Detection of forces acting on an oscillator by measuring the integral of its coordinate

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Operators exist whose eigenvalues do not depend on the initial state of the system but are changed by the system perturbation that takes place when the operator is measured. One such operator is the integral of the coordinate of harmonic oscillator over an integer number of periods. A method of measuring this operator is analyzed and it is shown that exact measurement is possible only at a definite correlation between the operators of the quantum read-out system.

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The general principle of observing the action on some system consists of comparing the results of the measurement of some operator of the system with its probable eigenvalues in the absence of the action. The eigenvalues of some operators are known exactly in the case of free evolution of the system, regardless of the state of the system. In the case of a harmonic oscillator, these operators include not only the operator of the coordinate differences measured in steps equal to an integer number of periods of the oscillations,[1] but also the difference between the integrals of the Heisenberg operator of the coordinate $\hat{q}(t)$ at an aribtrary integration time τ, provided that the integration intervals do not overlap and are shifted relative to one another by an integer number of oscillation periods. The same properties are possessed by the operator of the sum of integrals, if the integration intervals are shifted by an odd number of half-periods. A particular case of the latter operator is the integral over an integer number of oscillation periods:

$$\hat{J}=\int_t^{t+2n\pi/\omega}\hat{q}(\xi)\,d\xi$$

(ω is the natural frequency of the oscillator, $n=1, 2, \ldots$).

In the case of free evolution of the oscillator in the interval from t to $t+2n\pi/\omega$, the eigenvalue of $\hat{J}$ is zero. If the oscillator is subjected in the integration interval to some external action, $\hat{q}(t)$ is not a pure harmonic function and the integration result may not be zero. After measuring $\hat{J}$ in the absence of an external action, the oscillator turns out to be in the same state as in the case of free evolution. Measurement of $\hat{J}$, as well as of all the other operators indicated above, does not yield any information on the state of the oscillator, and determines only the fact that external forces act on the operator in the interval from t to $t+2n\pi/\omega$.

The difference or sum of operators pertaining to different instants of time can be measured by two methods. It is possible, as proposed in most earlier papers,[2-4] to measure directly the operators at different instants of time and calculate the difference (sum) of the measurement results. But there is also a possible method of forming the necessary difference (7) during the stage of interaction of the investigated system with a quantum readout system (QRS). The basic feature of this method should apparently be that the QRS acts in a specially organized manner twice on the system, at a time shift equal to the shift between the integration intervals. Measurement of the integral over a period of the oscillations can be represented as measurement of the sum of integrals over half the period at a zero shift of the integration intervals.

Confirming the properties of these operators is the following rather simply proved commutation relation of the harmonic-oscillator operators:

$$[\hat{J}(t_1,\tau),\hat{J}(t_2,\tau)]=\frac{2i\hbar}{m\omega^3}(\cos\omega\tau-1)\sin\omega(t_2-t_1), \tag{1}$$

$$\hat{J}(t,\tau)=\int_t^{t+\tau}\hat{q}(\xi)\,d\xi,$$

where m is the effective mass of the oscillator. It is assumed that $t_2\geqslant t_1+\tau$. Perturbation of the oscillator when it interacts with the QRS in the interval from t_1 to $t_1+\tau$ will not affect the change of the state of the QRS when it interacts again with the oscillator in the interval from t_2 to $t_2+\tau$ when the right-hand side of (1) is equal to zero.

It is known that for a nonperturbing measurement of a time-independent operator it suffices to have the Hamiltonian of the interaction of the QRS with the system depend on this operator and not depend on operators that do not commute with it.[4] However, there is no known interaction whose Hamiltonian would be a function of the integral of the coordinates, and such a Hamiltonian is in any case physically unreal.

We consider the process of measuring the integral when the interaction Hamiltonian is proportional to the product of the operators of the system coordinate and the QRS coordinate. Such a Hamiltonian appears, e.g., when a charged particle interacts with the electric field of a parallel-plate capacitor that forms a tank circuit with an inductor L.[5] The generalized coordinates are in this case the charge q in the capacitor and the particle coordinate y measured from the null equipotential plane. The duration of the interaction is determined by the component of the initial particle velocity along the capacitor plates. For the interaction time to be independent of q and to be a well defined quantity, the initial energy of the particle must be a large enough and definite quantity.

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The changes of the charge operator in the tank circuit and of the operator of the y-component of the particle momentum p_y during the time of the interaction will be described by the equations

$$L\frac{d^2\hat{q}}{dt^2}+\frac{\hat{q}}{C}=\alpha\hat{y}+F(t),\qquad \frac{d\hat{p}_y}{dt}=\alpha\hat{q}, \tag{2}$$

where $\alpha=e/CY$ is the coupling coefficient, C is the capacitance of the capacitor, e is the charge of the particle, Y is the distance between the plates, and $F(t)$ is the external force whose action is to be detected. We find from (2) that after the particle has interacted with the resonator for a time τ, the particle momentum is

$$\hat{p}_y(t)=\hat{p}_y(0)+\alpha\int_0^\tau\hat{q}(\xi)\,d\xi=\frac{1}{\omega_1^2+\lambda^2}[\mu\hat{y}(0)\omega_1\lambda(\omega_1\operatorname{sh}\lambda\tau-\lambda\sin\omega_1\tau)$$
$$+\hat{p}_y(0)(\lambda^2\cos\omega_1\tau+\omega_1^2\operatorname{ch}\lambda\tau)+\hat{q}(0)\alpha(\lambda\operatorname{sh}\lambda\tau+\omega_1\sin\omega_1\tau)$$
$$+\hat{\dot{q}}(0)\alpha(\operatorname{ch}\lambda\tau-\cos\omega_1\tau)]+\alpha\int_0^\tau\int_0^t g(t-t')F(t')\,dt'\,dt. \tag{3}$$

Here $\hat{q}(0)$, $\hat{\dot{q}}(0)$, $\hat{\dot{y}}(0)$, and $\hat{p}_y(0)$ are the operators at the instant of the start of the interaction of the particle with the oscillator, μ is the particle mass,

$$\lambda^2={}^1/_2\omega^2[(1+4\varepsilon^2)^{1/2}-1],\quad \omega_1^2={}^1/_2\omega^2[(1+4\varepsilon^2)^{1/2}+1],$$
$$\varepsilon^2=C\alpha^2/\mu\omega^2,\quad \omega^2=1/LC,$$

and $g(t)$ is the impulse response of the oscillator coupled to the particle.

As seen from (3), at $\lambda\neq 0$ there is no time $\tau>0$ at which $p_y(t)$ does not depend on $\hat{q}(0)$ and $\hat{\dot{q}}(0)$. The particle mass, however, can in principle be so large that one can assume, with a known error, that $\varepsilon^2=0$ and $\lambda=0$. In this case $\mu\lambda^2$ becomes a finite quantity equal to $C\alpha^2$. (If an electron is used, then $\varepsilon^2=10^{-12}$ at $C=1$ pF, $Y=0.1$ cm, and $\omega=10^{10}$ sec^{-1}.)

In this approximation we obtain from (3)

$$\hat{p}_y(t)=\hat{p}_y(0)+\hat{y}(0)C\alpha^2\left(\tau-\frac{1}{\omega}\sin\omega\tau\right)+\hat{q}(0)\frac{\alpha}{\omega}\sin\omega\tau$$
$$+\hat{\dot{q}}(0)\frac{\alpha}{\omega^2}(1-\cos\omega\tau)+\frac{\alpha}{\omega L}\int_0^\tau\int_0^t\sin\omega(t-t')F(t')\,dt'\,dt. \tag{4}$$

Consequently, at $\omega\tau=2n\pi$ the operator $p_y(t)$ is independent of the initial state of the oscillator. But this is still insufficient to impose in principle a limitation on the sensitivity to the force $F(t)$. If $\hat{p}_y(0)$ and $\hat{y}(0)$ are not correlated, the momentum-measurement result cannot be predicted, even at $\omega\tau=2n\pi$ and $F(t)=0$, with an error smaller than

$$\Delta p_y=[(\Delta p_y(0))^2+(\Delta y(0))^2(C\alpha^2\tau)^2]^{1/2}\geqslant(\hbar C\alpha^2\tau)^{1/2}. \tag{5}$$

The sensitivity to the force $F(t)=F_0\sin(\omega t+\varphi)$ at the optimal value $\varphi=\pi/2$ is limited to a level

$$F_0\geqslant 2(2n\pi\hbar\omega L)^{1/2}/\tau, \tag{6}$$

which exceeds by a factor of $(2n\pi)^{1/2}$ the quantum limit of the sensitivity when the instantaneous coordinate is tracked continuously over the time τ (Ref. 5). This limit can be overcome by excluding from the "output signal," i.e., from $\hat{p}_y(t)$, the operator $\hat{y}(0)$, which does not commute with $\hat{p}_y(0)$. This can be done if in the initial state of the particle there exists a correlation of $\hat{p}_y(0)$ with $\hat{y}(0)$ such that

$$\hat{p}_y(0)=\hat{p}_{0y}-\hat{y}(0)C\alpha^2(\tau-\omega^{-1}\sin\omega\tau), \tag{7}$$

where $\hat{p}_{y0}$ is that part of the operator $\hat{p}_y(0)$ which is not correlated with $\hat{y}(0)$.

We then obtain from (4) and (7) at $\omega\tau=2n\pi$

$$\hat{p}_y(t)=\hat{p}_{0y}+\frac{\alpha}{\omega L}\int_0^\tau\int_0^t\sin\omega(t-t')F(t')\,dt'\,dt. \tag{8}$$

In this case, by decreasing the ratio $\Delta p_{0y}/\alpha$, we can obtain an arbitrarily low threshold for the observation of the force, if it is assumed that the momentum $\hat{p}_y(t)$ can be measured with arbitrary accuracy.[6] A correlation between $\hat{p}_y(0)$ and $\hat{y}(0)$, corresponding to Eq. (7), can be obtained by passing the particle, prior to its entry into the capacitor, through a converging lens of suitable focal length. An ideal thin converging lens transforms the coordinate and momentum of the particle in accordance with the rules

$$\hat{y}_2=\hat{y}_1,\quad \hat{p}_{y2}=\hat{p}_{y1}-\gamma\hat{y}_2.$$

Here $\hat{y}_1$ and $\hat{p}_{y_1}$ are the operators of the coordinate reckoned from the optical axis and of the momentum conjugate to it directly in front of the lens; $\hat{y}_2$ and $\hat{p}_{y2}$ are the analogous operators directly behind the lens; γ is a number that depends on the focal length of the lens. If the particle enters the capacitor directly from behind the lens, the following correspondence exists between the operators of (7) and the operators $\hat{y}_{1,2}$ and $\hat{p}_{y1,2}$:

$$\hat{p}_y(0)=\hat{p}_{y2},\quad \hat{p}_{0y}=\hat{p}_{y1},\quad \hat{y}(0)=\hat{y}_2,\quad C\alpha^2(\tau-\omega^{-1}\sin\omega\tau)=\gamma.$$

The condition $\mu\to\infty$ makes it possible to exclude from the output signal the information on the initial state of the oscillator. Simple calculation shows that the perturbed state of the oscillator is likewise missing from the measurement result if $F(t)\equiv 0$. If relation (7) is satisfied and $\mu\to\infty$, this measurement method can therefore be regarded as an ideal nonperturbing method of measuring the integral of the coordinate.

The method of observing the action exerted on an oscillator by measuring the integral of its coordinates may offer a number of technical advantages over the method of measuring the "instantaneous" values of the coordinate at integer numbers of the half-periods of the oscillations,[2,3] since it does not require that the instrument be turned on pulsewise at exactly integer numbers of half-periods, and the specified sensitivity can be reached at a smaller coupling coefficient α. A shortcoming of this method is that the correlation between $\hat{p}_y(0)$ and $\hat{y}(0)$ can greatly increase the sensitivity only one test particle or one narrow packet with a given number of particles are simultaneously present in the resonator.

Let us estimate the possibility of using electrons in the described measurement scheme. The influence of $q(0)$ and $\dot{q}(0)$ on $p_y(t)$ can be neglected if the third and fourth terms of Eq. (3) are smaller than the uncertainty of the second term. If electrons are used and if $(L/C)^{1/2}=10\ \Omega$, we should have $Y^2>3\times10^{-2}n_0^{1/2}$ cm (n_0 is the average number of quanta in the resonator). If ω

$=10^{10}$ sec^{-1}, then we get $n_0 \approx 10$ at a temperature 4 K. Consequently it is possible to realize in principle the needed conditions at a reasonable distance Y between the capacitor plates. In a real experiment, the decisive factors are the influence of the extraneous fields and the imperfection of the electron optics.

The use in the initial state of the QRS of the correlation of its noncommuting operators is the general principle for excluding from the output signal one of these operators, independently of the physical nature and structure of the QRS. It can be shown that the need for using the correlation arises always when the Hamiltonian of the interaction of the system with the QRS is a function of the coordinate operator of the system or of any other operator which is not an integral of the motion.

We consider now the measurement of the quadrature component $\hat{X}_1 \equiv \hat{q}(0)/\omega$ of the operator of the oscillator coordinate by measuring the integral of the coordinate at a time $\tau = (2n+1)\pi/\omega$. In this case we obtain from (4), putting $F(t)=0$,

$$\hat{p}_y(t) = \hat{p}_y(0) + \hat{y}(0) C\alpha^2\tau + 2\alpha X_1/\omega. \quad (9)$$

If there is no correlation of $p_y(0)$ with $y(0)$, the error in the measurement of $\hat{X}_1$ is, even if $\hat{p}_y(t)$ is exactly measured,

$$\Delta X_1 \geqslant (^1/_4 \hbar C\tau\omega^2)^{1/2}.$$

The minimum value of τ is π/ω, therefore

$$\Delta X_1 \geqslant (^1/_4 \pi\hbar\omega C)^{1/2}.$$

Introduction of the correlation in accordance with relation (7) offers a possibility of lowering the error in the measurement of $\hat{X}_1$ to an arbitrarily low level. The calculation of the perturbation of the operator $\hat{X}_2$ shows that it satisfies the uncertainty relation

$$\Delta X_1 \Delta X_2 \geqslant \hbar/2\omega L.$$

Let us illustrate the principle of the use of the correlation with another example. The coordinate of the mechanical oscillator can be determined by measuring the delay time of a photon that is repeatedly reflected from the oscillator mass and from a certain immobile mirror. The photon path length depends not only on the initial state of the oscillator but also on the displacement of the oscillator mass under the influence of the momentum of the photon during the time from the first to the last reflection. Thus, the instant that the photon is registered by the receiver will depend not only on the instant of the photon emission, but also on its frequency. Since the uncertainty Δt of the instant of emission is connected with the frequency uncertainty by the relation $\Delta\omega\Delta t \geqslant \frac{1}{2}$, the total uncertainty in the delay time is limited in principle. In order for the total delay time of the photon to be independent of frequency, it is necessary to introduce a delay whose time is a required function of the frequency. The delay time will yield in this case information on that corrdinate value which would exist in a system that does not interact with the photon.

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