can find the expansion of $G_j(x_1, x_2)$ corresponding to (2.8). This is done in the Appendix, since the calculation can be instructive for other models. However, in our case this is not necessary, since, by virtue of the gauge invariance, $\Pi_{\mu\nu}$ can be recovered from the first term in the expansion of G_j, namely, from the function $G_j^{(0)}$, which satisfies the equation

$$[i\hat{\partial}_2 - m_j a(x_1)]G_j^{(0)}(x_1, x_2) = \delta(x_1 - x_2)$$

and is equal to

$$G_j^{(0)} = \frac{1}{(2\pi)^4}\int \frac{dp e^{ip(x_1-x_2)}}{p^2 - m_j^2 a^2(x_1)}[\hat{p} + m_j a(x_1)]. \tag{2.9}$$

By virtue of the gauge invariance and the transversality of $\Pi_{\mu\nu}$, Eq. (2.5) can be written in the form

$$\int dx_2 dk e^{ik(x_1-x_2)}\left[d^{-1}(k^2, \xi(x_1))\partial_\nu F_{\mu\nu}(x_2) + \partial_\nu\xi(x_1)\frac{\partial d^{-1}(k^2, \xi(x_1))}{\partial\xi}F_{\mu\nu}(x_2)\right] = 0, \tag{2.10}$$

where $\xi(x) = \ln a(x)$. It can be concluded from dimensional arguments that the result does not contain the second derivatives of ξ, since $\partial^2\xi$ must on dimensional grounds be multiplied by the potential A_μ, which contradicts gauge invariance. Note that the result depends on $\xi = \ln a$ but not on a, since a is always contained in the combination ma.

The function $d^{-1}(k^2, \xi)$ can be calculated in exactly the same way as the corresponding quantity in Minkowski space with the only difference that the cutoff parameter (or regulating mass) depends on the coordinates: $\Lambda \to \Lambda a(x)$. In other words, all the calculations are made as in the ordinary theory, but the renormalization constants are functions of the coordinates. This dependence of the renormalizations on the point of space is observable and, in particular, leads to the production of massless particles. In the lowest order in $\alpha = e^2/4\pi$

$$d^{-1}(k^2, \xi) = 1 - \frac{\alpha N_F}{3\pi}\ln\frac{(-k^2)}{\Lambda^2} + \frac{2\alpha N_F}{3\pi}\xi, \tag{2.11}$$

where $\Lambda^2 = (m_1^{C_1} m_2^{C_2})^{-2}$, and N_F is the number of charged fermions. Integrating in (2.10) and extracting the factor $(1 - (\alpha N_F/3\pi)\ln k_0^2/\Lambda^2)$ in the renormalization of the inverse photon propagator (k_0 is an arbitrary normalization point in the momentum space), we obtain

$$\left[1 + \frac{2\alpha N_F}{3\pi}\xi(x)\right]\partial_\nu F_{\mu\nu}(x) + \frac{2\alpha N_F}{3\pi}(\partial_\nu\xi(x))F_{\mu\nu}(x) = 0, \tag{2.12}$$

the wave equation for the single-particle wave function of a photon propagating in the metric (1.2) with allowance for the radiative corrections of order α. Because of the last term in this equation, a solution having the asymptotic behavior $\exp i\omega t$ in the limit $t \to -\infty$ will contain in the limit $t \to +\infty$ both positive and negative frequencies [see (2.1)], i.e., particle production will occur.

Note that in (2.12) we have omitted the nonlocal term

$$\frac{1}{(2\pi)^4}\frac{\alpha}{3\pi}\int dx_2 dk e^{ik(x-x_2)}\ln\frac{k_0^2}{(-k^2)}\partial_\nu F_{\mu\nu}(x_2), \tag{2.13}$$

which makes a contribution of second order in α on the iteration of (2.12). This term is important when higher corrections in α are taken into account. In particular, for strong gravitational fields, when $\alpha\xi \equiv \alpha\ln a \approx 1$, the leading logarithmic terms in d^{-1} can be summed. However, it is known that the result in this case is equal to the lowest order (2.11) of perturbation theory, and therefore Eq. (2.12) with the correction term (2.13) is also valid for $\alpha\ln a \approx 1$; at the same time, the correction terms are of order $\alpha^2\ln a$. One can also have a situation when $\alpha\ln a \ll 1$ but $\alpha\partial_\nu\ln a \gtrsim 1$. Then one can also use Eq. (2.12) provided $\alpha^2\partial_\nu\ln a \ll 1$.

Using the well-known results of quantum electrodynamics, we can take into account the corrections of order α^2 to d^{-1} (two-loop diagrams) and, accordingly, the corrections of the same order in Eq. (2.12) [with allowance for (2.13)].

Note also that Eq. (2.10) with obvious modifications holds not only in quantum electrodynamics but in any gauge theory. In particular, for the gauge group SU(N) with N_F generations of fermions belonging to the fundamental representation, Eq. (2.12) is as before valid if we make the substitution

$$\beta_0 = -\frac{2N_F}{3} \to \beta_N = \frac{11}{3}N - \frac{2N_F}{3}. \tag{2.14}$$

For appropriate choice of N and N_F, the coefficient β_N may vanish, so that particles will not be produced, but this is true only in the lowest order in α. Allowance for terms of higher order in α again leads to particle production.

3. PRODUCTION OF GAUGE BOSONS IN COSMOLOGY

In Sec. 2, we derived Eq. (2.12), which describes the propagation of a photon in conformally flat space. In accordance with the general theory,[11,12] to determine the probability of particle production it is necessary to calculate the coefficient $\beta_{\mathbf{k}}$ [see (2.1)], i.e., the negative-frequency part of the wave function. We note immediately that without allowance for the radiative corrections, i.e., when the second term in (2.12) is ignored, photon propagation is described by the free equation, so that $\beta_{\mathbf{k}} = 0$ and particles are not produced.

In what follows, we shall work in the Lorentz gauge, in which Eq. (2.12) with allowance for the additional term (2.13) has the form

$$[1 + \varkappa\xi(x)]\partial^2 A_\mu(x) + \varkappa\partial_\nu\xi(x)[\partial_\nu A_\mu(x) - \partial_\mu A_\nu(x)] + \frac{\varkappa}{2(2\pi)^4}\int dy dk e^{ik(x-y)}\ln\frac{k_0^2}{(-k^2)}\partial^2 A_\mu(y) = 0, \tag{3.1}$$

where $\xi(x) = \ln a(x)$ and $\varkappa = 2\alpha N_F/3\pi$.

A further simplification arises if $a(x)$ depends only on the conformal time η and not on the spatial coordinates. In this case, scalar and longitudinal photons are not produced by the gravitational field. Indeed, for the time component of the vector potential we have the free equation

$$\partial^2 A_0 = 0,$$

and the longitudinal part $A_\parallel$ is related to the scalar component by the gauge condition

$$i|\mathbf{k}|\tilde{A}_\parallel(\eta, \mathbf{k}) + \partial_\eta\tilde{A}_0(\eta, \mathbf{k}) = 0, \tag{3.2}$$

where $\tilde{A}_\mu$ is the Fourier transform of the potential A_μ with respect to the spatial coordinates:

$$\tilde{A}_\mu(\eta, \mathbf{k}) = \int d^3x e^{i\mathbf{k}\mathbf{x}} A_\mu(\eta, \mathbf{x}).$$

Note that $\tilde{A}_\parallel$ defined in accordance with Eq. (3.2) satisfies Eq. (3.1).

Because of this, we can ignore A_0 and $A_\parallel$, and restrict ourselves to "normal" transverse photons, i.e., ones described by a potential $\mathbf{A}_\perp$ such that $\mathbf{k}\cdot\mathbf{A}_\perp=0$. If the scale factor $a(x)$ depends on the spatial coordinates, this would be incorrect and it would be necessary to take into account not only the transverse but also longitudinal modes, i.e., the photon would appear to acquire a mass in the gravitational field.

Returning to the case of a homogeneous and isotropic metric $a=a(\eta)$ and assuming that the field is weak, i.e., $\alpha \ln a \equiv \alpha\xi \ll 1$ (but not necessarily $\alpha\partial_\eta\xi \ll 1$), we obtain from (3.1)

$$(\partial_\eta^2+\mathbf{k}^2)\tilde{A}_\perp+\varkappa\partial_\eta\xi\partial_\eta\tilde{A}_\perp=0. \tag{3.3}$$

This equation describes the production of photons in the considered gravitational field. In particular, in the lowest order in the field we readily obtain for the coefficient $\beta_\mathbf{k}$ [see (2.1)]

$$\beta_\mathbf{k}=\frac{2\alpha N_F}{3\pi}|\mathbf{k}|\int_{-\infty}^{+\infty} d\eta e^{2i|\mathbf{k}|\eta}\xi(\eta). \tag{3.4}$$

An explicit expression for $\beta_\mathbf{k}$ can also be readily obtained for short-wavelength photons; it is related trivially to the coefficient of transmission through the potential barrier in the quasiclassical limit.

Note that in the weak-field limit it is possible to speak of the local rate of particle production[14] in unit volume in unit time:

$$W=\frac{\alpha^2}{2^5\pi}\left(\frac{2N_F}{3}\right)^2|\partial_t^2\xi(t)|^2 \tag{3.5}$$

(in this approximation, the conformal time η is equal to the physical time t; the exact connection between them is $dt=a d\eta$).

If we use this formula near the cosmological singularity, where $\xi \sim \ln t$, we find that the rate of particle production is proportional to t^{-4}. Such behavior is well known for the case of an anisotropic metric.[5,14]

Comparing the energy density ρ of the produced particles with the cosmological energy density in a Friedmann cosmology, $\rho_c=(3/32\pi)(t_P t)^{-2}$, we find for their ratio

$$\frac{\rho}{\rho_c}\approx(\alpha\beta_N)^2\left(\frac{t_P}{t}\right)^2, \tag{3.6}$$

where t_P is the Planck time, β_N is determined by (2.14), and $\alpha \approx 0.02$ is the gauge coupling constant at energies of order t_P^{-1}. Unfortunately, ρ becomes of order ρ_c at $t \approx 0.1 t_P$, i.e., outside the region of applicability of formula (3.5), which is based on perturbation theory with respect to the gravitational field.

We note in conclusion that for $\partial_t\xi=\text{const}$, i.e., for $a=\exp Ht$, Eq. (3.3) shows that particle production does not occur (in the approximation described by this equation). However, this is due to the employed approximation and does not hold in higher orders in α.

4. REASONS FOR THE OCCURRENCE OF THE CONFORMAL ANOMALY AND THE POSSIBILITY OF PRODUCTION OF SCALAR PARTICLES

In the calculations made above, we used Pauli–Villars regularization, which is gauge invariant but conformally noninvariant. It may be asked to what extent the result we have obtained above is unambiguous or, in other words, whether there exists a regularization that would preserve both invariances. The answer to this question is negative, which can be seen best by considering the imaginary part of the amplitude for the transition of a virtual graviton into two photons, $A(g\to 2\gamma)=h_{\mu\nu}T_{\mu\nu}$, through a real electron–positron pair following the treatment in Ref. 15 of the anomaly in the Ward identity for the axial current (in this section, gravitation is considered in the lowest order of perturbation theory).

As we note above, conformal invariance is not violated at the level of the tree diagrams, and for this reason it also holds for the imaginary part $A(g\to 2\gamma)$. It would appear that, recovering the total amplitude from its imaginary part by means of a dispersion relation, one could also achieve conformal invariance for the amplitude as a whole. In reality, however, the situation is different, and conformal invariance is broken even for the imaginary part of the amplitude, and this shows that there is no way in which such breaking can be eliminated. The point is that, as a direct calculation shows, the imaginary part of the energy–momentum tensor of the photons contains, because of e^+e^- in the intermediate state, a term proportional to

$$\operatorname{Im} T_{\mu\nu}' \sim \frac{m^2}{q^4}(q_\mu q_\nu-q^2\eta_{\mu\nu})F_{\alpha\beta}^2,$$

where q is the graviton momentum, and m is the electron mass. It is obvious that $\operatorname{Sp}\operatorname{Im} T_{\mu\nu}'=0$ for $m=0$, so that formally the requirements of conformal invariance are observed. However, because of the pole at $q^2=0$ the transition to the limit must be made more accurately. It is readily seen that the dispersion integral

$$\int_{4m^2}^{\infty}\frac{dz m^2}{z^2(z-q^2)}$$

tends to q^{-2} as $m\to 0$, and this means that

$$\operatorname{Im} T_{\mu\nu}\to\delta(q^2)(q_\mu q_\nu-q^2\eta_{\mu\nu})F_{\alpha\beta}^2\cdot\text{const},\ m\to 0.$$

Thus, the theory contains a singularity corresponding to a massless scalar particle, and this necessarily leads to an anomaly in the trace of the energy–momentum tensor.

Taking into account this circumstance, let us consider the production by a gravitational field of scalar particles in the $\lambda\varphi^4$ theory. The energy–momentum tensor in such a theory has the form

$$T_{\mu\nu}=\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{2}(\partial_\alpha\varphi)^2\eta_{\mu\nu}+\left(\frac{m^2}{2}\varphi^2+\frac{\lambda}{4!}\varphi^4\right)\eta_{\mu\nu}+\xi(\eta_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)\varphi^2. \tag{4.1}$$

Using the equation of motion

$$\left(\partial^2+m^2+\frac{\lambda}{3!}\varphi^2\right)\varphi=0, \tag{4.2}$$

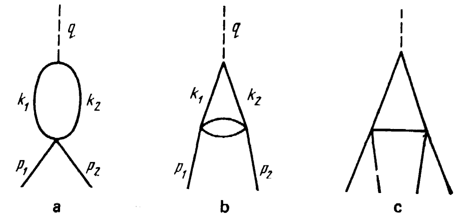

FIG. 1.

we readily see that the tensor $T_{\mu\nu}$ satisfies the conservation law $\partial_\mu T_{\mu\nu}=0$ and for $\xi=\frac{1}{6}$ and $m=0$ the condition $T_{\mu\mu}=0$ of a vanishing trace.

In accordance with what we have said above, we shall seek breaking of conformal invariance in the loop diagrams, analyzing their imaginary part with respect to the momentum q transferred to the gravitational field. In the first order in λ, only diagram a in Fig. 1 is relevant. Its imaginary part is

$$\operatorname{Im} A_{\mu\nu}^{(a)}={}^1/_2\lambda\int d\tau(k_1,k_2)\,[k_{1\mu}k_{2\nu}+k_{1\nu}k_{2\mu}$$

$$+(m^2-k_1k_2)\eta_{\mu\nu}+2\xi(\eta_{\mu\nu}q^2-q_\mu q_\lambda)\,]=\frac{\lambda}{96\pi}\operatorname{Im} g(q^2)\ (q^2\eta_{\mu\nu}-q_\mu q_\nu),\qquad(4.3)$$

where

$$\operatorname{Im} g(q^2)=\left(1-\frac{4m^2}{q^2}\right)^{1/2}\left(12\xi-3+\frac{q^2-4m^2}{q^2}\right),$$

and $d\tau(k_1,k_2)$ is the element of phase space. We can recover the total amplitude $A_{\mu\nu}^{(a)}$ from its imaginary part if we write down for $g(q^2)$ a dispersion relation, which for $\xi=\frac{1}{6}$ can be an unsubtracted relation. The subtractional constants are arbitrary if no additional requirements are imposed on the theory. If these constants are set equal to zero (which corresponds to a conformally invariant augmenting of the definition of the theory), no anomalies arise in $A_{\mu\nu}^{(a)}$. For $m=0$, this amplitude vanishes. Therefore, the theory can be formulated in such a way that no particles are produced in the first order in λ.

In the second order in λ, diagrams b and c in Fig. 1 are relevant. The second of them, which corresponds to transition of a graviton into four scalar particles, obviously cannot lead to an anomaly, since it diverges only logarithmically. For the imaginary part of the quadratically diverging diagram b we can write down the following representation after charge renormalization:

$$\operatorname{Im} A_{\mu\nu}^{(b)}=\frac{\lambda^2}{32\pi^2}\int_{4m^2}^{\infty}\frac{ds}{s}\left(1-\frac{4m^2}{s}\right)^{1/2}\int d\tau(k_1,k_2)$$

$$\times\frac{(p_1-k_1)^2}{s-(p_1-k_1)^2}\,[{}^1/_6(q_\mu q_\nu-q^2\eta_{\mu\nu})-{}^1/_2 r_\mu' r_\nu'\,],$$

where $r'=k_1-k_2$ and we have set $\xi=\frac{1}{6}$. Integrating over the phase space, we readily obtain

$$\operatorname{Im} A_{\mu\nu}^{(b)}=\frac{\lambda^2}{32\pi^2}(q^4-4m^2q^2)^{-1/2}\int_{4m^2}^{\infty}ds\left(1-\frac{4m^2}{s}\right)^{1/2}$$

$$\times\left[(q^2g_{\mu\nu}-q_\mu q_\nu)f_1\left(\frac{s}{q^2-4m^2}\right)+r_\mu r_\nu f_2\left(\frac{s}{q^2-4m^2}\right)\right],\qquad(4.4)$$

where $r=p_1-p_2$,

$$f_1(x)=\frac{2m^2}{3q^2x}-{}^1/_6\ln(1+x^{-1})-{}^1/_2\left(1-\frac{4m^2}{q^2}\right)[2x(1+x)\ln(1+x^{-1})-1-2x],$$

$$f_2(x)={}^3/_2+3x-{}^1/_2\ln(1+x^{-1})-3x(1+x)\ln(1+x^{-1}).$$

It can be seen that $\operatorname{Im}A_{\mu\nu}^{(b)}$ does not acquire a singularity of the form m^2q^{-4}, which in the limit $m\to0$ goes over into $\delta(q^2)$ (or into q^{-2} in the total amplitude). Therefore, in the calculation of $A_{\mu\nu}^{(b)}$ by means of the dispersion relation we can always choose the subtraction constants to make $A_{\mu\mu}^{(b)}$ vanish for $m=0$. In fact, this is true for any amplitude for transition of a graviton into two scalar particles and is based solely on dimensional arguments. To see this, we note that the general structure of the imaginary part of the amplitude is

$$\operatorname{Im} A_{\mu\nu}=\operatorname{Im} g_1\left(\frac{m^2}{q^2}\right)(\eta_{\mu\nu}q^2-q_\mu q_\nu)+\operatorname{Im} g_2\left(\frac{m^2}{q^2}\right)r_\mu r_\nu,$$

where the functions g_1 and g_2 are dimensionless. If for $\operatorname{Im}g_i$ we have the condition $3\operatorname{Im}g_1(0)=\operatorname{Im}g_2(0)$, which is a consequence of the vanishing of $T_{\mu\mu}$ for $m=0$, then it is obvious that this condition can also be recovered for the real parts by means of a suitable polynomial subtraction.

Thus, for scalar particles it is always possible to make a regularization which ensures that the matrix elements of the trace of the energy–momentum tensor vanish. As a result, scalar particles will not be produced by the field (1.2).

I thank A. I. Vaĭnshteĭn, V. I. Zakharov, and Ya. B. Zel'dovich for helpful discussions.

APPENDIX

Explicit calculation of the polarization operator of a photon in conformally flat space

In the lowest order in α, the operator $\Pi_{\mu\nu}$ is determined by Eq. (2.6), so that the problem reduces to determination of the functions G_j using Eqs. (2.7) (in what follows, we shall omit the subscript j). We shall seek $G(x_1,x_2)$ in the limit $x\equiv x_1-x_2\to0$ in the form of the series

$$G(x_1,x_2)=\sum_{\lambda=0}G^{(\lambda)}(x_1,x_2),\qquad(\text{A.1})$$

$$[i\hat\partial_2-ma(x_1)\,]G^{(0)}(x_1,x_2)=\delta(x_1-x_2),\qquad(\text{A.2})$$

$$[i\hat\partial_2-ma(x_1)\,]G^{(\lambda)}(x_1,x_2)=m[a(x_2)-a(x_1)\,]G^{(\lambda-1)}(x_1,x_2).\qquad(\text{A.3})$$

The solution of Eq. (A.2) is determined by (2.9). For known $G^{(0)}$, the functions $G^{(\lambda)}$ can be found in accordance with the equations

$$G^{(\lambda)}(x_1,x_2)=m\int dyG^{(0)}(x_1,x_1+x_2-y)\,[a(y)-a(x_1)\,]G^{(\lambda-1)}(x_1,y).\qquad(\text{A.4})$$

The difference $a(x_2)-a(x_1)$ must be expanded in the series (2.8), and it clear from dimensional arguments that the final results can contain terms of not higher than the second or the square of the first derivative of $a(x_1)$. Therefore, we shall be interested in only $G^{(1)}$ and $G^{(2)}$, and in the latter function take into account only the contribution $\partial_\mu a(x_1)$ (but not $\partial_\mu\partial_\nu a$).

We introduce the notation

$$\Pi_{\mu\nu}^{(\lambda\sigma)}(x_1,x_2)=\sum_j C_j\,\mathrm{Sp}\,\gamma_\mu G_j^{(\lambda)}(x_2,x_1)\gamma_\nu G_j^{(\sigma)}(x_1,x_2).\qquad(\text{A.5})$$

Here, $\Pi_{\mu\nu}^{(00)}$ is known, and its contribution to Eq. (2.5) is

(for one fermion species)

$$\frac{\alpha}{3\pi}\sum_{j=0} C_j\int dx_2\,\frac{dk}{(2\pi)^4}e^{ik(x_1-x_2)}\ln\frac{(-k^2)}{m^2a^2(x_1)}\,(\partial_\mu\partial_\nu-\eta_{\mu\nu}\partial^2)A_\nu(x_2).$$

It is only this contribution that was needed to recover Eq. (2.12). One can however show directly that gauge invariance holds. This is done in the following calculations.

The contribution to Eq. (2.5) from $\Pi_{\mu\nu}^{(10)}$ is (we do not write out the sum over the different regularizers $j=1,2$)

$$\Delta_\mu^{(10)}=\int dx[A_\nu(x_1)+x_\sigma\partial_\sigma A_\nu(x_1)]\int dy[y_\alpha\partial_\alpha M+{}^1/_2 y_\alpha y_\beta\partial_\alpha\partial_\beta M] \tag{A.6}$$

$$\times\int\frac{dp_1dp_2dk}{(2\pi)^{12}}\exp\{ip_1(y-x)-ip_2y+ikx\}\frac{\mathrm{Sp}\,\gamma_\nu(p_1+M)(p_2+M)\gamma_\mu(k+M)}{(p_1^2-M^2)(p_2^2-M^2)(k^2-M^2)},$$

where $M=ma(x_1)$; the function $A_\nu(x_2)$ in (2.5) was expanded in powers of $x=x_2-x_1$, and it was only necessary to retain the first two terms.

Replacing multiplication by x_α and y_α by differentiation of the last factor (in the square brackets) with respect to $\partial/\partial k_\alpha$ and $\partial/\partial p_{2\alpha}$, respectively, noting further that the integration with respect to $\partial x\partial y$ gives $\delta(k-p_1)\times\delta(p_1-p_2)$, and calculating the trace of the product of the γ matrices, we obtain

$$\Delta_\mu^{(10)}=4M\partial_\alpha M\partial_\sigma A_\nu(x_1)\int\frac{dp}{(2\pi)^4}(p^2-M^2)^{-3}$$
$$\times\Big[\eta_{\alpha\nu}\eta_{\mu\sigma}+\eta_{\nu\sigma}\eta_{\mu\alpha}-\eta_{\mu\nu}\eta_{\alpha\sigma}-\frac{p^2}{p^2-M^2}(2\eta_{\alpha\mu}\eta_{\nu\sigma}$$
$$+\eta_{\mu\sigma}\eta_{\alpha\nu}-\eta_{\mu\nu}\eta_{\alpha\sigma})+\frac{p^2(p^2+M^2)}{(p^2-M^2)^2}\eta_{\mu\nu}\eta_{\alpha\sigma}+\frac{p^4}{(p^2-M^2)^2}(-{}^4/_3\eta_{\alpha\sigma}\eta_{\mu\nu}+{}^2/_3\eta_{\alpha\nu}\eta_{\mu\sigma}$$
$$+{}^2/_3\eta_{\alpha\mu}\eta_{\nu\sigma})\Big]-2M\partial_\alpha\partial_\beta M\cdot A_\nu(x_1)\int\frac{dp}{(2\pi)^4}(p^2-M^2)^{-4}$$
$$\times\Big[-p^2(\eta_{\alpha\mu}\eta_{\beta\nu}+\eta_{\alpha\nu}\eta_{\beta\mu})-2\eta_{\alpha\beta}\eta_{\mu\nu}(p^2+M^2)$$
$$+{}^4/_3\frac{p^4}{p^2-M^2}(\eta_{\alpha\beta}\eta_{\mu\nu}+\eta_{\alpha\mu}\eta_{\beta\nu}+\eta_{\alpha\nu}\eta_{\beta\mu})\Big].$$

The integration with respect to dp can be readily performed; as a result, as one would expect, the terms proportional to $\partial_\alpha\partial_\beta M$ disappear, and we obtain

$$\Delta_\mu^{(10)}=\frac{1}{12\pi^2}\partial_\sigma A_\nu(x_1)\partial_\alpha(\ln a(x_1))(\eta_{\sigma\mu}\eta_{\alpha\nu}-\eta_{\mu\nu}\eta_{\alpha\sigma}).$$

We have here summed over j [see (A.5)] and used the condition $C_1+C_2=-1$. It is easy to see that $\Delta_\mu^{(10)}=\Delta_\mu^{(01)}$; this leads to a doubling of the considered contribution.

It can be verified similarly that the contributions of $\Pi_{\mu\nu}^{(11)}$, $\Pi_{\mu\nu}^{(20)}$ and $\Pi_{\mu\nu}^{(02)}$ vanish. In the expansion of $A_\nu(x_2)$ [see (A.6)] it is necessary to take only the zeroth term $A_\nu(x_1)$, and in the Green's functions $G^{(\lambda)}(x_1,x_2)$ only the terms quadratic in $\partial_\mu a(x_1)$. After a fairly long calculation it can be shown that zero is finally obtained, as one would expect on the basis of the requirements of gauge invariance.

1) To fix the gauge, it is necessary to add to the Lagrangian the term $-1/2(D_\mu A^\mu-2A^\lambda\partial_\lambda\ln a)^2$ which ensures the condition $\eta^{\mu\lambda}\partial_\lambda A_\mu\equiv\partial_\mu A_\mu=0$ [here, D_μ is the covariant derivative in the metric (2.1)].

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Translated by Julian B. Barbour