# Exact Dicke superradiance theory: Bethe wavefunctions in the discrete atom model

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(Submitted 31 May 1983; resubmitted 27 July 1983)
Zh. Eksp. Teor. Fiz. **86**, 819–825 (March 1984)

We prove the complete integrability of the Dicke superradiance model with discrete atoms. We construct the Bethe wavefunctions of the system for problems with infinite or finite intervals. We obtain a general expression for the probability of decay of an initial excited state of an atomic subsystem into eigenstates of the model. We study the transition to a previously-considered model of a continuous medium.

**1.** Recently the complete integrability of a quantum model of Dicke superradiance of extended, essentially one-dimensional systems was proved in Refs. 1 by means of the quantal inverse scattering method (see Refs. 2,3). One of the basic approximations used in constructing the model is the continuous medium approximation. We relax in the present paper that restriction and show that the "field + discrete atoms" model is also completely integrable.

**2.** Bearing in mind a subsequent comparison of the results for continuous and discrete media we first of all construct explicitly the wavefunctions for the continuous medium model on an infinite interval $-\infty < x < \infty$. The model studied in Ref. 1 is defined by the Hamiltonian

$$H=-i\int_{-\infty}^{\infty} dx\varepsilon^+(x)\,\partial_x\varepsilon(x)-\varkappa^{1/2}\int_{-\infty}^{\infty} dx[\varepsilon^+(x)s^-(x)+s^+(x)\varepsilon(x)] \tag{1}$$

and the commutation relations for the electromageic field operators:

$$[\varepsilon(x),\ \varepsilon^+(y)]=\delta(x-y) \tag{2}$$

and the spin operators describing the resonance medium:

$$[s^i(x),\ s^k(y)]=ie^{ikl}s^l(x)\,\delta(x-y). \tag{3}$$

Here $i=1,2,3$; $s^{\pm}=s^1\pm is^2$, and $\varkappa$ is the interaction constant. Such a model was proposed in the framework of classical field theory to describe superradiance in Ref. 4 and was later studied in a number of papers (see Ref. 5).

One sees easily that the elementary single-particle excitation of the model (1) to (3) has the following wavefunction:

$$|\lambda\rangle=\int_{-\infty}^{\infty} dx e^{ikx}r^+(x,\lambda)\,|0\rangle. \tag{4}$$

In this expression the state $|0\rangle$ is the vacuum of the model which has the properties

$$\varepsilon|0\rangle=0,\quad s^-|0\rangle=0,\quad s^3|0\rangle=-{}^1/_2n|0\rangle;$$

the operator

$$r^+(x,\lambda)=\varepsilon^+(x)-(\varkappa^{1/2}/\lambda)\,s^+(x)$$

is the operator creating an excitation, the parameter

$$k=\lambda-\varkappa n/\lambda$$

plays the role of the momentum of the state, while $n$ is the linear density of the medium.

The wavefunction of the $N$-particle state is constructed from the single-particle wavefunction (4) using the Bethe Ansatz:[6]

$$|\lambda_1,\ldots,\lambda_N\rangle=\int_{-\infty}^{\infty} dx_1\ldots dx_N\exp\Big(i\sum_{i=1}^N k_ix_i\Big)$$
$$\times\prod_{i<j}^N\Big[1+\frac{i\varkappa}{\lambda_i-\lambda_j}\operatorname{sign}(x_i-x_j)\Big]\,r^+(x_1,\lambda_1)\ldots r^+(x_N,\lambda_N)\,|0\rangle, \tag{5}$$

$$H|\lambda_1,\ldots,\lambda_N\rangle=\Big(\sum_{i=1}^N\lambda_i\Big)|\lambda_1,\ldots,\lambda_N\rangle. \tag{6}$$

One can check the validity of Eq. (6) by direct calculation using the obvious relation

$$s^+(x)=\frac{1}{\varkappa^{1/2}}\frac{\lambda\mu}{\lambda-\mu}[r^+(x,\lambda)-r^+(x,\mu)]. \tag{7}$$

We note that the state (5) is created through the action of the off-diagonal element $B(\lambda)$ of the monodromy matrix of the auxiliary quantal spectral scattering problem studied in Ref. 1:

$$|\lambda_1,\ldots,\lambda_N\rangle=B(\lambda_1)\ldots B(\lambda_N)\,|0\rangle. \tag{8}$$

Such a construction of the Bethe states was first used by Sklyanin[7] when studying the quantal non-linear Schrödinger equation.

**3.** The Hamiltonian of the model with discrete atoms which are situated at the points $\{x_a\}$ has the form

$$H_d=-i\int_{-\infty}^{\infty} dx\varepsilon^+(x)\,\partial_x\varepsilon(x)-\varkappa^{1/2}\sum_a[\varepsilon^+(x_a)s_a^-+\varepsilon(x_a)s_a^+], \tag{9}$$

where the atomic operators $s_a^i$ satisfy the commutation relations

$$[s_a^i, s_b^k]=ie^{ikl}s_a^l\delta_{ab}. \tag{10}$$

When studying the model (9), (10) we meet with a singularity typical of models with a linear spectrum of bare particles and a delta-function-shape interaction (see, e.g., Ref. 8). This singularity manifests itself already in the wavefunctions of the single-particle excitations $|\lambda\rangle$ ($H_d|\lambda\rangle=\lambda|\lambda\rangle$). Writing the state $|\lambda\rangle$ in the form

$$|\lambda\rangle=\int_{-\infty}^{\infty} dxf(x)\,\varepsilon^+(x)\,|0\rangle+\sum_a c_as_a^+|0\rangle, \tag{11}$$

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we have the following set of Schrödinger equations for the coefficients $f(x)$ and $c_a$:

$$i\partial_x f(x)+\lambda f(x)=-\varkappa^{1/2}\sum_a c_a\delta(x-x_a), \tag{12}$$

$$\varkappa^{1/2}\int_{-\infty}^{\infty} dx f(x)\delta(x-x_a)=-\lambda c_a. \tag{13}$$

It follows from Eq. (12) that the function $f(x)$ is discontinuous in the points $x=x_a$ and the integral on the left-hand side of Eq. (13) is thus undetermined. The necessary regularization is obtained by replacing $\delta(x-x_a)$ by a function $u(x-x_a)$ and subsequently taking the limit as $u(x)\to\delta(x)$.[8] One checks easily that the regularization corresponds to the substitution:

$$f(x_a)={}^1/_2[f(x_a+0)+f(x_a-0)]. \tag{14}$$

Taking what we have said into account we have for $x\neq x_a$

$$f(x,\lambda)=\exp\left[i\lambda x+\sum_a\theta(x-x_a)\ln\frac{1-i\varkappa/2\lambda}{1+i\varkappa/2\lambda}\right], \tag{15}$$

$$c_a=-(\varkappa^{1/2}/\lambda)f(x_a). \tag{16}$$

Using (14) to (16) we rewrite Eq. (11) in the form

$$|\lambda\rangle=\int_{-\infty}^{\infty}dx f(x,\lambda)\left[\varepsilon^+(x)-\frac{\varkappa^{1/2}}{\lambda}\sum_a s_a^+\delta(x-x_a)\right]|0\rangle. \tag{17}$$

The structure of the single-particle state (17) is completely analogous to the structure of the single-particle state (4) of the continuous model. One checks easily by direct calculations that the $N$-particle excited states of the model with discrete atoms are described by Bethe wavefunctions:

$$|\lambda_1,\ldots,\lambda_N\rangle=\int_{-\infty}^{\infty}dx_1\ldots dx_N\prod_i f(x_i,\lambda_i)$$
$$\times\prod_{i<j}\left[1+\frac{i\varkappa}{\lambda_i-\lambda_j}\operatorname{sign}(x_i-x_j)\right]\prod_i r_d^+(x_i,\lambda_i)|0\rangle,$$
$$r_d^+(x,\lambda)=\varepsilon^+(x)-\frac{\varkappa^{1/2}}{\lambda}\sum_a s_a^+\delta(x-x_a). \tag{18}$$

Here

$$H_d|\lambda_1,\ldots,\lambda_N\rangle=\left(\sum_{i=1}^N\lambda_i\right)|\lambda_1,\ldots,\lambda_N\rangle,$$

$$N_d|\lambda_1,\ldots,\lambda_N\rangle=N|\lambda_1,\ldots,\lambda_N\rangle,$$

where the excitation number operator $N_d$ has the form

$$N_d=\int_{-\infty}^{\infty}dx\varepsilon^+(x)\varepsilon(x)+\sum_a(s_a^3+{}^1/_2). \tag{19}$$

The transition to the case of a continuous medium occurs in the limit $\varkappa/\lambda\ll 1$; then

$$f(x,\lambda)\to\exp\,[i(\lambda-\varkappa n/\lambda)x].$$

The eigenvalues $\lambda$ are determined by the relation $\lambda\sim\varkappa n/\lambda$ and the condition $\varkappa/\lambda\ll 1$ then takes the form $n\varkappa^{-1}\ll 1$, which corresponds to a large number of atoms in the size of the packet.

4. Above we considered the problem on an infinite interval. In the case of the problem on a finite interval $-L/2\leqslant x\leqslant L/2$ with periodic boundary conditions the eigenfunctions of the system are as before determined by Eqs. (5) for a continuous medium and by Eqs. (18) for the model with discrete atoms (with the appropriate limits for the region of integration). The parameters $\{\lambda_i\}$ satisfy a set of transcendental equations:

a) continuous medium

$$e^{i\lambda_i L}=\prod_{j\neq i}^{N}\frac{\lambda_i-\lambda_j-i\varkappa}{\lambda_i-\lambda_j+i\varkappa},\quad i=1,\ldots,N, \tag{20}$$

b) discrete medium

$$e^{i\lambda_i L}\left(\frac{\lambda_i-i\varkappa/2}{\lambda_i+i\varkappa/2}\right)^M=\prod_{j\neq i}^{N}\frac{\lambda_i-\lambda_j-i\varkappa}{\lambda_i-\lambda_j+i\varkappa},\quad i=1,\ldots,N, \tag{21}$$

where $M$ is the number of atoms and $N$ the number of elementary excitations of the model.

5. The results obtained in Ref. 1 by the quantal inverse scattering method, just as the results of the present paper in which we constructed explicitly the Bethe wavefunctions of the model, are still insufficient to completely interpret the known experimental data on the effect of superradiance extended systems.[4,5] In particular, there remain unexplained such important features of the kinetics of the superradiance process as the delay time of a Dicke pulse or the structure of the experimentally observed state.

At present one can distinguish in the extensive literature devoted to the theory of superradiance two basic approaches. The first, to which most authors adhere, regards the oberved state of the system as the result of a dynamic decay of the initial excited state of the atomic subsystem. In the second approach the observed state is regarded as a thermodynamic-equilibrium phase formed as the result of fast relaxation processes between states in a narrow spectral range near resonance.

We shall not discuss here in any detail the second approach. We note merely that the transcendental Eqs. (20) and (21) obtained above are the basis for constructing exact thermodynamic functions of the extended Dicke model.

The first—dynamic—treatment of the superradiance phnomenon seems to us to be more realistic and therefore we formulate in what follows the problem of the kinetics of Dicke superradince of extended systems in the framework of the dynamic approach for the "field + discrete atoms" model.

Let initially at $t=0$ only the atomic subsystem be in an excited state. The wavefunction of the model then has at $t=0$ the form

$$|\Phi_0\rangle=\frac{1}{(m!)^{1/2}}s_1^+\ldots s_m^+|0\rangle, \tag{22}$$

where $m$ is the number of excited atoms. This state is normalized to unity:

$$\langle\Phi_0|\Phi_0\rangle=1,$$

and is not, as can easily be verified, an eigenstate of the model. As time goes on it will therefore decay into a set of eigenstates determined by the Bethe wavefunctions (18) where the parameters rapidities $\{\lambda_i\}$ satisfy the set of transcendental Eqs. (21). Our problem is to calculate the probabilities for the transition of the initial state (22) into the possible eigenstates (18), (21).

As the number $N_d$ of elementary excitations of the model (19) is conserved:

$$[N_d, H_d]=0,$$

only the probabilities for decay into the $m$-particle Bethe state are non-vanishing. In the quantal inverse scattering method the $m$-particle eigenstate is given by Eq. (8) (see also Ref. 1) so that to determine the probability amplitude for the decay

$$A(\{x_a\}\mid\{\lambda_i\})=\langle 0|B^+(\lambda_1)\dots B^+(\lambda_m)s_1^+\dots s_m^+|0\rangle \quad (23)$$

we should establish the commutation relations between the operators $B^+(\lambda_i)$ and $s_a^+$ and afterwards commute the $m$ operators $B^+(\lambda_i)$ with the $m$ operators $s_a^+$. The explicit form of the Bethe wavefunctions found in the present paper enables us to avoid solving that complicated problem.

We note that the operator part

$$\prod_{i=1}^{m} r_d^+(x_i,\lambda_i)|0\rangle$$

of the Bethe wavefunction (18) has only one term containing $m$ oprators $s_a^+$:

$$(-1)^m\frac{\varkappa^{m/2}}{\lambda_1\dots\lambda_m}\prod_{i=1}^{m}\sum_{a=1}^{M}s_a^+\delta(x_i-x_a)|0\rangle.$$

In the scalar product of the Bethe and the initial states there is therefore a contribution from only a single term, while all other contributions vanish.

We thus have

$$A(\{x_a\}\mid\{\lambda_i\})=(-1)^m\frac{\varkappa^{m/2}}{(m!)^{1/2}}\|\lambda_1,\dots\lambda_m\|^{-1/2}\left(\prod_{i=1}^{m}\lambda_i\right)^{-1}$$
$$\times\int_{-L/2}^{L/2}\prod_{i=1}dx_i\prod_{i=1}^{m}f(x_i,\lambda_i)\prod_{i<j}^{m}\left[1+\frac{i\varkappa}{\lambda_i-\lambda_j}\operatorname{sign}(x_i-x_j)\right]$$
$$\times\sum_{\{x_a\}}\prod_{i=1}^{m}\delta(x_i-x_a), \quad (24)$$

where $f(x,\lambda)$ is given by Eqs. (14) and (15) while $\|\dots\|$ is the norm, recently calculated by Korepin,[9] of the Bethe wavefunctions.

The probability for the decay of the initial state into the final Bethe state is given by the square of the modulus of expression (24):

$$W(\{\lambda_i\})=\langle\langle|A(\{x_a\}\mid\{\lambda_i\})|^2\rangle\rangle, \quad (25)$$

where the rates $\{\lambda_i\}$ satisfy the set of transcendental Eqs. (21) while the angular brackets $\langle\langle\dots\rangle\rangle$ denote averages over the initial arrangement of the excited atoms. We note here to avoid misunderstandings that the use of periodic boundary conditions in deriving (24) is physically justified if we study superradiant decay in a system such as a ring resonator.

If the size of the system $L$ is sufficiently large the rapidities $\{\lambda_i\}$ which are solutions of the set of transcendental Eqs. (21) are, with exponential accuracy in the complex plane, all gathered in "strings" and the possible number of particles in each string $l^\alpha$ lies between 1 and $m$, i.e., $1\leqslant l^\alpha\leqslant m$.[2,3,1] One can classify the Bethe state by the number of strings of a given kind $\nu_l$ and the magnitude of the real part of the rapidities $\lambda_l^{(\alpha)}$ which is common to all which occur in a given string and

$$\sum_\alpha l^\alpha\nu_l=m.$$

If we are not interested in the spectral distribution of the strings in the final state of the system and evaluate the probability for the decay into a type of state characterized by the size $l$ and the number $\nu_l$ of the strings, we must sum expression (25) over all possible values of the real parts $\lambda_l^{(\alpha)}$.

The physical quantity which is directly measured experimentally is the intensity of the radiation which can be expressed in terms of the current density of the elementary excitations of the system.[5] The operator $\hat\rho(x)$ of the number of excitations has clearly the form

$$\hat\rho(x)=\varepsilon^+(x)\varepsilon(x)+\sum_{a=1}^{M}\left(s_a^3+\frac{1}{2}\right)\delta(x-x_a).$$

The total number of excitations

$$N_d=\int_{-L/2}^{L/2}dx\,\hat\rho(x)$$

is then the same as expression (19). Using the commutation relations (2) and (10) we can check that the operators $\hat\rho(x)$ satisfies the continuity equation

$$\hat\rho_t+\hat j_x=0,$$

where the current operator $\hat j(x)$ is the same as the photon-number density operator

$$\hat j(x)=\varepsilon^+(x)\varepsilon(x). \quad (26)$$

(We remember that we are working with a system of units in which the light velocity $c=1$.)

If initially at $t=0$ the state of the system $|\Phi_0\rangle$ is given by Eq. (22), it takes at all later times the form

$$|\Phi(t)\rangle=\exp(-iH_dt)|\Phi_0\rangle$$
$$=\sum_{\{\lambda_i\}}\bar A(\{x_a\}|\{\lambda_i\})\exp(-iE[\{\lambda_i\}]t)|\lambda_1,\dots,\lambda_m\rangle, \quad (27)$$

where

$$E[\{\lambda_i\}]=\sum_{i=1}^{m}\lambda_i$$

is the energy of the state while the summation is over all possible sets of solutions of Eqs. (21).

At the initial instant the current in the system vanishes, i.e.,

$$\langle\Phi_0|\hat j(x)|\Phi_0\rangle=0,$$

while at time $t$ it is given by the expression

$$j(x,t) = \langle \Phi(t) \mid \hat{j}(x) \mid \Phi(t) \rangle$$

$$= \sum_{\{\lambda_i\}} \sum_{\{\bar{\lambda}_j\}} \ll \bar{A}(\{x_a\} \mid \{\bar{\lambda}_j\}) A(\{x_a\} \mid \{\lambda_i\}) \gg$$

$$\times \langle \{\bar{\lambda}_j\} \mid \varepsilon^+(x)\, \varepsilon(x) \mid \{\lambda_i\} \rangle \exp[it(E[\{\bar{\lambda}_j\}] - E[\{\lambda_i\}])]. \quad (28)$$

The complete solution of the superradiance problem thus does require not only calculating the decay probability amplitude (24) but also calculating the matrix elements of the operator $\varepsilon^+\varepsilon$ in the Bethe states. The solution of such problems is at present one of the basic problems in the theory of quantal completely integrable models. We hope, however, that the problem of evaluating the decay probability of the initial state will be solved in the near future.

The authors express their gratitude to V. M. Agranovich and I. V. Lerner for their interest in this paper and for useful discussion. We are also pleased to thank P. B. Wiegmann for a number of valuable discussions and critical remarks.

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Translated by D. ter Haar