Peculiarities of the dynamics of discotics

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The nonlinear dynamics of discotic liquid crystals (discotics) is investigated. It turns out that the reactive part of the hydrodynamic equations, which determines in particular the interaction vertices of the hydrodynamic modes, is subject to weak renormalization. However, allowance for fluctuational corrections leads to the appearance of a divergent contribution to the kinetic coefficients. A distribution function is constructed in which the weakly fluctuating variables not related to the order parameter are effectively excluded from consideration. The diagram technique for calculating the fluctuation effects is used to show that in the leading approximation of perturbation theory the imaginary part of the discotic mode spectrum behaves like $|k_\parallel|^{\sqrt{2}}$ for $k_\parallel \gg k_\perp$ and like $|\omega|^{\sqrt{2}}$ for $k_\parallel \sim k_\perp$. Fluctuations of the order parameter also lead to strong renormalization of the attenuation of first sound, the long-wavelength behavior of which changes from k^2 to $k^{\sqrt{2}}$.

INTRODUCTION

Discotic liquid crystals consist of filaments of molecules which are rigidly connected to each other. These filaments can slide against each other in the manner of a liquid, and in a plane perpendicular to the filaments they form a two-dimensional crystal lattice. The linear elasticity theory and the hydrodynamics of such systems are known.[1] In the present paper we consider nonlinear effects in the dynamics of discotics, which turn out to modify substantially the character of the relaxation effects in these liquid crystals.

When one considers the hydrodynamics of discotics, one must take into account, in addition to the variables characterizing isotropic liquids, a two-component vector **u** which describes the displacement of the filaments in a perpendicular direction. We are thus get for **u** to a two-dimensional elasticity theory that has no elastic modulus for variation of **u** along the filaments. This leads to anomalously large fluctuations of the vector **u** (Ref. 2) which do not affect the static properties of discotics. By this the situation differs from the case of smectics, where the fluctuations of the displacement vector **u** (which for smectics has only one component) lead to a logarithmic renormalization of the static elastic moduli.[3–5]

When one studies the nonlinear properties of discotics it is more convenient to use in place of the displacement vector of the filaments two scalar functions: W_1 and W_2, the meaning of which is that the pair of equations $W_\alpha(t,\mathbf{r}) = \text{const}$ defines the position of the filament. The energy density of the discotic can be represented as an expansion with respect to the gradients of W_α. For a crystal which is isotropic in a plane perpendicular to the filaments the leading terms of the expansion of the energy density have the form

$$E^{(W)} = -{}^1/_2(\beta_1+\beta_2)l^{-2}\nabla W_\alpha \nabla W_\beta + {}^1/_2\varkappa \nabla^2 W_\alpha \nabla^2 W_\alpha + {}^1/_8\beta_1(\nabla W_\alpha \nabla W_\alpha)^2 + {}^1/_4\beta_2(\nabla W_\alpha \nabla W_\beta)(\nabla W_\alpha \nabla W_\beta), \quad (1)$$

where β_1, β_2, and $\varkappa$ are the elastic moduli, and summation over the Greek indices is understood from 1 to 2. We note that in the approximation in which terms of second and fourth order in ∇W_α appear (and only these terms are essential in the consideration of nonlinear effects) the same expression (1) will be valid for two-dimensional crystals having a sixfold axis. It is exactly this case of hexagonal symmetry which is characteristic for real discotics, so that the expression (1) is of a quite universal nature.

To the minimal value of the energy density (1) correspond the equilibrium values $W_1 = x/l$, $W_2 = y/l$, corresponding to alignment of the filaments of the discotic along the z axis, so that the filaments are defined by the equations $x = \text{const}$, $y = \text{const}$. As can be seen from the structure of the equilibrium equations, the quantity l defines the distance between the filaments. For the description of the deformation of the filaments one may introduce the vector **u**:

$$W_1 = (x-u_1)/l, \quad W_2 = (y-u_2)/l. \quad (2)$$

For small deviations from equilibrium u_1 and u_2 coincide with the displacements of the filaments along the x and y axes, respectively, in agreement with the standard notation of the linear theory.[1] Expanding Eq. (1) with respect to **u**, it is easy to find the harmonic part of the energy density

$$E^{(2)} = {}^1/_2 l^{-2}[(\beta_2+\beta)l^{-2}(\nabla_\alpha u_\alpha)^2 + \beta_2 l^{-2}(\varepsilon_{\alpha\beta}\nabla_\alpha u_\beta)^2 + \varkappa(\nabla^2 u_\alpha)^2]. \quad (3)$$

Here $\beta = \beta_1 + \beta_2$, ∇_α is a vector with the components ∇_x and ∇_y, and the matrix $\varepsilon_{\alpha\beta}$ is:

$$\varepsilon_{\alpha\beta} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}.$$

Thus, β corresponds to the longitudinal elastic modulus of the filaments, β_2 corresponds to the transverse modulus, and $\varkappa$ corresponds to the Frank bending modulus of the linear theory.[1]

Anomalously large fluctuations in discotics manifest themselves in the dynamics of these systems. A consideration of the fluctuations of the discotic variable leads to the appearance of a contribution to the kinetic coefficients which diverges for low frequencies. This contribution can be calculated with the help of the diagram technique for the interaction of long-wavelength fluctuations developed in Refs. 6 and 7. In many respects this discussion resembles the investigation of the dynamics of smectic liquid crystals,

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which was carried out by the present authors in Refs. 8 and 9. In particular, we shall make use of the representation for the effective action given in Ref. 9, as well as the procedure described there for the elimination of weakly fluctuating variables.

THE EFFECTIVE ACTION FOR THE ORDER PARAMETER OF DISCOTICS

Taking account of all the hydrodynamic variables of the discotic, the energy density E is a function of the following variables:

$$E=E(\rho,\ \sigma,\ \mathbf{j},\ \nabla_i W_\alpha,\ \nabla_i\nabla_k W_\alpha). \quad (4)$$

Here ρ is the mass density, σ is the specific entropy, $\mathbf{j}$ is the momentum density, and W_α is the discotic variable introduced in the Introduction. The thermodynamic identity for the energy density has the form

$$dE=\mu d\rho+T\rho d\sigma+\mathbf{v}d\mathbf{j}+\psi_{\alpha k}\nabla_k dW_\alpha+\nabla_k(\psi_{\alpha ik}\nabla_i dW_\alpha), \quad (5)$$

where μ is the chemical potential, $\mathbf{v}$ is the velocity, T is the temperature, and $\psi_{\alpha i}$ and $\psi_{\alpha ik}$ are the variables which are thermodynamic conjugates to $\nabla_i W_\alpha$ and $\nabla_i\nabla_k W_\alpha$. The pressure P is defined as follows:

$$P=\mu\rho+\mathbf{v}\mathbf{j}-E. \quad (6)$$

Finally, the nondissipative nonlinear equations for discotics (cf., e.g., Ref. 10) have the following form:

$$\frac{\partial\rho}{\partial t}+\nabla\mathbf{j}=0,\quad \frac{\partial\sigma}{\partial t}+\mathbf{v}\nabla\sigma=0,$$
$$\frac{\partial W_\alpha}{\partial t}+\mathbf{v}\nabla W_\alpha=0,\quad \frac{\partial j_i}{\partial t}+\nabla_k T_{ik}=0. \quad (7)$$

Here the stress tensor is

$$T_{ik}=[P+\nabla_m(\psi_{\alpha mn}\nabla_n W_\alpha)]\delta_{ik}+\rho v_i v_k+\psi_{\alpha k}\nabla_i W_\alpha-\nabla_i\psi_{\alpha kn}\nabla_n W_\alpha. \quad (8)$$

We now consider the expression for the energy density of a discotic corresponding to the expansion (1)

$$E=\mathbf{j}^2/2\rho+\varepsilon+E^{(W)}, \quad (9)$$

where $E^{(W)}$ is given by Eq. (1), and the quantities $\varepsilon, \beta_1, \beta_2, l, \varkappa$ are functions of ρ and σ. Calculating the pressure according to Eq. (6), we find

$$P=\rho\partial\varepsilon/\partial\rho-\varepsilon+P^{(W)}. \quad (10)$$

Here the discotic part of the pressure is

$$P^{(W)}=\frac{\beta\gamma}{l^2}(\nabla W_\alpha\nabla W_\alpha-2l^{-2})+\frac{1}{8}\left(\rho\frac{\partial\beta}{\partial\rho}-\beta\right)[(\nabla W_\alpha)^2-2l^{-2}]^2$$
$$+\frac{1}{8}\left(\rho\frac{\partial\beta_2}{\partial\rho}-\beta_2\right)\{[(\nabla W_1)^2-(\nabla W_2)^2]^2+4(\nabla W_1\nabla W_2)\}$$
$$+\frac{1}{2}\left(\rho\frac{\partial\varkappa}{\partial\rho}-\varkappa\right)(\nabla^2 W_\alpha)^2. \quad (11)$$

In Eq. (11) we have introduced the notation:

$$\gamma=(\partial\ln l/\partial\ln\rho)_\sigma. \quad (12)$$

The discotic contribution to the stress tensor is

$$T_{ik}^{(W)}={}^1/_2\beta(\nabla W_\alpha\nabla W_\alpha-2l^{-2})\nabla_i W_\beta\nabla_k W_\beta$$
$$+{}^1/_2\beta_2(2\nabla_i W_\alpha\nabla_j W_\alpha\nabla_j W_\beta\nabla_k W_\beta-\nabla_j W_\beta\nabla_j W_\beta\nabla_i W_\alpha\nabla_k W_\alpha) \quad (13)$$
$$-\varkappa\nabla_k\nabla^2 W_\alpha\nabla_i W_\alpha-\varkappa\nabla_i\nabla^2 W_\alpha\nabla_k W_\alpha$$
$$+[P^{(W)}+\nabla_n(\varkappa\nabla^2 W_\alpha\nabla_n W_\alpha)]\delta_{ik}.$$

Before going on to nonlinear effects we note that in the linear approximation[1] the equations (7) describe three modes of the acoustic type. The first of these is related to oscillations of the density and represents ordinary longitudinal sound, the velocity c of which is determined by the compressibility $c^2=\rho\partial^2\varepsilon/\partial\rho^2$. The remaining two modes are related to the deformation of the lattice of filaments and have the following frequency spectrum:

$$\omega_l^2=(a_l k_\perp^2+bk^4)\frac{k_\parallel^2}{k^2},\quad \omega_t^2=(a_t k_\perp^2+bk^4). \quad (14)$$

Here

$$a_l=(\beta+\beta_2)\rho^{-1}l^{-4},\quad a_t=\beta\rho^{-1}l^{-4},\quad b=\varkappa\rho^{-1}l^{-2},$$

$k_\parallel$ and $k_\perp$ are respectively the components of the wave vector along the filaments and in a plane perpendicular to them. The first expression (14) corresponds to a longitudinal wave in which the displacement vector $\mathbf{u}$ is parallel to $\mathbf{k}_\perp$, and the second one corresponds to a transverse wave in which $\mathbf{u}$ is perpendicular to $\mathbf{k}_\perp$. Note that the relations (14) have been obtained taking into account the smallness of the ratios a_l/c^2 and a_t/c^2, for which the following estimate holds

$$a/c^2\sim 10^{-3}. \quad (15)$$

The estimate (15) shows that the spontaneous symmetry breaking in discotics is weak.

As was shown in Refs. 6 and 7, the generating functional for the (equal-time) correlators of the hydrodynamic quantitites φ_a (among which for discotics are comprised ρ, σ, $\mathbf{j}$, W_α) with the fluctuations taken into account, can be written in the form[9]:

$$Z(\tilde{m},\tilde{y})=\int D\tilde{\varphi}D\tilde{p}\exp\left[\int dt\,d^3r\,d^2\theta\exp(i\theta\bar{\theta})\right.$$
$$\left.(L+\tilde{p}_a\tilde{y}_a+\tilde{m}_a\tilde{\varphi}_a)\right]. \quad (16)$$

The expression (16) contains

$$\tilde{y}_a=y_a+i\bar{\theta}\nu_a,\quad \tilde{m}_a=m_a+i\bar{\nu}_a\theta,$$
$$\tilde{p}_a=p_a+\bar{\psi}_a\theta,\quad \tilde{\varphi}_a=\varphi_a+\bar{\theta}\psi_a. \quad (17)$$

Here θ, $\bar{\theta}$ are Grassmann (anticommuting) variables for which the Berezin integral is defined so that

$$\int d^2\theta\,\theta\bar{\theta}=1.$$

The functional expansion of Z defined in Eq. (16) with respect to the variables $y, m, \nu, \bar{\nu}$ yields[6,7] the correlators of the quantities $\varphi, p, \psi, \bar{\psi}$.

The reactive part of the Lagrange density L is defined by the nondissipative hydrodynamic equations $\partial\varphi_a/\partial t+F_a(\varphi)=0$ (for discotics these are the equations (7)) and has the form[9]

$$L_r=\tilde{p}_a\left[\frac{\partial \varphi_a}{\partial t}+F_a(\varphi)\right]. \quad (18)$$

The dissipative part of the Lagrange density is determined by the kinetic terms in the hydrodynamic equations and has the form

$$L_d=i\tilde{p}_a\Sigma_{ab}\tilde{\varphi}_b+\frac{i}{2}\tilde{p}_a\Pi_{ab}\tilde{p}_b. \quad (19)$$

The self-energy operator Σ and the polarization operator Π are related by an equation of the type of the fluctuation-dissipation theorem.[6,7]

In the system considered here the discotic variable W_α fluctuates strongly and the other hydrodynamic variables ρ, σ, $\mathbf{j}$ are weakly fluctuating. Therefore one can integrate explicitly with respect to the latter in Eq. (16), retaining in the exponential only the quadratic part in the weakly fluctuating variables. As a result in Eq. (16) there remain only the integrations with respect to the variables $\tilde{p}_\alpha$ and $\tilde{W}_\alpha$, with the integrand preserving its structure; the exponential contains now the following Lagrange density[9]:

$$L=\tilde{p}_\alpha\partial\tilde{W}_\alpha/\partial t+i\tilde{p}_\alpha\Sigma_{\alpha\beta}\tilde{W}_\beta+{}^1/_2 i\tilde{p}_\alpha\Pi_{\alpha\beta}\tilde{p}_\beta -i(\rho^{-1}\tilde{p}_\alpha\nabla_i\tilde{W}_\alpha+\tilde{m}_i)B_{ik}^{-1}(F_k+\tilde{y}_k). \quad (20)$$

Here $F_k(W)=\nabla_n T_{kn}^{(W)}$, and the differential operator B_{ik} will be described below. In the derivation of (20) we have ignored the (unrenormalized) dissipative part (19) of the Lagrange density, which is related to weakly fluctuating variables, since for low frequencies the kinetic terms in the hydrodynamic equations are much smaller than the reactive terms.

The differential operator B_{ik} which occurs in Eq. (20) has the following definition:

$$B_{ik}=i\delta_{ik}\frac{\partial}{\partial t}-ic^2\left(\frac{\partial}{\partial t}\right)^{-1}\nabla_i\nabla_k+iF_{i,k}^{(W)},$$
$$F_{i,k}^{(W)}=-\Phi_i\left(\frac{\partial}{\partial t}\right)^{-1}\nabla_k. \quad (21)$$

The last term in B_{ik} is expressed in terms of the differential operator Φ_i defined as follows:

$$\nabla_k\delta T_{ik}^{(W)}=\Phi_i\delta\rho.$$

Making use of the explicit expression (13) we obtain for the leading terms (terms that do not vanish at equilibrium):

$$F_{i,k}^{(W)}=-\frac{2\beta\gamma}{\rho l^2}\nabla_n\left(\nabla_i\tilde{W}_\beta\nabla_n\tilde{W}_\beta+\frac{2}{l^2}\delta_{in}\right)\left(\frac{\partial}{\partial t}\right)^{-1}\nabla_k \quad (22)$$

THE EFFECTIVE ACTION FOR THE DISCOTIC VARIABLE

Before starting any concrete calculations we give an estimate for the characteristic wave vector k corresponding to the discotic mode. The real part of its oscillation spectrum is determined by the relations (14) from which for a given frequency ω we obtain for the components of a characteristic wave vector the estimates

$$k_\perp\sim\omega/\sqrt{a},\quad k_\parallel^2\sim\omega/\sqrt{b}. \quad (23)$$

This implies that for low frequencies the following inequality is true:

$$k_\parallel\gg k_\perp. \quad (24)$$

Thus, in the intermediate integration over the discotic variable one may set $k\simeq k_\parallel$.

We now consider the expression (21) for the discotic intermediate frequency. The inequaltiy $a/c^2\ll 1$, Eq. (15), guarantees that the last term in (21) is small compared to the second term, and on account of the estimate (23) the same inequality guarantees the smallness of the first term in Eq. (21) compared to the second term. In this situation we find for the inverse of the operator B_{ik} the Fourier representation

$$B_{ik}^{-1}\approx\omega^{-1}(\delta_{ik}-k_ik_k/\mathbf{k}^2). \quad (25)$$

Thus, for intermediate discotic frequencies the operator B_{ik}^{-1} becomes an orthogonal projector. In this situation one may omit from the quantity $F_k(W)=\Delta_n T_{kn}^{(W)}$ the last term in (13). To leading order one may neglect the derivatives of $\Delta_i\tilde{W}_\alpha$, which occur in Eq. (20), and carry out the substitution

$$\nabla_i\tilde{W}_\alpha\nabla_i\tilde{W}_\beta\to l^{-2}\delta_{\alpha\beta}.$$

As a result of all this we obtain the following Lagrange density for the discotic variable:

$$L_{disc}=\tilde{p}_\alpha\frac{\partial\tilde{W}_\alpha}{\partial t}-\frac{1}{2}(a_l-a_t)\tilde{p}_\alpha'\nabla_k\left(\frac{\partial}{\partial t}\right)^{-1}\times\{[(\nabla W_\beta)^2-2l^{-2}]\nabla_k\tilde{W}_\alpha\}$$
$$-\frac{a_t}{2}\tilde{p}_\alpha'\nabla_k\left(\frac{\partial}{\partial t}\right)^{-1}(2\nabla_j\tilde{W}_\alpha\nabla_j\tilde{W}_\beta\nabla_k\tilde{W}_\beta-\nabla_j\tilde{W}_\beta\nabla_j\tilde{W}_\beta\nabla_k\tilde{W}_\alpha)$$
$$+b\tilde{p}_\alpha'\left(\frac{\partial}{\partial t}\right)^{-1}\nabla^4\tilde{W}_\alpha+i\tilde{p}_\alpha\Sigma_{\alpha\beta}\tilde{W}_\beta+\frac{i}{2}\tilde{p}_\alpha\Pi_{\alpha\beta}\tilde{p}_\beta. \quad (26)$$

Here

$$\tilde{p}_\alpha'=(\delta_{\alpha\beta}-\nabla_\alpha\nabla_\beta\nabla_\perp^{-2})\tilde{p}_\beta. \quad (27)$$

The unrenormalized values of the self-energy function $\Sigma_{\alpha\beta}$ and of the polarization operator $\Pi_{\alpha\beta}$ in Eq. (26), which have their origin in the traditional kinetic terms of the hydrodynamic equations,[6,7] exhibit the following frequency dependence:

$$\Pi_{\alpha\beta}=\text{const},\quad \Sigma_{\alpha\beta}\propto\omega^2.$$

However, even the first correction to $\Sigma_{\alpha\beta}$ and $\Pi_{\alpha\beta}$, necessitated the interaction of the fluctuations carries a lower power of the frequency than the unrenormalized quantities, i.e., this correction exceeds the unrenormalized terms at hydrodynamic frequencies. Thus, for the discotic variable the domain of applicability of traditional hydrodynamics disappears, and the expressions for $\Sigma_{\alpha\beta}$ and $\Pi_{\alpha\beta}$ must be derived from an equation which takes account of the self-interaction of the discotic mode to leading order of perturbation theory. As a result one obtains

$$\Sigma_{\alpha\beta}=-2ig|k_\parallel|^{1+\sqrt{2}}\left[\frac{k_\parallel^2k_\alpha k_\beta}{k^2k_\perp^2}+\frac{\varepsilon_{\alpha\delta}k_\delta\varepsilon_{\beta\gamma}k_\gamma}{k_\perp^2}\right], \quad (28)$$

$$\Pi_{\alpha\beta}=\tau\,\frac{|k_\parallel|^{1+\sqrt{2}}}{\omega^2}\left[\frac{k_\parallel^4 k_\alpha k_\beta}{k^4 k_\perp^2}+\frac{\varepsilon_{\alpha\delta}k_\delta\varepsilon_{\beta\gamma}k_\gamma}{k_\perp^2}\right]. \quad (29)$$

Here g and τ are constants for which an expression will be derived below.

It follows from the structure of the Lagrange function (27) that the following pair correlators referring to the discotic variable are non-zero:

$$G_{\alpha\beta}(t_1-t_2,\ \mathbf{r}_1-\mathbf{r}_2)=-\langle W_\alpha(t_1,\ \mathbf{r}_1)p_\beta(t_2,\ \mathbf{r}_2)\rangle$$
$$=i\langle\psi_\alpha(t_1,\ \mathbf{r}_1)\bar{\psi}_\beta(t_2,\ \mathbf{r}_2)\rangle, \quad (30)$$

$$D_{\alpha\beta}(t_1-t_2,\ \mathbf{r}_1-\mathbf{r}_2)=-\langle W_\alpha(t_1,\ \mathbf{r}_1)W_\beta(t_2,\ \mathbf{r}_2)\rangle. \quad (31)$$

Expanding (27) to second order we find that the pair correlators decompose into a longitudinal and a transverse part, so that in the Fourier representation we have

$$G_{\alpha\beta}=G_l\frac{k_\alpha k_\beta}{k_\perp^2}+G_t\frac{\varepsilon_{\alpha\delta}k_\delta\varepsilon_{\beta\gamma}k_\gamma}{k_\perp^2}, \quad (32)$$
$$D_{\alpha\beta}=D_l\frac{k_\alpha k_\beta}{k_\perp^2}+D_t\frac{\varepsilon_{\alpha\delta}k_\delta\varepsilon_{\beta\gamma}k_\gamma}{k_\perp^2}.$$

In these relations

$$G_{l,\,t}=\omega[\omega^2-(a_{l,\,t}k_\perp^2+bk^4-2ig\omega|k_\parallel|^{1+\sqrt{2}})\zeta_{l,\,t}]^{-1}, \quad (33)$$

$$D_{l,\,t}(\omega,\mathbf{k})=G_{l,\,t}(\omega,\mathbf{k})\,\tau\,\frac{|k_\parallel|^{1+\sqrt{2}}}{\omega^2}\zeta_{l,\,t}G_{l,\,t}(-\omega,-\mathbf{k}). \quad (34)$$

For the longitudinal correlators G_l and D_l one should substitute $\zeta_l=k_\parallel^2k^{-2}$, and for the transverse correlators one should substitute $\zeta_t=1$.

We note that the expressions (33) and (34) imply the relation

$$D_{l,t}(\omega,\mathbf{k})=-\frac{\tau\zeta_{l,t}}{4g\omega^2}[G_{l,t}(\omega,\mathbf{k})+G_{l,t}(-\omega,-\mathbf{k})]. \quad (35)$$

Making use of this relation and of the fact that the function $G(\omega)$ is holomorphic in the upper half-plane, it is easy to calculate the integral (a similar calculation holds for the transverse mode)

$$-\int\frac{d\omega}{2\pi}D_l(\omega,k)=\frac{\tau}{4g(a_lk_\perp^2+bk^4)}.$$

This integral yields the equal-time correlator $\langle W_l(\mathbf{k})W_l(-\mathbf{k})\rangle$. On the other hand, this correlator could be computed starting from Eq. (3). Comparing the two expressions we find the ratio

$$\tau/g=4T/\rho l^2. \quad (36)$$

If one takes into account the explicit expressions (28)–(34), the relation (36) guarantees the validity of a fluctuation-dissipation theorem for the discotic variable.

FLUCTUATION CONTRIBUTIONS TO THE OSCILLATION SPECTRUM OF DISCOTICS

As already mentioned, the unrenormalized values of the polarization operator $\Pi_{\alpha\beta}$ and of the self-energy function $\Sigma_{\alpha\beta}$ are unimportant. Thus, for $\Pi_{\alpha\beta}$ and $\Sigma_{\alpha\beta}$ one needs to take into account only the expressions given by the diagrams represented in Figs. 1 and 2, where the dashed line

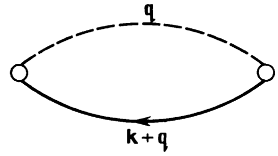

FIG. 1.

corresponds to the D function and the solid line to the G function. The triple interaction vertices which occur in these diagrams can be obtained by expanding the expression (26) with respect to the deviation of W_α from its equilibrium value. Important for us will be the singular contribution to $\Pi_{\alpha\beta}$ and $\Sigma_{\alpha\beta}$ coming from denominators which appear when the lines on the diagrams of Fig. 1 and 2 are either both longitudinal or both transverse. In this situation the expression for the leading part of the interaction vertex has the form (the expression corresponds to the definitions of Ref. 6 and 7)

$$V_\alpha=ial\omega_1^{-1}(k_{1\parallel}k_{2\parallel}k_{3\alpha}+k_{1\parallel}k_{2\alpha}k_{3\parallel}+k_{1\alpha}k_{2\parallel}k_{3\parallel}).$$

Here the subscript 1 refers to the variable P_α and the subscripts 2 and 3 to the variables δW_α; the index α in V_α should be contracted with the index of the external variable of the diagram; for the longitudinal lines $a=a_l$, for the transverse lines $a=a_t$.

We now substitute V_α into the diagrams of Figs. 1 and 2 and obtain in the leading approximation the following expressions:

$$\Pi_{l,t}(\omega,\mathbf{k})=\frac{l^2k_\parallel^2}{\omega^2}\zeta_{l,t}^2\int\frac{d\nu\,d^3q}{(2\pi)^4}q_\parallel^2q_\perp^2$$
$$\times[a_l^2D_l(\nu,\mathbf{q})D_l(\omega-\nu,\mathbf{k}-\mathbf{q})+a_t^2D_t(\nu,\mathbf{q})D_t(\omega-\nu,\mathbf{k}-\mathbf{q})], \quad (37)$$

$$\Sigma_{l,t}(\omega,\mathbf{k})=-\frac{2l^2k_\parallel^2}{\omega}\zeta_{l,t}\int\frac{d\nu\,d^3q}{(2\pi)^4}\left(q_\parallel-\frac{k_\parallel}{2}\right)^2\left(\mathbf{q}_\perp-\frac{\mathbf{k}_\perp}{2}\right)^2$$
$$\times\frac{1}{\nu}[a_l^2G_l(\nu,\mathbf{q})D_l(\omega-\nu,\mathbf{k}-\mathbf{q})+a_t^2G_t(\nu,\mathbf{q})D_t(\omega-\nu,\mathbf{k}-\mathbf{q})]. \quad (38)$$

Making use of the explicit expressions (33) and (34) as well as of the symmetry properties of the integrand in (38) one can show that the following relation holds for the integrals (37) and (38)

$$\operatorname{Im}\Sigma_{l,t}=-(2g/\tau)\zeta_{l,t}^{-1}\Pi_{l,t}\omega^2. \quad (39)$$

The real part $\operatorname{Re}\Sigma_{\alpha\beta}$ gives a small correction to the real part of the spectrum and is unimportant for us. Thus everything reduces to an analysis of the integral (37); this analysis is relegated to the Appendix. As a result we obtain for $\Pi_{\alpha\beta}$ an expression having the structure of Eq. (29), and for the

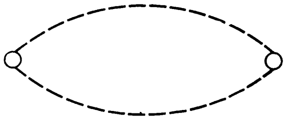

FIG. 2.

determination of the constant τ we obtain an equation which together with Eq. (36) yields

$$g^{1+\sqrt{2}}=-\frac{b^{(\sqrt{2}-1)/2}}{8\sqrt{2}\,\pi\cos(\pi\sqrt{2}/2)}\frac{T}{\rho}. \qquad (40)$$

The quantity τ is easily determined from Eq. (36).

We now consider the contribution of the fluctuations to the spectrum of the weakly fluctuating quantities. Such a contribution appears only in the spectrum of acoustic oscillations which are determined by the poles of the Green's function $G_{ik}(\omega)$, where

$$G_{ik}(t_1-t_2,\ \mathbf{r}_1-\mathbf{r}_2)=-\langle j_i(t_1,\ \mathbf{r}_1)p_k(t_2,\ \mathbf{r}_2)\rangle. \qquad (41)$$

The expression for this function, as well as the expression for the pair correlator

$$D_{ik}(t_1-t_2,\ \mathbf{r}_1-\mathbf{r}_2)=-\langle j_i(t_1,\ \mathbf{r}_1)j_k(t_2,\ \mathbf{r}_2)\rangle \qquad (42)$$

can be obtained by expanding Eq. (16) with respect to the variables y_i and m_i. The corresponding Lagrange density is given by Eq. (20). It follows from this expression that the unrenormalized value of G_{ik} is determined by the function B_{ik}^{-1}, whereas the fluctuation corrections are collected into the self-energy function Σ_{ik}:

$$G_{ik}^{-1}=\omega\delta_{ik}-c^2\omega^{-1}k_ik_k-\Sigma_{ik}. \qquad (43)$$

In Σ_{ik} one must first of all take into account the contribution coming from the term (22). In addition in G_{ik} there appears a contribution related to the diagrams with intermediate single[1)] $G_{\alpha\beta}$ lines. Such diagrams appear in the expansion of Z taking account of Eq. (20) in the pairing of the factor p_β with the factor ∇W_α from F_k. Retaining in the corresponding expression the part which does not vanish at equilibrium and joining it to (22), we obtain

$$\mathrm{Re}\,\Sigma_{ik}=\frac{\beta}{\rho l^4\omega}(2\gamma k_i+k_{i\perp})(2\gamma k_k+k_{k\perp})+\frac{\beta_2}{l^4\omega}\delta_{ik}{}^{\perp}k_\perp{}^2. \qquad (44)$$

The expression (44) yields a correction to the real part of the sound oscillation spectrum which is simply due to the presence of the discotic part of the stress tensor $T_{ik}^{(W)}$ in the equation for j_i. Therefore $\mathrm{Re}\Sigma_{ik}$ is not fluctuational in origin.

The fluctuations determine the imaginary part of Σ_{ik}. This imaginary part is due to loop diagrams of the type represented in Fig. 1. First, such diagrams appear in pairings of W_α from Eq. (22) and p_α with a pair of W_α from F_k [see Eq. (20)]. Second, one must take into account a loop in the diagrams of the type mentioned above with a single intermediate $G_{\alpha\beta}$ line (in this case the single $G_{\alpha\beta}$ line enters into the diagram of Fig. 1 from the right). As a result of this (taking account of footnote 1) we obtain the following expression

$$\Sigma_{ik}=-4\rho^{-2}l^{-6}\omega^{-1}[\beta^2(\gamma k_i+{}^1/_2k_{i\perp})(\gamma k_k+{}^1/_2k_{k\perp})+{}^1/_8\beta_2{}^2k_\perp{}^2\delta_{ik}]\int\frac{d\nu\, d^3q}{(2\pi)^4}\frac{(\mathbf{q}-\mathbf{k}/2)^4}{\nu}$$
$$\times[G_l(\nu,\mathbf{q})D_l(\omega-\nu,\mathbf{k}-\mathbf{q})+G_t(\nu,\mathbf{q})D_t(\omega-\nu,\mathbf{k}-\mathbf{q})]. \qquad (45)$$

In addition to the mentioned diagrams we also consider diagrams for the pair correlator (42) which can be collected into the following expression:

$$D_{ik}(\omega,\ \mathbf{k})=G_{in}(\omega,\ \mathbf{k})\Pi_{nm}(\omega,\ \mathbf{k})G_{km}(-\omega,\ -\mathbf{k}). \qquad (46)$$

The diagram for the polarization operator Π_{mn} is analogous to the one in Fig. 2, and is due to the pairing of two pairs of W from F_k [see Eq. (20)]. We obtain in the leading approximation

$$\Pi_{ij}=\frac{2}{l^4}\left[\beta^2\left(\gamma k_i+\frac{1}{2}k_{i\perp}\right)\left(\gamma k_j+\frac{1}{2}k_{j\perp}\right)+\frac{1}{8}\beta_2{}^2k_\perp{}^2\delta_{ij}{}^{\perp}\right]$$
$$\times\int\frac{d\nu\, d^3q}{(2\pi)^4}q^4[D_l(\nu,\mathbf{q})D_l(\omega-\nu,\mathbf{k}-\mathbf{q})+D_t(\nu,\mathbf{q})D_t(\omega-\nu,\mathbf{k}-\mathbf{q})]. \qquad (47)$$

Taking account of the explicit expressions (33), (34), the relation (36), and the symmetry of the integrand in (45) one can derive

$$\mathrm{Im}\,\Sigma_{ik}=-(1/2\rho T)\Pi_{ik}. \qquad (48)$$

Recalling the structure of (43) and (46) we see that the relation (48) is a fluctuation-dissipation theorem. The analysis of the integral which occurs in Eq. (47) is carried out in the Appendix.

CONCLUSION

We have thus shown that a consideration of the fluctuation corrections radically changes our notions about the spectrum of discotics. As far as the discotic longitudinal and transverse modes are concerned, their dispersion laws are defined by the poles of the Green's function (33) and have the form

$$\omega_{l,\,t}=-ig\zeta_{l,\,t}|k_\parallel|^{1+\sqrt{2}}+[\zeta_{l,\,t}(a_{l,\,t}k_\perp{}^2+bk^4)]^{1/2}. \qquad (49)$$

The damping decrement of these modes which is determined by the first term in Eq. (49) manifestly exceeds the unrenormalized value for characteristic wave vectors.

The case $k_\parallel\sim k_\perp$ requires a separate discussion. In this situation the damping decrement (49) becomes smaller than the unrenormalized value which is proportional to $k_\perp^2$. The leading contributions of the fluctuations to Σ and Π can again be obtained respectively from diagrams of the type of Fig. 1 and 2, in which one must take into account terms proportional to $k_\perp^2$. We obtain for the longitudinal and transverse parts of the polarization operator

$$\Pi_l=\left(\frac{1}{2}a_l{}^2+a_la_t+\frac{3}{4}a_t{}^2\right)\frac{k_\parallel{}^2}{l^2k^2a_l}(I_{2l}+I_{2t}), \qquad (50)$$
$$\Pi_t=\frac{a_t}{4l^2}\frac{k_\parallel{}^2}{k^2}(I_{2l}+I_{2t}).$$

The explicit form of the integral I_2 can be found in the Appendix [Eq. (A7)]. In this limit the expression for Σ is related to (50) in the same manner as the expression (38) is related to (37), and consequently the fluctuation-dissipation theorem (39) holds. The analysis of the integral which occurs in Eq. (50) given in the Appendix shows that in the case $k_\perp\sim k_\parallel$ ω and $k_\perp$ are cutoff factors in the integral, so that one must use the expression (A10) for the integral (A7). As a result of this we obtain for Σ and Π the estimate

$$-\frac{2Tk^2}{\rho l^2\omega^2k_\parallel{}^2}\mathrm{Im}\,\Sigma=\Pi\sim10^{-2}\frac{\tau^2}{l^2bg^{1+\sqrt{2}}|\omega|^{2-\sqrt{2}}}. \qquad (51)$$

Thus, $\Pi \sim |\omega|^{\sqrt{2}-2}$ diverges for small frequencies and exceeds the unrenormalized $\Pi = \text{const}$, so that in the case $k_\perp \sim k_\parallel$ too the attenuation of discotic modes $\text{Im}\,\Sigma \sim |\omega|^{\sqrt{2}}$ is of fluctuational origin.

The sound spectrum is determined by the poles of the Green's function (43). The real part of the discotic contribution to the sound spectrum (44) yields an anisotropic contribution to the velocity of sound:

$$\delta c = \frac{\beta}{2l^4 c\rho}\left(2\gamma + \frac{k_\perp^2}{k^2}\right)^2 + \frac{\beta_2}{2l^4 c\rho}\frac{k_\perp^4}{k^4}, \quad (52)$$

which on account of Eq. (15) is small. The attenuation of sound is determined by the imaginary part of the self-energy function Σ_{ik} which can be determined from Eqs. (47) and (48). For sound $\omega = ck$, and therefore on account of the estimates the cutoff factor in the integral (47) is the frequency ω, so that we can make use of the expression (A11) for the integral (A7). As a result we obtain for the longitudinal part of the self-energy function

$$-\frac{1}{2}\text{Im}\,\Sigma_\parallel = \frac{1}{Tk^2}\left[\beta^2\left(\gamma k^2 + \frac{1}{2}k_\perp^2\right)^2 + \frac{1}{8}\beta_2^2 k_\perp^4\right]$$
$$\times \frac{(\sqrt{2}-1)|\omega|^{\sqrt{2}-2}}{128\cdot 2^{1/2}\pi\sin(\pi\sqrt{2}/2)}\frac{1}{b}\left(\frac{1}{\beta_2} + \frac{1}{\beta+\beta_2}\right)\frac{\tau^2}{g^{1+\sqrt{2}}}. \quad (53)$$

The expression (53) determines directly the damping decrement of the sound waves.

We see that the fluctuation contribution (53) to sound attenuation behaves proportionally to $\omega^{\sqrt{2}}$, i.e., decreases slower than the unrenormalized contribution, which is proportional to ω^2. Thus, for small frequencies, the fluctuation contribution to the attenuation of sound manifestly exceeds the unrenormalized one. However, the expression (53) contains both a numerical smallness, as well as a smallness related to the fact that the thermal energy is much smaller than the sound energy $\rho c^2 l^3$. Therefore, for realistic frequencies such as the ones used in experiments, both contributions may compete, fact which should be taken into account in treating experimental data.

Note added in proof (26 December 1983): Recently there has appeared a short note by S. Ramaswamy and J. Toner [Phys. Rev. **A28**, 3159 (1983)] devoted to the dynamics of discotics. These authors have taken into account only the first correction of perturbation theory and therefore have obtained an incorrect result for the contribution of fluctuations to the kinetic coefficients: $\sim\omega^{-1/2}$ in place of $\omega^{\sqrt{2}-2}$.

APPENDIX

Consider the integral which occurs in Eq. (38)

$$I_1 = \int \frac{d\nu\, d^3q}{(2\pi)^4} q_\parallel^2 q_\perp^2 D(\nu, \mathbf{q}) D(\omega-\nu, \mathbf{k}-\mathbf{q}). \quad (A1)$$

It is more convenient to carry out the integration with respect to v by shifting the integration path into the upper half-plane, after which the integration reduces to taking the residues of the integrand (it is convenient to make use of the representation (35)). It will be seen below that

$$q \gg k, \quad (A2)$$

where q is a characteristic wave vector in Eq. (A1). Therefore the dependence on k and ω in the integral so obtained should be retained only in the singular denominators, which have the form

$$G^{-1}(\nu, \mathbf{q}) - G^{-1}(-\omega-\nu, -\mathbf{k}-\mathbf{q}).$$

In the imaginary part of this difference one may neglect the dependence on k and ω, and in the real part one must take into account only the dependence on $k_\parallel$ as the main cutoff factor. As a result we obtain

$$I_1(k_\parallel) = \frac{i\tau^2}{32g^2}\int\frac{d^3q}{(2\pi)^3}\frac{q_\perp^2}{\eta^3}[bq_\parallel k_\parallel - ig\eta|q_\parallel|^{\sqrt{2}-1}]^{-1}. \quad (A3)$$

Here

$$\eta^2 = aq_\perp^2 + bq_\parallel^4. \quad (A4)$$

Changing over to the coordinates $x = b^{1/4}q_\parallel/\eta^{1/2}$, $\xi = \eta^{1\sqrt{2}}$, we find

$$I_1(k_\parallel) = \frac{\sqrt{2}}{4\pi^2 a^2 b|k_\parallel|}\int_0^1 dx(1-x^4)x^{2\sqrt{2}-3}$$
$$\times\int_0^\infty \frac{d\xi\,\xi^{\sqrt{2}-1}}{x^{2\sqrt{2}-2}}\left(\frac{1}{1-iA\xi} - \frac{1}{1+iA\xi}\right), \quad (A5)$$
$$A = g[|k_\parallel|x^{2-\sqrt{2}}b^{(2+\sqrt{2})/4}]^{-1}.$$

The integral with respect to ξ is easily done if one represents it as an integral over a contour which surrounds the positive half-axis. Deforming this contour, we find that the integral reduces to residues at points and a divergent integral over an infinite contour which has to be thrown away, corresponding to the subtraction from (A1) a constant which diverges but does not depend on ω and on k. After the integration with respect to x, we obtain, finally,

$$I_1(k_\parallel) = \frac{i\sqrt{2}\,b^{(\sqrt{2}-1)/2}|k_\parallel|^{\sqrt{2}-1}}{4\pi a^2\cos(\pi\cdot\sqrt{2}/2)g^{\sqrt{2}}}. \quad (A6)$$

We also consider the integral which occurs in Eq. (47):

$$I_{2l,t} = \int\frac{d\nu\,d^3q}{(2\pi)^4}q^4 D_{l,t}(\nu, \mathbf{q}) D_{l,t}(\omega-\nu, \mathbf{k}-\mathbf{q}). \quad (A7)$$

The integral with respect to the frequency ν in (A7) is calculated similar to (A1). The inequality (A2) holds again, so that the dependence on ω, k need be retained only in singular denominators. Omitting in (A7) the dependence on $k_\parallel$ and expanding to first order in $\mathbf{k}_\perp$ and ω, we obtain as a result of integration over the frequency ν:

$$I_2 = \frac{i\tau^2}{32g^2}\int\frac{d^3q}{(2\pi)^3}\frac{q_\parallel^4}{\eta^4}$$
$$\times\left[\left(\omega - \frac{a}{\eta}\mathbf{k}_\perp\mathbf{q}_\perp + 2ig|q_\parallel|^{1+\sqrt{2}}\right)^{-1} + (\omega\to-\omega)\right]. \quad (A8)$$

We make the change of variables:

$$\sqrt{b}q_\parallel^2 = \eta\cos\psi, \quad \sqrt{a}q_\perp = \eta\sin\psi. \quad (A9)$$

The integration in (A8) with respect to the variables η and ψ can be done explicitly, yielding

$$I_2=\frac{\tau^2(\sqrt{2}-1)}{64\pi^2 2^{\sqrt{2}}abg^{1+\sqrt{2}}\sin(\pi\sqrt{2}/2)}\frac{1}{|\omega|^{2-\sqrt{2}}}$$
$$\times\int_0^{2\pi}\frac{d\theta}{h^2\cos^2\theta}\left[\frac{|1+h\cos\theta|^{\sqrt{2}}-1}{\sqrt{2}}\right.$$
$$\left.-\frac{|1+h\cos\theta|^{\sqrt{2}-1}\operatorname{sgn}(1+h\cos\theta)-1}{\sqrt{2}-1}\right]. \quad \text{(A10)}$$

Here $h=a^{1/2}k_\perp/\omega$. In the case $k_\perp$ the integration with respect to θ in (A10) is trivial and we obtain for the integral I_2

$$I_2=\frac{(\sqrt{2}-1)\tau^2|\omega|^{\sqrt{2}-2}}{64\cdot 2^{\sqrt{2}}\pi\sin(\pi\sqrt{2}/2)g^{1+\sqrt{2}}ab}. \quad \text{(A11)}$$

[1)]These lines have the frequency $\omega\sim ck$ and therefore on account of the inequality (15) we can neglect in Eq. (33) the terms with the wave vector, so that on intermediate lines $G_{\alpha\beta}\simeq\delta_{\alpha\beta}/\omega$.

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