On the theory of the thermal resistance of the intermediate state of superconductors

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The thermal conductivity (across the layers) of the intermediate state of a superconductor at low temperatures ($T \ll \Delta$) is calculated. It is shown that the effective thermal resistance of a N–S boundary differs from the Andreev resistance for two reasons. First, because only the quasiparticles with energies $\varepsilon > \Delta$ can cross the N–S boundary, the energy in a normal-metal layer of thickness equal to the diffusion length for the energy relaxation is transported precisely by such quasiparticles, and the thermal conductivity in this layer coincides with the thermal conductivity of the superconductor. If the diffusion length exceeds the thickness of the normal layer, the entire layer conducts heat like a superconductor. Secondly, because of the state of nonequilibrium of the quasiparticle distribution function near the boundary, the boundary thermal resistance itself differs from the Andreev resistance by a quantity of the order of unity.

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1. INTRODUCTION

The thermal resistance of the intermediate state at low temperatures was first investigated by Andreev.[1] In doing this, he assumed that the entire temperature jump is concentrated at the boundary between the normal and superconducting phases, and that the distribution functions are practically equilibrium distribution functions in the entire volume of the sample. In the present paper we investigate the problem in its general formulation, in which the distribution functions are of essentially nonequilibrium character. The results for certain relations between the domain dimensions and the relaxation lengths then differ from the corresponding results obtained by Andreev. The boundary thermal resistance is found to consist of two parts. The first part is the boundary resistance proper. Its magnitude differs from the Andreev resistance by a coefficient of the order of unity, that depends on the relation between the layer thicknesses and the diffusion lengths (the diffusion lengths L_N and L_S for the energy relaxations in the normal and superconducting phases, respectively, and the diffusion length $\mathscr{L}_S$ for the establishment of the equilibrium quasiparticle number in the superconductor). This coefficient is connected with the state of nonequilibrium, and becomes equal to unity when the assumptions made by Andreev[1] are valid.

The second part is the resistance of that diffusion-length-thick region of the normal domain which is contiguous to the boundary. This region, in which the heat flux is transported by the electrons with energy Δ, and whose thermal conductivity therefore coincides with the thermal conductivity of the superconductor, arises because of the fact that only the electrons with energy higher than Δ pass through the boundary and the distribution function relaxes in terms of energy over a diffusion length. Besides the boundary contributions, there are, naturally, contributions due to the finite thermal conductivities of the normal and superconducting domains.

The final answer has the following form:

$$\frac{a_N+a_S}{\varkappa_i}=\frac{a_N-2L_N^*}{\varkappa_N}+\frac{a_S+2L_N^*}{\varkappa_S}+W_A\Phi(L_A, L_T, L_Q). \tag{1}$$

Here $\varkappa_i$ is the sought thermal conductivity of the intermediate state,

$$W_A=\frac{2\pi^{3/2}l}{9\varkappa_N}\, e^{\Delta/T}\left(\frac{\Delta}{T}\right)^{-3/2} \Big/ \int_0^1 f(\mu)\,\mu\, d\mu$$

is the boundary thermal resistance obtained by Andreev,

$$w(\varepsilon, \mu)=\left(\frac{\varepsilon-\Delta}{\Delta}\right)^{1/2} f(\mu)$$

is the coefficient of transmission of the quasiparticles through the $N-S$ boundary, and μ is the cosine of the angle of incidence at the boundary[1] for $0<\varepsilon-\Delta \ll \Delta$. Further,

$$\varkappa_S=\frac{6}{\pi^2}\varkappa_N\left[\left(\frac{\Delta}{T}\right)^2+2\frac{\Delta}{T}+2\right]e^{-\Delta/T}$$

is the electronic thermal conductivity of the superconductor, $\varkappa_N$ is the thermal conductivity of the normal metal, l is the mean free path, a_N and a_S are the thicknesses of the normal and superconducting layers,

$$L_N^*=L_N \operatorname{th}(a_N/2L_N),$$

$L_N=[D\tau_\varepsilon^N(\Delta)]^{1/2}$ is the diffusion length for the energy relaxation, $D=lv_F/3$, $\tau_\varepsilon^N(\Delta)\sim\hbar\Theta_D^2/\Delta^3$, Θ_D is the Debye temperature (in energy units), Δ is the superconducting gap,

$$L_A=2l\left(\frac{\Delta}{T}\right)^{1/2} \Big/ 3\int_0^1 f(\mu)\,\mu\, d\mu,$$

$L_T=L_N^*+L_S^*$ is the effective thermalization length,

$$L_S^*=L_S \operatorname{th}\frac{a_S}{2L_S}, \quad L_S=(D\tau_\varepsilon^S)^{1/2}, \quad \tau_\varepsilon^S\sim\frac{\hbar\Theta_D^2}{T^3}\left(\frac{\Delta}{T}\right)^{1/2}$$

$$L_Q=\mathscr{L}_S \operatorname{th}\frac{a_S}{2\mathscr{L}_S}-L_S^*, \quad \mathscr{L}_S=(D\tau_S^>)^{1/2}, \quad \tau_S^>\sim\frac{\hbar\Theta_D^2}{\Delta^3}\frac{\Delta}{T}e^{2\Delta/T}\gg\tau_\varepsilon^S,$$

and $\tau_S^>$ is the quasiparticle-number relaxation time in the superconductor.[2,3]

The dependence of Φ on L_A is shown in Fig. 1 in the case in which $L_Q>L_T$. For $L_Q<L_T$ the L_A dependence of Φ

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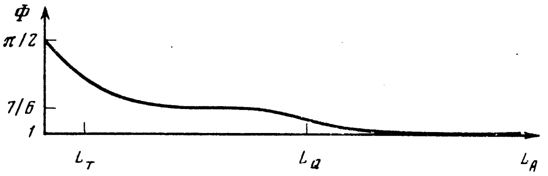

FIG. 1. Form of the function $\Phi(L_A, L_T, L_Q)$.

does not exhibit a plateau. The exact formula is given in Sec. 3. It is assumed that the inequality $l \ll L_{N,S} \ll \mathscr{L}_S$ is fulfilled. The introduction of the energy-independent relaxation times τ_ε^N, τ_ε^S, and $\tau_\varepsilon^>$ is a crude approximation, but all the asymptotic forms of the function Φ do not depend on this approximation.

Why does the Andreev boundary thermal resistance change when the relaxation lengths change? If the transmission coefficient tends to zero, then the distribution functions on both sides of the boundary should tend to the equilibrium distribution function. The thermal resistance of the boundary is then given by the Andreev expression. But as will be shown below, the criterion for its validity is the smallness of w not in comparison with unity, but in comparison with some quantity that is itself small compared to unity. Indeed, there arises near the boundary in the presence of a heat flux a region in which the distribution functions are nonequilibrium distribution functions, and whose contribution to the thermal resistance can compete with the boundary contribution because of the large dimension of the region. Therefore, the distribution function is readjusted in such a way as to decrease the total resistance as much as possible, as is required by the principle of minimal entropy-generation rate. Then we may, depending on the relation between the relaxation lengths, find it advantageous to lose out on the boundary thermal resistance by sharply decreasing at its expense the contribution of the boundary regions. This is precisely the situation described by the function Φ when the criterion for the existence of a state of nonequilibrium contains, besides the transmission coefficient, the relaxation lengths, or the layer thicknesses if they are smaller than the relaxation lengths.

2. KINETIC EQUATION AND THE BOUNDARY CONDITIONS

In the absence of external fields the kinetic equation for the quasiparticles has the same form in the S and N regions:

$$\mu l \operatorname{sign}\xi \frac{\partial n}{\partial x} = \bar{n} - n. \tag{2}$$

Here we have taken into account only the collisions (described by the mean free path l) with the impurities, $\xi = p^2/2m - \varepsilon_F$, n is the quasiparticle distribution function, $\mu = p_x/p$, and the bar denotes averaging over the directions of the momenta.

We find it convenient to represent the distribution function in the form:

$$n = n_0 + \frac{\partial n_0}{\partial \varepsilon}(Q + P \operatorname{sign}\xi), \tag{3}$$

where Q and P are even functions of ξ with a characteristic variation scale $\xi \sim \Delta$. Substituting (3) into (2), we obtain the system of equations:

$$\mu l \frac{\partial P}{\partial x} = \bar{Q} - Q, \tag{4}$$

$$\mu l \frac{\partial Q}{\partial x} = -P. \tag{5}$$

We have taken into account the fact that Q is an even, while P is an odd, function of μ.

If the domain thickness is sufficiently large, then the inelastic collisions should be taken into account at large (larger than l) distances from the boundary, where $Q = \bar{Q}$. In the N region the energy relaxation is described by the diffusion equation

$$\frac{\partial^2 Q_N}{\partial x^2} + \frac{Q_N^T - Q_N}{L_N^2} = 0; \quad Q_{N,S}^T \frac{\partial n_0}{\partial \varepsilon} = -\frac{\varepsilon}{T}\frac{\partial n}{\partial \varepsilon}(T(x) - T), \tag{6}$$

where T is the mean temperature of the sample.

The quantity $T(x)$ is determined from the condition

$$\int d^3p \frac{\partial n_0}{\partial \varepsilon} |\xi| (Q_N^T - Q_N) = 0. \tag{7}$$

This equation describes the fact that the phonons effecting the energy relaxation do not, as a result of their low velocity, transport energy in space, but only redistribute it among the quasiparticles at each point, striving to establish local equilibrium with the given energy. For simplicity, L_N is assumed to be energy independent. As we shall see below, Eq. (6) describes the relaxation of the quasiparticles having energies ranging from $\varepsilon \sim \Delta$ near the boundary to $\varepsilon \sim T$ in the interior of the normal domain (if its thickness turns out to be sufficient). Therefore, the phonons participating in such a process also have energy $\hbar\omega \sim \Delta$, and $\tau_N^\varepsilon \sim \hbar\Theta_D^2/\Delta^3$. The situation is more complex in superconductor.

As has been demonstrated,[2,3] there are two relaxation lengths in a superconductor; the first one L_S is the distance over which there gets established the Fermi function ($Q_S^{T,\nu}$) with quasiparticle chemical potential given (together with the temperature) by the conditions for energy and quasiparticle-number conservation:

$$\int d^3p \frac{\partial n_0}{\partial \varepsilon} |\xi| (Q_S^{T,\nu} - Q_S) = 0, \tag{8}$$

$$\int d^3p \frac{\partial n_0}{\partial \varepsilon} \frac{|\xi|}{\varepsilon} (Q_S^{T,\nu} - Q_S) = 0, \tag{9}$$

where

$$Q_S^{T,\nu} \frac{\partial n_0}{\partial \varepsilon} = \left[-\nu(x) - \frac{\varepsilon}{T}(T(x) - T)\right] \frac{\partial n_0}{\partial \varepsilon},$$

and corresponds to the appearance of a local-equilibrium Fermi function with a given quasiparticle chemical potential $\nu(x)$ and temperature $T(x)$. This is due to the following: The quasiparticles, being scattered by the low-frequency phonons, acquire the distribution function $\exp[-(\nu - \varepsilon)/T]$. There arise as a result of the generation and recombination processes high-frequency phonons that follow the quasiparticles, and have the distribution function[2,3] $\exp[-(2\nu - \hbar\omega)/T]$. Each of such phonons is equivalent to two quasiparticles, and the number of quasiparticles changes only in the processes in which such phonons are absorbed by quasiparticles, which then give the obtained energy to the low-frequency phonons. The probability for this process is proportional to the square of the quasiparticle concentration, and is exponentially small. Therefore, the diffusion length connected with the variation of the quasiparticle number is large.

There gets established over this length $\mathscr{L}_S \gg L_S$ a distribution function (Q_S^T) with temperature such that

$$\int d^3p \frac{\partial n_0}{\partial \varepsilon}|\xi|(Q_S^T - Q_S) = 0. \tag{10}$$

Therefore, the diffusion equation in the superconductor has the form

$$\frac{\partial^2 Q_S}{\partial x^2} + \frac{Q_S^{T,\nu} - Q_S}{L_S^2} + \frac{Q_S^T - Q_S}{\mathscr{L}_S^2} = 0. \tag{11}$$

To derive for Eqs. (4), (5), (6), and (11) boundary conditions corresponding to the Andreev reflection, let us write the condition for balance of the fluxes at the $N-S$ boundary:

$$n_N(v<0) = n_N(v>0)(1-w) + w n_S(v<0),$$
$$n_S(v>0) = n_S(v<0)(1-w) + w n_N(v>0),$$

or

$$n_N(v>0) - n_N(v<0) = n_S(v>0) - n_S(v<0) = w[n_N(v>0) - n_S(v<0)].$$

Here $n(v>0)$ denotes the number of quasiparticles moving from the left to the right at some angle to the $N-S$ boundary, while $n(v<0)$ denotes the number of quasiparticles moving from the right to the left at the same angle; $n(v>0)$ and $n(v<0)$ thus describe quasiparticles that differ only in the sign of ξ.

Similarly, at the $S-N$ boundary

$$n_N(v>0) - n_N(v<0) = n_S(v>0) - n_S(v<0) = w(n_S(v>0) - n_N(v>0)).$$

Substituting the expression (3), we obtain

$$P_N = P_S = \pm \frac{w}{2(1-w)} \operatorname{sign} \mu (Q_N - Q_S)_b. \tag{12}$$

The upper sign corresponds to the $N-S$ boundary; the lower sign, to the $S-N$ boundary. From Eq. (4) it follows that the quantity $\int_{-1}^{1} P(\mu)\mu d\mu$ does not depend on the coordinate; from (12) we have the relation

$$\int_{-1}^{1} P(\mu)\mu d\mu = \int_0^1 \frac{w}{1-w}(Q_N - Q_S)_b \mu d\mu. \tag{13}$$

On the other hand, Eqs. (4) and (5) can be represented in the form $Q = Q_0 + \tilde{Q}$, where $\tilde{Q}$ vanishes at points lying at distances greater than l from the boundary, while Q_0 behaves in (4) and (5) like a constant, changing only over distances $L_{N,S} \gg l$.

If $w \ll 1$, i.e., if $T \ll \Delta$, we can replace Q by Q_0 in (13). Indeed, since the right-hand side of (13) is the only inhomogeneity in the system of equations, $\tilde{Q} \sim w(Q_{0N} - Q_{0S})_b \ll Q_{0N} - Q_{0S}$, i.e., $\tilde{Q}$ can be neglected. Then

$$\int_{-1}^{1} P(\mu)\mu d\mu = \left(\frac{\varepsilon - \Delta}{\Delta}\right)^{1/2} (Q_{0N} - Q_{0S})_b \int_0^1 f(\mu)\mu d\mu. \tag{14}$$

Using (8), multiplying by μ, and integrating over the angles, we find that far from the boundary

$$\left.\frac{\partial Q}{\partial x}\right|_{l \ll x \ll L_{N,S}} = \frac{\bar{w}}{l}(Q_{0S} - Q_{0N}) = g_0(\varepsilon), \tag{15}$$

$$\bar{w} = \left(\frac{\varepsilon - \Delta}{\Delta}\right)^{1/2} \frac{3}{2} \int_0^1 f(\mu)\mu d\mu.$$

The relation (15) is one of the boundary conditions for Eqs. (5) and (11). Further, let us write down the boundary conditions, obtained in Ref. 2, for the temperature and the chemical potential in the superconductor:

$$\left.\frac{\partial T}{\partial x}\right|_b = \frac{(\Delta/T)^2 + 2\Delta/T + 2}{\varkappa_S}\left[\left(\frac{\Delta}{T} + 1\right) JT - q\right], \tag{16}$$

$$\left.\frac{\partial \nu}{\partial x}\right|_b = \frac{(\Delta/T)^2 + 2\Delta/T + 2}{\varkappa_S}\left[\left(\frac{\Delta}{T} + 1\right) q - \left(\left(\frac{\Delta}{T}\right)^2 + 2\frac{\Delta}{T} + 2\right) JT\right], \tag{17}$$

where q and J are respectively the energy and quasiparticle-number fluxes passing through the boundary:

$$q = \varkappa_N \frac{6}{\pi^2} \int_\Delta^\infty e^{-\varepsilon/T} \frac{\varepsilon}{T} g_0(\varepsilon) \frac{d\varepsilon}{T}, \tag{18}$$

$$J = \frac{\varkappa_N}{T} \frac{6}{\pi^2} \int_\Delta^\infty e^{-\varepsilon/T} g_0(\varepsilon) \frac{d\varepsilon}{T}, \tag{19}$$

$g_0(\varepsilon)$ being the yet-to-be-determined unknown energy function (15).

3. COMPUTATION OF THE THERMAL CONDUCTIVITY

To find the thermal conductivity, we must solve Eqs. (6) and (11) with the appropriate boundary conditions. The solution to Eq. (6) with the boundary condition (15) in the N region has the form

$$\partial Q_N/\partial x = g_N(x) = g_N^T + (g_0(\varepsilon) - g_N^T) \operatorname{ch}(x/L_N) \operatorname{ch}^{-1}(a_N/2L_N), \tag{20}$$

$$g_N^T = \frac{\partial Q_N^T}{\partial x} = -\frac{\varepsilon}{T}\nabla T_N = \frac{\varepsilon}{T}\frac{q}{\varkappa_N}. \tag{21}$$

The relation (18) guarantees the fulfillment of the energy conservation law (7). The solution to Eq. (11) in the S region is

$$g_S(x) = \partial Q_S/\partial x = g_S^T + (g_S^{T,\nu}|_b - g_S^T) \operatorname{ch}(x/\mathscr{L}_S) \operatorname{ch}^{-1}(a_S/2\mathscr{L}_S)$$
$$+ (g_0(\varepsilon) - g_S^{T,\nu}|_b) \operatorname{ch}(x/L_S) \operatorname{ch}^{-1}(a_S/2L_S), \tag{22}$$

$$g_S^T = -\frac{\varepsilon}{T}\nabla T_S \frac{\varepsilon}{T}\frac{q}{\varkappa_S}, \tag{23}$$

$$g_S^{T,\nu}|_b = -\left.\frac{\partial \nu}{\partial x}\right|_b - \frac{\varepsilon}{T}\left.\frac{\partial T}{\partial x}\right|_b. \tag{24}$$

Let us now consider half of the period of the structure (see Fig. 2). The total change that occurs in the distribution function over this distance is connected with the mean temperature gradient by the relation

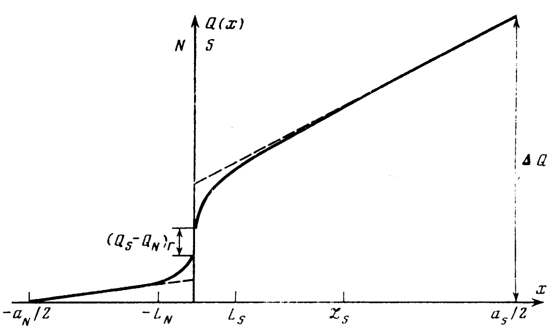

FIG. 2. Dependence of the distribution function on the coordinate.

$$\Delta Q=-\frac{\varepsilon}{T}\Delta T=\frac{\varepsilon}{T}\frac{a_N+a_S}{2}\frac{q}{\varkappa_i}, \qquad (25)$$

where $\varkappa_i$ is the sought thermal conductivity of the intermediate structure:

$$\Delta Q=\int_{-a_N/2}^{0} g_N(x)dx+\int_{0}^{a_S/2} g_S(x)dx+(Q_S-Q_N)_b. \qquad (26)$$

Expressing now $(Q_S-Q_N)_b$ in terms of $g_0(\varepsilon)$ with the aid of (15), and substituting the found $g_{N,S}$ values, we obtain the following equation:

$$q\frac{\varepsilon}{T}\left[\frac{a_N+a_S}{2\varkappa_i}-\frac{a_N-2L_N^*}{2\varkappa_N}-\frac{a_S-2L_S^*}{\varkappa_S}\right]-g_0(\varepsilon)(L_N^*+L_S^*)$$
$$-(\mathscr{L}_S^*-L_S^*)\left(g_S^{T,v}|_b-\frac{\varepsilon}{T}\frac{q}{\varkappa_S}\right)=\frac{l^*}{\overline{w}(\varepsilon)}g_0(\varepsilon) \qquad (27)$$

(for the definitions of the asterisked quantities, see the Introduction).

Solving Eq. (27) simultaneously with (24), (17), (18), and (19), we find $\varkappa_i$. For this purpose, we solve (27) for $g_0(\varepsilon)$:

$$g_0(\varepsilon)=\frac{A\varepsilon/T+B\Delta/T}{L_T+L_A y^{-1/2}},\quad y=\frac{\varepsilon-\Delta}{T}, \qquad (28)$$

$$A=A_0+L_Q\frac{\Delta/T+1}{\varkappa_S}\left[\left(\left(\frac{\Delta}{T}\right)^2+2\frac{\Delta}{T}+2\right)JT-\left(\frac{\Delta}{T}+1\right)q\right], \qquad (29)$$

$$A_0=q\left[\frac{a_N+a_S}{2\varkappa_i}-\frac{a_N-2L_N^*}{2\varkappa_N}-\frac{a_S-2L_S^*}{2\varkappa_S}\right], \qquad (30)$$

$$B=-\frac{L_Q}{\varkappa_S}\frac{T}{\Delta}\left(\left(\frac{\Delta}{T}\right)^2+2\frac{\Delta}{T}+2\right)\left[\left(\left(\frac{\Delta}{T}\right)^2+2\frac{\Delta}{T}+2\right)JT-\left(\frac{\Delta}{T}+1\right)q\right]. \qquad (31)$$

We used (24), (16), and (17).

Now, substituting (28) into (18) and (19), we obtain

$$A\left(\frac{\Delta^2}{T^2}I_0+2\frac{\Delta}{T}I_1+I_2\right)+B\frac{\Delta}{T}\left(\frac{\Delta}{T}I_0+I_1\right)=\frac{q}{\varkappa_S}\left(\frac{\Delta^2}{T^2}+2\frac{\Delta}{T}+2\right), \qquad (32)$$

$$A\left(\frac{\Delta}{T}I_0+I_1\right)+B\frac{\Delta}{T}I_0=\frac{J}{\varkappa_S}\left(\frac{\Delta^2}{T^2}+2\frac{\Delta}{T}+2\right), \qquad (33)$$

$$I_n=\int_0^\infty\frac{y^n e^{-y}dy}{L_T+L_A y^{-1/2}}. \qquad (34)$$

Solving the system of linear equations (29)–(33) for the unknowns A, B, A_0, $\varkappa_i$, and J, we obtain for $\varkappa_i$ the expression (1) with

$$\Phi(L_A,L_T,L_Q)=\frac{\pi^{1/2}}{2}L_A^{-1}\left\{\frac{I_0-2I_1+I_2+L_Q^{-1}}{I_0I_2-I_1^2+I_0L_Q^{-1}}-L_T\right\}. \qquad (35)$$

Using the asymptotic forms of the expression (34), we find that for $L_A\gg L_T$

$$\Phi=(^7/_8\pi^{1/2}+L_A/L_Q)/(^6/_8\pi^{1/2}+L_A/L_Q),$$

and that if, moreover, $L_A\gg L_Q$, then $\Phi=1$. For $L_A\ll L_T$, we have $\Phi=\pi/2$. The function $\Phi\geq 1$, and decreases with increasing boundary contribution to the thermal resistance. For $L_A\gg L_T$, L_Q, the boundary contribution is greater than all the other contributions, and the value of the coefficient Φ attached to it is minimal, and equal to unity.

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