Interaction regimes of nonlinear waves in a low-viscosity region

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The evolution of a two-component acoustic signal in a nonlinear low-viscosity medium is considered. A perturbation method that is applicable at all stages of the wave propagation is used to obtain a representation of the velocity field and to analyze the scale and regimes of the interactions between the components. The statistical characteristics of a regular signal interacting with weak small-scale noise are investigated. It is shown that in the case of a signal interacting with large-scale noise the concept of turbulent viscosity can be used to describe the evolution of the average velocity.

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1. INTRODUCTION

The evolution of acoustic waves of finite amplitude is of great theoretical and applied interest in connection with the study of sound propagation in liquids and gases both under natural conditions and in the laboratory.[1] Propagation of waves traveling in the same direction is described in this case by the well-known Burgers equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where ν is the viscosity coefficient. (In acoustic applications t can have the meaning of a spatial coordinate.)

Equation (1) can also be used to describe the propagation of three-dimensional fronts, if the diffraction effects develop later than the onset of the action of the nonlinearity. It is then possible to take into account, within the framework of nonlinear geometric acoustics, the inhomogeneity of the medium and the divergence of the propagating waves.[2]

The wave-propagation process described by (1) is characterized by two time scales: the wave-breaking time $\tau_0 = la^{-1}$, where a and l are the amplitude and the scale of the perturbation at $t=0$, and the attenuation time $\tau_{\rm att} = l^2\nu^{-1}$. The ratio of these times gives the values of the acoustic Reynolds number which shows the degree to which the nonlinear effects can develop. At large Reynolds numbers there is formed an ensemble of sawtooth waves that attenuate because of the energy dissipation on the steep fronts. The running Reynolds number, which can be treated as the ratio of the wave scale to the diffusion length $l^d = (\nu t)^{1/2}$, decreases in this case, so that at $t\sim\tau_{\rm att}$ we have $\mathrm{Re} = l^2(\nu t)^{-1}\sim 1$.

At small Reynolds numbers, the random solutions of (1) can be investigated by expansion in harmonics. At large Re, however, coherent interaction takes place between a large number of harmonics, so that this approach is not effective. Evolution of a random perturbation leads in this case to development of a strong turbulence, for which the phase-randomization hypothesis is not valid.

A more adequate method of investigation is to change over to the Riemann equation that follows from (1) at $\nu=0$ and whose solution shows that the sought velocity u is constant along the characteristics (rays) described by the equation $x = x_0 + v(x_0)t$, where $v(x_0)$ is the initial perturbation of the velocity field.

Until recently most studies were devoted to the statistics of random waves in the region prior to the onset of discontinuities, where the Riemann solution is single valued or, in other words, one ray arrives at each point. Also considered were various particular cases pertaining to the region where the discontinuities are formed. The results of these studies are reviewed in Ref. 2.

At $t\gg\tau_0$, where the investigation is made difficult by the fact that the Riemann wave is multiply valued, it becomes necessary to use the principle of selecting the dominant solution (ray) realized in the produced shock wave. According to this principle, which has been formulated for different cases in Refs. 3–6, the dominant ray is the one that has at the given point the least action:

$$s(x)=\varphi(x_0)+\frac{(x-x_0)^2}{2t},\quad \varphi(x_0)=\int_{-\infty} v(x_0)\,dx_0.$$

This form of the selection principle allows us to investigate the wave propagation with the aid of a numerical experiment that reduces in the main to the construction of realizations of a random process.

Analytic results can be obtained in cases when the number of competing rays is small or large. In the latter case, the rays that become dominant in the course of time are those connected with even deeper minima of the initial action, as a result of which the scale of the sawtooth waves increases up to the instant when the running Reynolds number becomes of the order of unity and the waves go over into the stage of exponential damping. This regime, which can be called one-dimensional acoustic turbulence, was investigated for a pure noise wave in Refs. 3–5.

The problem of the interaction of two components of an initial perturbation (primarily of a regular signal with stationary noise) was posed and solved for individual ranges of parameters and separate stages of propagation, including the stage prior to the breaking of the Riemann wave,[7–9] the stage of random sawtooth waves,[1] and for several cases pertaining to the intermediate stage.[8,9]

The present paper is an attempt to obtain a general description of the evolution of a two-component wave on the

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basis of a perturbation method that requires smallness of the ratio of the amplitudes or of the scales of the components. A perturbation method that takes into account the competition of the rays and the selection of the dominant solution makes it possible to analyze from a unified viewpoint various interaction regimes and find their characteristic scales. On the basis of this method, the present paper presents also a statistical description of the interaction of a regular signal with weak small-scale noise and with a large-scale noise. For the latter, we investigate the applicability of the concept of turbulent viscosity when describing the evolution of the average field.

2. RAY DESCRIPTION OF THE PROPAGATION OF A NONLINEAR ACOUSTIC WAVE

We consider the ray method of describing the evolution of a nonlinear acoustic wave specified at $t=0$ in the form $u=v(x)$ with parameters $l, a, \tau = la^{-1}$ such that the Reynolds number is $\mathrm{Re}=alv^{-1}\gg 1$.

It is known that the solution of an exact Burgers equation can be obtained in the form of a ratio of two integrals by using the Hopf-Cole substitution.[1] Calculating the integrals by the method of steepest descent, we arrive at an expression for the velocity field in the form[10]

$$u(x,t)=\left(\sum_m u_m \exp[-s_m(2\nu)^{-1}]\right)\left(\sum_m \exp[-s_m(2\nu)^{-1}]\right)^{-1}, \tag{2}$$

where each term of the sum is connected with a stable branch of the solution of the Riemann equation. The quantities u_m and s_m are solutions of a characteristic system of ordinary differential equations (partial values of the velocity and of the action) and are of the form

$$u_m=v(x_m),\qquad s_m=\varphi(x_m)+(x-x_m)^2/2t,\qquad \varphi=\int_{-\infty}^{+\infty} v(x)\,dx,$$

where $\{x_m\}$ is the set of points x_0 satisfying the equation

$$v(x_0)=(x-x_0)/t \tag{3}$$

subject to the additional condition $1+v_1'(x_m)t>0$ that requires the action s to have a local minimum at the given x_m.

At $t<\tau$ Eq. (3) has one solution and (2) contains one term. At $t>\tau$ the solution of (3) is multiplying valued, i.e., many rays arrive at a given point. At large running Reynolds numbers, however, we can take into account in (2) almost everywhere (except for narrow fronts) only one term—the dominant partial solution satisfying the selection rule

$$\varphi(x_m)+(x-x_m)^2/2t=\min. \tag{4}$$

Taken together, (3) and (4) specify the mapping $x_0=x_0(x,t)$, and at $t\gg\tau$, when the regime of sawtooth waves sets in the x axis is mapped on an assembly of small vicinities of points x_m satisfying the condition $\varphi(x_m)=\min$. These vicinities are the regions of emergence of the beams of rays that dominate on a scale L and have a width $L\tau t^{-1}$.

If the initial perturbation is regarded as a noise or as a regular signal with indeterminate phase, x_0 becomes indeterminate at a fixed x. This indeterminacy can be characterized by a turbulent-diffusion length L_d.

Over times shorter than the signal breaking time, the diffusion length is determined directly from (3):

$$L_d=at\quad (t<\tau). \tag{5}$$

In multiray propagation, the diffusion length depends on the displacement of the dominant ray. It follows from (4) that this displacement is limited by the condition that the regular increase of the action s be offset by its possible decrease due to variastion of φ. The concrete value of L_d differs in different cases. For a regular periodic signal with zero dc component we have $L_d=l$. If v is a stationary noise, the L_d is given approximately by $(t>\tau)$

$$L_d^2/2t=[D_\varphi(L_d)]^{1/2},\quad D_\varphi(L_d)=\langle[\varphi(x+L_d)-\varphi(x)]^2\rangle \tag{6}$$

and depends on the value of following parameter, which is conserved as the wave propagates:

$$J=\int_{-\infty}^{+\infty}\langle u(x)u(x+y)\rangle dy=\int_{-\infty}^{+\infty}\langle v(x)v(x+y)\rangle dy=\text{const}.$$

If $J=0$, then φ is also a stationary noise and $D_\varphi(x)\sim a^2l^2$ at $x>l$. If $J\neq 0$, then $D_\varphi(x)\sim Jx$ at $x>l$. We have thus for stationary noise $(t>\tau)$

$$L_d=(alt)^{1/2}\ (J=0),\qquad L_d=(Jt^2)^{1/3}\ (J\neq 0). \tag{7}$$

The wave-breaking time is characterized by equality of the diffusion length and of the initial scale. At $t>\tau$ the scale of the wave is $L\sim L_d$.

3. PERTUBATION METHOD AND PRINCIPAL INTERACTION REGIMES

We consider the evolution of the initial velocity, specified at $t=0$ in the form of a sum of two components: $v=v_1+v_2$, for each of which are defined parameters l, a, and τ such that $\mathrm{Re}\gg 1$ and that one of the relations $l_1l_2^{-1}\gg 1$ or $l_1l_2^{-1}\sim 1$, $a_1a_2^{-1}\gg 1$ is satisfied.

Under these assumptions the ray equation (3) breaks up, apart from small terms, into two:

$$v_1(\tilde{x}_0)=(x-\tilde{x}_0)/t, \tag{8a}$$

$$v_2(x_0)=(\tilde{x}_0-x_0)/\tilde{t},\quad \tilde{t}=t/[1+tv_1'(\tilde{x}_0)]. \tag{8b}$$

For the partial values u_m and s_m we have in the same approximation:

$$\begin{aligned} u_m&=v_1(\tilde{x}_m)+v_2(x_m)\tilde{t}/t\approx v_2(x_m)+v_1[\tilde{x}_m-v_2(x_m)\tilde{t}],\\ s_m&=\varphi_1(\tilde{x}_m)+\varphi_2(x_m)+\frac{(x-\tilde{x}_m)^2}{2t}+\frac{(\tilde{x}_m-x_m)^2}{2\tilde{t}}\\ &=s_{1m}(x,\tilde{x}_m,t)+s_{2m}(\tilde{x}_m,x_m,\tilde{t}), \end{aligned} \tag{9}$$

where $\{\tilde{x}_m\}$ is the set of solutions (8a), and x_m is the solution (8b) that satisfies the selection condition at a given $\tilde{x}_m$:

$$\varphi_2(x_m)+(\tilde{x}_m-x_m)^2/2\tilde{t}=\min. \tag{10}$$

Substituting (9) in (2) we obtain a general representation for the velocity field of a two-component signal in terms of the parameters of the individual components; this representation is valid both before and after the breaking of the field. It follows from this representation that everywhere except in the vicinities of the shock fronts, the following formula is valid:

$$u(x,\ t)=v_1(\tilde{x}_0-v_2(x_0)\tilde{t})+v_2(x_0), \tag{11}$$

where $\tilde{x}_0$ is an element of the set $\{\tilde{x}_m\}$ and satisfies the condition

$$s_1(\tilde{x}_0,\ x,\ t)+s_2(\tilde{x}_0)=\min, \tag{12}$$

where $s_2(\tilde{x}_0)=s_2(\tilde{x}_0,x_0,\tilde{t})$, and the functions $x_0(\tilde{x}_0),\tilde{t}(\tilde{x}_0)$ are given by (8b) and (10).

In the argument of each of the functions v_i in (11) the action of the other component v_j gives rise to an indeterminacy that can be described by a length L_{ij}. The time at which L_{ij} (the relative length of the turbulent diffusion) becomes equal to the spatial scale v_i is the time of interaction of v_i with v_j. It describes the destruction of one component by the action of the other.

We consider now in greater detail the features of the interaction between the components at various ratios of the problem parameters. We assume for simplicity that one of the components is a regular periodic signal with zero dc component, and the other is a similar signal or a stationary noise with a one-scale correlation function. We note that a number of the results that follow are valid also outside the boundaries of these restrictions. At $t\ll\tau_1$ Eq. (8a) has one solution and is independent of $t\tau_2^{-1}$,

$$u(x,\ t)=u_1[x-u_2(\tilde{x},\ \tilde{t})\tilde{t}]+u_2(\tilde{x}_0,\ \tilde{t}), \tag{13}$$

where u_1 and u_2 are the solutions of the Burgers equation for each of the components.

An expression close to (13) was obtained in (9) for the region v_2 ahead of the breaking. It can be easily seen that (13) describes phase modulation of the small-scale component by the large-scale one. The modulation is characterized by the same length as the proper length of the turbulent diffusion for v_1, defined by (5): $L_{21}=a_1t$. The interaction time at which the modulation index reaches unity and destruction of the coherent (but not spectral) structuer of v_2 takes place can be obtained from the equality $L_{21}=L_2$. If v_2 is a signal or a noise with $a_2<a_1$, we have for the interaction time

$$\tau_{21}=l_2/a_1. \tag{14}$$

At $\tau_2<t$ and $\tau_2<\tau_1$ there can be established in the system sawtooth wavles of scale L_2, modulated by u_1.

At $t<\tau_1$ the effect of v_2 on v_1, as follows from (13), is also characterized by a proper length L_d for the component v_2. At $t>\tau_1$ the presence of the component v_2 changes the value of the action connected with each of the competing rays that emerge from the vicinities of the points $\tilde{x}_m$, and therefore changes the conditions of the competition. The interaction length that represents the interval in which is located the point of emergence of the dominant ray is specified by a condition that follows from (12):

$$\begin{gathered}L_{21}=|\tilde{x}_i-\tilde{x}_j|,\\ |s_1(x,\ \tilde{x}_i)-s_1(x,\ \tilde{x}_j)|=|s_2(\tilde{x}_i)-s_2(\tilde{x}_j)|=\Delta s_2.\end{gathered} \tag{15}$$

If v_2 is the signal, the right-hand side of (15) takes the form

$$\begin{gathered}\Delta s_2=l_2^2\tilde{t}^{-1}\ (\tau_1>\tau_2),\\ \Delta s_2=a_2l_2\ (\tau_1<\tau_2),\end{gathered} \tag{16}$$

whence, if v_1 is also a signal, follow the interaction scales:

$$\begin{gathered}L_{12}=(a_2l_2t)^{1/2},\ \tau_{12}=l_1^2l_2^{-1}a_2^{-1}\ (\tau_2>\tau_1),\\ L_{12}=l_2t^{1/2}\tau_1^{-1/2},\quad \tau_{12}=\tau_1l_1^2l_2^{-2}\quad (\tau_2<\tau_1).\end{gathered} \tag{17}$$

If v_1 is a noise, the regular small-scale component has little influence on it.

If v_2 is a noise, we have for the right-hand side of (15) $\Delta s_2=D_{\varphi_2}(L_{12})$, and the interaction length with the signal v_1 is specified just as at $t<\tau_1$, by expression (7). From the equation $L_{12}=L_1=l_1$ we obtain the interaction time

$$\tau_{12}=l_1^2(l_2a_2)^{-1}(J_2=0),\quad \tau_{12}=l_1^{4/3}J_2^{-1/3}\quad (J_2\neq0). \tag{18}$$

If $l_2a_2>l_1a_1$ or $J_2>l_1a_1^2$, the signal v_1 is destroyed at $t\sim\tau_{12}<\tau_1$ prior to its breaking, because the growth of the scale of the noise waves prevents the establishment of sawtooth waves of scale l_1.

If v_1 is a regular signal or noise, but $\tau_{12}<\tau_1$, breaking of v_1 takes place at $t\sim\tau_1$. In this case each produced saw is connected with a bundle of rays that emerge from the vicinity $\delta\tilde{x}_m\sim L_1\tau_1/t$ of the point $\tilde{x}_m[\varphi_1(\tilde{x}_m)=\min]$. At $t>L_1L_2^{-1}$, when $\delta x_m\sim L_2$, modulation by v_2 affects only the positions of the centers and of the boundaries of the individual saws.

Using the foregoing analysis, we can describe the basic types of interaction and self-action of the two components of the initial perturbation (v_1 and v_2):

1. Self-action of the small-scale component with a characteristic time τ_2, which manifests itself at $\tau_2<\tau_1$.

2. Destruction of the coherent structure of the small-scale component (phase modulation) over times τ_{21}, which takes place if v_2 is a regular signal or a noise with $\tau_{21}<\tau_2$.

3. Self-action of the large-scale component at $t\sim\tau_1$, which appears if v_2 is a signal or a noise with $\tau_{12}>\tau_1$.

4. Destruction of the spectral structure (scale) of the small-scale component, which takes place at $t=\tau_1L_1L_2^{-1}$ in the case when the conditions indicated in Item 3 are satisfied.

5. Destruction of the structure of the large-scale signal on account of modulation by the small-scale noise at $t=\tau_{12}$, and further transfer of energy towards the lower frequencies all the way to a scale that is in whole-number ratio with l_1 and l_2 (if v_1 and v_2 are regular and $l_1l_2^{-1}$ is rational), or until the running Reynolds number reaches the value unity.

Upon interaction of a regular periodic signal and stationary noise, the following principal regimes are observed:

a) Interaction of the large-scale signal v_1 with the weak small-scale noise v_2 ($\tau_1<\tau_{12}$). In this case the modulation of the noise by the signal destroys the structure of the noise and causes establishment of sawtooth waves with the scale of the signal at $t=\tau_1$. At $t=\tau_2$ the residual modulation by the noise blurs the structure of the signal and an acoustic-turbulence regime with growing scale sets in.

b) Interaction of a large-scale signal v_1 with a strong small-scale noise v_2 ($\tau_1>\tau_{12}$). In this case the noise is weakly modulated by the signal during the intial stage. After the breaking of the noise, its scale begins to grow, and this leads to destruction of the signal prior its breaking at τ_{12}.

c) Interaction of a small-scale signal v_2 with a large-scale noise v_1. The destruction of the coherent structure of the signal takes place at $t = \tau_{21}$, and its final blurring and transition into the regime of acoustic turbulence take place at $t \sim \tau_1 L_1 l_2^{-1}$.

The general expressions obtained above for the velocity field allow us to calculate the statistical characteristics of a signal interacting with a noise. In the sections that follow we shall dwell on two problems pertaining to regimes a) and c), in which the approach employed yields some new results.

4. INTERACTION OF A SIGNAL WITH A WEAK SMALL-SCALE NOISE

If the signal, v_1 interacts with a nosie v_2, with $l_2 \lesssim l_1$ and $\tau_2 \gg \tau_1$ it follows from the general relations (9) that

$$\begin{aligned} u_m &= v_1(\tilde{x}_m) + v_2(\tilde{x}_m)[1 - v_1'(\tilde{x}_m)\tilde{t}], \\ s_m &= \varphi_1(\tilde{x}_m) + \varphi_2(\tilde{x}_m) + (x_1 - \tilde{x}_m)^2/2t. \end{aligned} \tag{19}$$

Prior to the breaking of v_1, the statistical characterisitcs can be calculated with the aid of the probability density of the Riemann-wave parameters. At $t \gtrsim \tau_1$ it is necessary for such a calculation to use in addition the selection rule. At $\tau_1 < t < \tau_{12}$, at any point of observation, we can confine ourselves to allowance for the competition of only two rays, for which $\tilde{x}_m < x < \tilde{x}_{m+1}$. From the selection rule it follows that a ray emerging from the point $\tilde{x}_m$ is dominant if

$$\varphi_1(\tilde{x}_m) + \varphi_2(\tilde{x}_m) + \frac{(\tilde{x}_m)^2}{2t} - \frac{\tilde{x}_m x}{t} < \varphi_1(\tilde{x}_{m+1}) + \varphi_2(\tilde{x}_{m+1}) + \frac{(\tilde{x}_{m+1})^2}{2t} - \frac{x\tilde{x}_{m+1}}{t}. \tag{20}$$

From (19) and (20) follows a formula from the probability density of the velocity at the point (x,t):

$$\begin{aligned} W(u,x,t) &= \int \delta\left(u - \frac{x - \tilde{x}_m}{t} - v_m\right) \mathcal{W}_0(v_m)\,dv_m \\ &+ \int \delta\left(u - \frac{x - \tilde{x}_{m+1}}{t} - v_{m+1}\right) \mathcal{W}_0(v_{m+1})\,dv_{m+1}, \\ \mathcal{W}_0(v_i) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{\psi}^{\infty} W_0(v_i, v_j, \varphi_i, \varphi_j)\,d\varphi_j\,d\varphi_i\,dv_j, \\ \psi &= \varphi_i + \varphi_1(\tilde{x}_i) + \frac{(x - \tilde{x}_i)^2}{2t} - \varphi_1(\tilde{x}_j) - \frac{(x - \tilde{x}_j)^2}{2t}, \end{aligned} \tag{21}$$

where W_0 is the joint probability density of the parameters v and φ at the points $\tilde{x}_m$ and $\tilde{x}_{m+1}$ for the noise v_2.

At $t \gg \tau_1$ and at a Gaussian function W_0 we obtain from (21) a formula for the average velocity on the interval $0 < x < l_1$ in the form

$$\begin{aligned} \langle u \rangle &= \frac{x}{t} - \frac{l_1}{t}\int_{-\infty}^{y} \exp\left[-\frac{z^2}{2}\right] dz \frac{1}{(2\pi)^{1/2}}; \\ y &= \frac{(x - l_1/2)\,l_1}{t[D_2(l_1)]^{1/2}}, \quad D_2(x) = \int_0 (x - z)\langle v_2(x) v_2(x+z)\rangle dz. \end{aligned} \tag{22}$$

Thus, the average velocity comprises sawtooth waves having the scale of the signal and blurred on account of the residual noise modulation by the front. The noise field takes the form of successive narrow random pulses located at the positions of the fronts of the regular signal.

At $t = \tau_{12}$, defined by (18), the residual noise modulation destroys the waves of scale l_1 and a turbulent regime sets in.[10]

5. INTERACTION OF SIGNAL WITH LARGE-SCALE NOISE

For a signal v_2 interacting with a large-scale noise v_1, expression (13) is valid and yields for the probability density of the velocity u the formula

$$W(u) = \int \delta[u - v_1 - u_2(x - v_1 t)]\,W_0(v_1)\,dv_1, \tag{23}$$

where W_0 is the one-point probability density of v_1.

From (23) following expressions for the moments

$$\begin{aligned} \langle u \rangle &= \int u_2(x - v_1 t)\,W_0(v_1)\,dv_1, \\ \langle u^2 \rangle &= \langle v_1^2 \rangle + \int [u_2^2(x - v_1 t) + 2v_1 u_2(x - v_1 t)]\,W_0(v_1)\,dv_1. \end{aligned} \tag{24}$$

Expression (24) for the Gaussian function W_0 was obtained in (11), but without indication of the limits of its applicability. At $t < \tau_2$ it coincides with the corresponding expression for the Riemann wave. Equations (23) and (24) for the region $t > \tau_2$ could also be obtained by likewise using the selection rule for the dominant ray and taking into account the competition between the neighboring rays, as was done in the preceding section.

It follows from (24) that at $t > l_2/a_1$ the average field attenuates. If

$$W_0 = (2\pi)^{-1/2} \exp[-v_1^2/2a_1^2],$$

we get at $t \gg \tau_{21}$

$$\langle u \rangle \sim \exp[-2\pi^2 a_1^2 t^2 l_2^{-2}], \quad \langle u^2 \rangle = a_1^2 + (2\pi l_2)^{-1} \int_0^{l_2} u_2^2(x)\,dx. \tag{25}$$

At $t \ll \tau_1$, when the competition of the rays must be taken into account, the selection procedure turns out to be different in the following cases:

1) $\tau_2 > \tau_1$, when competition takes place between rays with random value of the action; this competition can be described by the method of Ref. 10.

2) $\tau_1 > \tau_2$, when at $t > \tau_2$ there arrive at the point rays with correlated action, so that the method of Ref. 10 leads to not quite a correct result.

In the latter situation it is necessary to apply in succession the selection rules, first in the form (10) and then in the form (12). Expression (10) is a form of a local rule for selecting the dominant ray from among a set of correlated rays. We note, however, that for the most interesting cases the probability density can be obtained also by simpler methods. At $t \ll \tau_1$ it is given by expression (23), which takes the necessary restrictions into account automatically. At $t \gg \tau_1$, as can be seen from the equations of Sec. 3, acoustic turbulence sets in and has the same parameters as in the case of propagation of pure noise. It was erroneously stated in Ref. 10 that this regime has a weak dependence on the signal parameters.

We discuss in conclusion the question of the applicability of the concept of turbulent viscosity for the description of the evolution of the average field in the case of joint propagation of a signal and large-scale noise.[10,11]

Averaging the Burgers equation and discarding the viscosity term, which is small under our assumptions, we have

$$\frac{\partial\langle u\rangle}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}\langle u^2\rangle=0. \tag{26}$$

The condition for closing the equation for $\langle u\rangle$ by introducing turbulent viscosity ν_T is the relation

$$\frac{1}{2}\frac{\partial\langle u^2\rangle}{\partial x}=\langle u\rangle\frac{\partial\langle u\rangle}{\partial x}-\nu_T\frac{\partial^2\langle u\rangle}{\partial x^2}. \tag{27}$$

For large-scale noise $l_1 \gg l_2$ at $t \ll \tau_1$, when the moments of the velocity are described by (24), the required connection is obtained in two cases:

1) The signal is weakly perturbed by the noise, $t \ll l_2/a_1$. From a Taylor expansion of $u_2(x - v_1 t)$ we then obtain (27), where $\nu_T = a_1{}^2 t$. In this case, however, the influence of the noise on the signal is certainly weak.

2) The signal perturbs the noise regime weakly, as is the case at $a_1 \gg a_2$, as well at $t \gg l_2 a_1$.

Equation (24) leads then to the relation

$$\frac{1}{2}\frac{\partial\langle u^2\rangle}{\partial x}=\int v_1 u_2'(x-v_1t)\,W_0(v_1)\,dv_1$$

$$=-\int F(y)\langle u(x-yt)\rangle dy, \tag{28}$$

$$F(y)=(2\pi)^{-1}\int[\ln \tilde{W}(\varkappa)]'\varkappa\exp[-i\varkappa y]d\varkappa,$$

$$\tilde{W}(\varkappa)=\int W_0(v_1)\exp[i\varkappa v_1]dv_1.$$

From (28) we obtain the nonlocal equation

$$\partial\langle u\rangle/\partial t=\int F(y)\langle u(x-yt)\rangle dy, \tag{29}$$

which goes over, if W_0 is a Gaussian function, into the diffusion equation

$$\partial\langle u\rangle/\partial t=\nu_T\partial^2\langle u\rangle/\partial x^2,\quad \nu_T=a_1{}^2t. \tag{30}$$

The turbulent viscosity was obtained in this form[11] by a method that did not make it possible to set the limits of applicability of the result.

For a signal interacting with small-scale noise, turbulent viscosity was introduced in Ref. 10 under the condition $J=0$ for $t \gg \tau_n$, $\tau_s > \tau_n$, and $t \gg \tau_{sn}$, $t_s < \tau_n$ in the form

$$\nu_T=\frac{a_n l_n}{2}(\ln\psi)^{-1/2}\sim a_n l_n, \tag{31}$$

where ψ can depend on t and on the parameters of the signal and the noise.

We note that if upon interaction of a signal with a noise (assuming $J=0$) we describe the evolution of the average velocity by an equation with turbulent viscosity, defined by (30) or (31), this will lead to a qualitatively correct picture. The diffusion length and interaction time introduced above are connected with ν_T by the formulas

$$L_{sn}=(\nu_T t)^{1/2},\quad \tau_{sn}=l_c^2\nu_T^{-1}.$$

At $t \gg \tau_{sn}$ the turbulent viscosity leads to an incorrect value of the damping decrement. During the stage when the average velocity constitutes sawtooth waves of scale l_s, which manage to establish themselves at large values of the turbulent Reynolds number.

$$\mathrm{Re}_T\sim\frac{a_s}{a_n}\quad(l_s<l_n),\qquad \mathrm{Re}_T\sim\frac{a_s l_s}{a_n l_n}\quad(l_s>l_n),$$

the turbulent viscosity gives a correct estimate of the static broadening of the fronts: $\Delta\sim\nu_T t l_s{}^{-2}$. As for the structure of the average field near the front, it is correctly described only in the case of small-scale noise.

The indicated scheme of interaction of the components of the initial perturbation is applicable in the main to a large class of signals that have two or several characteristic scales. It is possible to treat in this manner the interaction of a bounded wave packet with noise, the evolution of a signal having a random amplitude and phase modulation, and others.

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