Quasiclassical equations of the theory of superconductivity for contiguous metals and the properties of constricted microcontacts

A. V. Zaĭtsev

Institute of Radio Engineering and Electronics, Academy of Sciences of the USSR

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Quasiclassical equations describing nonstationary and nonequilibrium phenomena in structures containing boundaries between metals are derived, and the boundary conditions (at the metal interface) for these equations are found. The boundary conditions are derived for an arbitrary transparency (and arbitrary shape) of the potential barrier, whose appearance at the boundary may be due to a difference in the parameters of the contiguous metals, the presence of a dielectric layer between the metals, etc. In the case of tunnel junctions the obtained boundary conditions allow one to easily derive for the current an expression that coincides with those obtained on the basis of the method of tunneling Hamiltonians. The general results are used to study the properties of different types of constrictions (N_1cN_2, ScN, S_1cS_2) with allowance for electron reflection at the metal interface. The case of "pure" constrictions (whose characteristic dimension $a \ll l_{1,2}$, where l_j is the mean free path) is studied in greatest detail. Expressions are found for the resistance of the N_1cN_2 junction and the boundary resistance R_b (in this structure), which determines the potential jump $V_b = R_b I$ at the boundary. A general expression is derived for the current in the ScN junction, and the influence of the boundary transparency on the shape of the current–voltage characteristic $I(V)$, the dependence $\sigma(V) = dI/dV$, and the magnitude of the excess current is analyzed. For the S_1cS_2 junctions, a general relation connecting the current with the phase difference at zero voltage potential is found, and the excess current is compared with the critical current for different potential-barrier transparencies. The case of "dirty" constrictions ($l_j \ll a$) is briefly discussed.

It is well known that an effective method of solving the problems of the theory of superconductivity is provided by the quasiclassical equations for the Green Functions. These equations (based, naturally, on the more general Gor'kov equations[1]), which explicitly take account of the fact that the characteristic scale of the spatial variation of all the macroscopic quantities substantially exceeds the interatomic distance, were first derived for the stationary and equilibrium case by Eilenberger.[2] The quasiclassical equations describing the nonstationary and nonequilibrium processes in superconductors were derived by Éliashberg[3] and Larkin and Ovchinnikov.[4]

The equations obtained in Refs. 2–4 do not, as a rule, allow us to study structures containing boundaries between metals. The point is that, for a number of reasons (differences between the parameters of the contiguous metals, the presence of a dielectric layer or of a gap between the metals, and other factors), the electrons can undergo reflection from the boundary. As a consequence, at some distance from the boundary, the Green function $\check{G}(\mathbf{r},\mathbf{r}')$ can no longer be considered to be a slowly varying function of the resultant coordinate $(\mathbf{r}+\mathbf{r}')/2$, as is done in Refs. 2–4.

In the present paper we derive a set of quasiclassical equations that allow us to investigate the nonequilibrium and nonstationary phenomena that occur in structures containing one or several parallel metal interfaces. These equations contain, besides the matrix function $\check{g}$, the equation for which is similar to the one obtained in Refs. 3 and 4, a matrix function $\check{\mathscr{G}}$ describing the waves reflected from the boundaries. The functions $\check{g}$ and $\check{\mathscr{G}}$ on the two sides of a boundary are matched with the aid of boundary conditions. The latter are derived for an arbitrary shape and arbitrary transparency of the potential barrier $U(z,\rho)$ (see (2)), which varies smoothly in the contact plane. A closed boundary condition containing only the function $\check{g}$ is found for the case in which the distance between neighboring boundaries is much greater than the mean free path. From this boundary condition it follows, in particular, that the quasiclassical functions $\check{g}$ undergo a jump at the boundary if the coefficient of transmission through the boundary $D \neq 1$. This result is valid also for a boundary at which metals with different parameters are in direct contact (with no dielectric layer or a gap between them). We note that the question of the boundary conditions for the Éliashberg equations has been analyzed for this particular case by Ivanov *et al.*,[5] who arrive at the wrong conclusion that the quasiclassical functions are continuous at boundaries that, on the scale of interatomic distances, are sharp.

The boundary condition obtained allows us to easily derive for the current in a tunnel junction a general expression that, in the case of a junction formed by homogeneous metals, leads to the same results given by the tunneling-Hamiltonian method.[6–8]

The theory constructed is used in §2 to study constricted microcontacts. We know that of greatest interest are superconducting constrictions in which the Josephson effects are manifested.[9] Their theoretical study, which was originally begun with the use of the Ginzburg-Landau equations by Aslamazov and Larkin,[10] has in recent years been carried out with the aid of a more general method: the method of

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quasiclassical equations for the Green functions. The stationary properties of ScS junctions (where S is a superconductor and c is a constriction) have been studied with the use of the Eilenberger equations by Kulik and Omel'yanchuk,[11] who have constructed a theory of the stationary Josephson effect in constrictions whose characteristic dimension a satisfies the conditions

$$l \ll a \ll (D/\Delta)^{1/2}, \tag{1a}$$

$$a \ll l,\ v_F/\Delta, \tag{1b}$$

where l is the mean free path. A theory of superconducting constrictions in the presence of voltage potentials at the junctions has been constructed (with the aid of the equations obtained in Refs. 3 and 4) by Artemenko, Volkov, and the present author[12] (the case (1a)) and by the present author[13] (the case (1b)). Information about the earlier investigations, based mostly on the Ginzburg–Landau equations, can be found in Likharev's review articles.[9]

Besides superconducting constrictions, junctions of the type ScN (where N is a normal metal) have in recent years been studied intensively both experimentally[14–18] and theoretically.[19,13] It turns out that the current-voltage characteristic (CVC) of such structures, which is nonlinear in the region $V \lesssim \Delta$, has in the region of high voltage potentials $V \gg \Delta$, as in the case of ScS junctions, an excess current, i.e., approaches an asymptote that is shifted relative to the ohmic straight line $I = V/R_N$ by an amount that does not depend on V. Notice that S and N metals with different parameters were used in the experiments reported in Refs. 14–18. At the same time, the model considered in the theoretical investigations[19,13] is one in which the Fermi velocities in the contiguous metals are assumed to be equal.

In §2 we investigate the properties of different types of constrictions: N_1cN_2, ScN, and S_1cS_2. Here we take account of electron reflection at the metal interfaces, which, as has already been noted, may be due to the differences between the parameters of the contiguous metals, the presence of a dielectric layer or a gap between the metals, the presence of defects localized in the vicinity of the boundary, etc. General relations are obtained which are not connected with any assumptions about the shape and transparency of the potential barrier. The greatest attention is given to the case of "pure" ($l_{1,2} \gg a$) contacts, the "dirty" limit being discussed briefly at the end of the section.

Let us note that the properties of the ScN junction have been analyzed with allowance for the potential barrier $U = U_0\,\delta(z)$ at the metal interfaces by Blonder *et al.*[20] But these authors make a number of intuitive assumptions (e.g., the possibility of computing the coefficient of reflection in the model of a stepwise varying gap), and, moreover, use a one-dimensional contact model in which, in particular, the dependence of the excitation distribution function and the reflection coefficient on the direction of the momentum is neglected.

§1. QUASICLASSICAL EQUATIONS FOR CONTIGUOUS METALS AND THE BOUNDARY CONDITIONS FOR THEM. CONSEQUENCES FOR TUNNEL JUNCTIONS

Our aim is to derive equations that describe nonstationary and nonequilibrium phenomena in structures containing one or several mutually parallel boundaries (such as the junctions SNS, SS_1S, etc.) whose spacing is significantly greater than the interatomic distance. Let us consider one of such boundaries. We shall assume that the properties of the metals vary in a thin layer (of thickness 2δ) near the plane $z = 0$. We shall describe the boundary as sharp if $\delta \sim p_{F1(2)}^{-1}$ and as smooth in the case when $\delta \gg p_{F1(2)}^{-1}$; and we shall label the quantities pertaining to the metal on the left (right) of the boundary in question by the subscript 1(2). There will arise at the metal interface a potential barrier $U(z, \rho)$ (ρ is a vector in the $z = 0$ plane) that varies in the transition layer of thickness 2δ. The potential should, in the region $|z| > \delta$, where it can be assumed to be constant, satisfy the following well-known condition, which guarantees the constancy of the electrochemical potential μ in the system:

$$\varepsilon_{F1} + U(z < -\delta) = \varepsilon_{F2} + U(z > \delta) = \mu.$$

The general expression for the potential satisfying this condition can be represented as follows:

$$U(z, \boldsymbol{\rho}) = U_0(z, \boldsymbol{\rho}) + \mu - u(z, \boldsymbol{\rho}), \tag{2}$$

where U_0 is an arbitrary delta function ($U_0 = 0$ for $|z| > \delta$), while the function $u(z, \rho)$, which can, without loss of generality, be assumed to be a monotonic function of z (the nonmonotonic part can always be included in U_0), is equal to ε_{F1} for $z < -\delta$ and ε_{F2} for $z > \delta$. The appearance of the potential barrier U_0 is most often due to the presence of a dielectric layer or of a gap between the metals; it may also be due to the presence of defects or a thin impurity layer localized near the boundary. We shall say that the metals are in direct contact if a potential barrier of the form U_0 does not occur at the boundary. Below we shall assume that the characteristic distance a over which the potential U changes in the boundary plane satisfies the quasiclassicality condition $a \gg p_{Fj}^{-1}$. We shall consider the transition-layer thickness 2δ to be small compared to $v_{Fj}/\bar{\varepsilon}$, l_j, and a, where $\bar{\varepsilon} = \max(T, \Delta, V, \omega)$ and V is the voltage potential at the junction.

We shall, in constructing the theory, proceed from a general equation that, in the Keldysh procedure,[21] can be written as follows (Ref. 4)[1)]:

$$\left(i\check{\tau}_z \frac{\partial}{\partial t} + \frac{1}{2m}\frac{\partial^2}{\partial \mathbf{r}^2} + \check{\Delta} - \Phi - U - \check{\Sigma} + \mu \right) \check{G}(\mathbf{r}, \mathbf{r}'; t, t')$$

$$= \check{1}\delta(\mathbf{r} - \mathbf{r}')\,\delta(t - t'). \tag{3}$$

Here the Green function $\check{G}$ and the self-energy part $\check{\Sigma}$ are matrices having the form

$$\check{G} = \begin{pmatrix} \hat{G}^R & \hat{G} \\ \hat{0} & \hat{G}^A \end{pmatrix}, \quad \check{\Sigma} = \begin{pmatrix} \hat{\Sigma}^R & \hat{\Sigma} \\ \hat{0} & \hat{\Sigma}^A \end{pmatrix},$$

where $\hat{G}^{R(A)}$ and $\hat{G}$ are 2×2 matrices formed from the ordinary Green functions and the Gor'kov functions (see Ref. 4). Furthermore,

$$\check{\tau}_z = \begin{pmatrix} \hat{\tau}_z & \hat{0} \\ \hat{0} & \hat{\tau}_z \end{pmatrix}, \quad \check{\Delta} = \begin{pmatrix} \hat{\Delta} & \hat{0} \\ \hat{0} & \hat{\Delta} \end{pmatrix}, \quad \hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix},$$

$$\check{\Sigma}\check{G} = \int dt_1 \check{\Sigma}(t, t_1)\,\check{G}(t_1, t'),$$

where Δ is the order parameter and $\hat{\tau}_z$ is the Pauli matrix. In (3) Φ is the electric potential that arises upon the passage of current, and U is the "equilibrium" electric potential given by the expression (2); we set the electron charge to be equal to unity. Going over to the Fourier representation in terms of the coordinate $\boldsymbol{\rho} - \boldsymbol{\rho}'$, and taking account of the quasiclassical nature of the variation of $\check{G}$ as a function of $\boldsymbol{\rho}_c = (\boldsymbol{\rho} + \boldsymbol{\rho}')/2$, we obtain for $\check{G}(z,z') \equiv \check{G}(z,z',\boldsymbol{\rho}_c,\mathbf{p}_\parallel;t,t')$, where $\mathbf{p}_\parallel$ is the momentum in the contact plane, the equation (we shall henceforth drop the subscript c on ρ_c)

$$\left(i\check{\tau}_z \frac{\partial}{\partial t} + \frac{1}{2m}\frac{\partial^2}{\partial z^2} + i\frac{\mathbf{v}_\parallel}{2}\frac{\partial}{\partial \boldsymbol{\rho}} + \check{\Delta} - \check{\Sigma} - \Phi - U - \frac{p_\parallel^2}{2m} + \mu \right) \check{G}(z,z') = \check{1}\delta(z-z')\delta(t-t'). \tag{4}$$

Let us represent $\check{G}(z,z')$ in the form of a sum of terms in which the rapidly oscillating parts have been explicitly separated. From (4), as well from the equation conjugate to it, it is clear that there will be four such terms. For example, for z, $z' < -\delta$

$$\check{G} = \check{G}_{11}\exp[ip_{z1}(z-z')] + \check{G}_{22}\exp[-ip_{z1}(z-z')] + G_{12}\exp[ip_{z1}(z+z')] + \check{G}_{21}\exp[-ip_{z1}(z+z')], \tag{5}$$

where $p_{z1} = (p_{F1}^2 - p_\parallel^2)^{1/2}$; p_{z1} in (5) should be replaced by p_{z2} when z, $z' > \delta$. Substituting (5) into (4), and neglecting the second derivatives, we obtain for the smoothly varying functions $\check{G}_{kn}(z,z')$ the equations

$$\left(i\check{\tau}_z \frac{\partial}{\partial t} - (-1)^k i v_{z1}\frac{\partial}{\partial z} + i\frac{\mathbf{v}_\parallel}{2}\frac{\partial}{\partial \boldsymbol{\rho}} + \check{\Delta} - \check{\Sigma} - \Phi \right)\check{G}_{kn}(z,z') = \check{0}, \quad z \neq z'. \tag{6}$$

Similarly, from the equation that is conjugate to (4), we shall have

$$\check{G}_{kn}(z,z') \times \left(-i\check{\tau}_z \frac{\partial}{\partial t'} + (-1)^n i v_{z1}\frac{\partial}{\partial z'} - i\frac{\mathbf{v}_\parallel}{2}\frac{\partial}{\partial \boldsymbol{\rho}} + \check{\Delta} - \check{\Sigma} - \Phi \right) = \check{0}, \quad z \neq z'. \tag{6'}$$

When $z,z' > \delta$, we should replace v_{z1} by v_{z2} in (6) and (6'). The values of the functions $\check{G}_{kn}(z,z')$ for $z < z'$ and $z > z'$ are matched with the aid of conditions that follow from (4):

$$\left[\frac{\partial}{\partial z}\check{G}(z,z') \right]_{z=z'+0} - \left[\frac{\partial}{\partial z}\check{G}(z,z') \right]_{z=z'-0} = 2m\cdot\check{1}\delta(t-t'),$$

$$\check{G}(z+0,\ z) = \check{G}(z-0,\ z), \qquad |z| > \delta.$$

Substituting (5) into them, we obtain

$$\check{G}_{kn}(z+0,\ z) - \check{G}_{kn}(z-0,\ z) = i\cdot\check{1}\delta(t-t')\delta_{kn}(-1)^k/v_{z1(2)}. \tag{7}$$

It can be seen from (6) and (6') that it makes sense to reduce the number of symbols by introducing functions $\check{g}$ and $\check{\mathcal{G}}$ that depend on a variable p_{zj} that can assume both positive and negative values:

$$\check{g}(z,z',p_{zj}) = \begin{cases} 2|v_{zj}|i\check{G}_{11}(z,z') - \operatorname{sign}(z-z'), & p_{zj} > 0, \\ 2|v_{zj}|i\check{G}_{22}(z,z') + \operatorname{sign}(z-z'), & p_{zj} < 0, \end{cases}$$

$$\check{\mathcal{G}}(z,z',p_{zj}) = 2|v_{zj}|i \begin{cases} \check{G}_{12}(z,z'), & p_{zj} > 0, \\ \check{G}_{21}(z,z'), & p_{zj} < 0. \end{cases} \tag{8}$$

As follows from (7), the thus defined functions are continuous at the point $z = z'$. From (6) and (6') we obtain for

$$\check{g}(\mathbf{R},\ \mathbf{p}_{Fj}) = \check{g}(z,\ z,\ \boldsymbol{\rho},\ p_{zj},\ \mathbf{p}_\parallel),$$

$$\check{\mathcal{G}}(\mathbf{R},\ \mathbf{p}_{Fj}) = \check{\mathcal{G}}(z,\ z,\ \boldsymbol{\rho},\ p_{zj},\ \mathbf{p}_\parallel),$$

equations having the form

$$\mathbf{v}_{Fj}\frac{\partial}{\partial \mathbf{R}}\check{g} + [\check{K},\check{g}]_- = \check{0}, \tag{9a}$$

$$v_{zj}\frac{\partial}{\partial z}\check{\mathcal{G}} + [\check{K},\check{\mathcal{G}}]_+ = \check{0}, \tag{9b}$$

where

$$\check{K}(t,t') = \left[\check{\tau}_z\frac{\partial}{\partial t} - i\check{\Delta}(t) + i\Phi(t) \right]\delta(t-t') + i\check{\Sigma}(t,t'),$$

$$[a,\ b]_\pm = ab \pm ba.$$

The first of the equations of the system has, as it should, the same form as the equations obtained in Refs. 3 and 4. The appearance of the second function, which does not occur in the equations derived in Refs. 3 and 4, is due to electron reflection from the metal interfaces. The system (9) is valid for $|z| > \delta$; therefore, to match the functions in the regions $z < -\delta$ and $z > \delta$, we must have boundary conditions, which we now proceed to obtain.

Let us take into account the fact that $\partial\check{G}/\partial t \sim \partial\check{G}/\partial t' \sim \bar{\varepsilon}\check{G}$, where $\bar{\varepsilon} \sim \max\ (T,\Delta,V,\omega)$. For $z \ll z^* \equiv \min(l_j,\ v_{Fj}/\bar{\varepsilon},\ a)$, Eq. (4) and its conjugate reduce to

$$H(z)\check{G}(z,\ z') = H(z')\check{G}(z,\ z') = \check{0},$$

$$H(z) \equiv \frac{1}{2m}\frac{\partial}{\partial z^2} - U(z,\boldsymbol{\rho}) + \mu - \frac{p_\parallel^2}{2m}, \tag{10}$$

where $z \neq z'$. In the region $|z|$, $|z'| > \delta$ we can write the solutions to (10) in the form

$$\check{G} = A_\pm^j\exp[ip_{zj}(z-z')] + \bar{A}_\pm^j\exp[-ip_{zj}(z-z')] + B^j\exp[ip_{zj}(z+z')] + \bar{B}^j\exp[-ip_{zj}(z+z')], \quad zz' > 0,$$

$$\check{G} = A^{jk}\exp[i(p_{zj}z - p_{zk}z')] + \bar{A}^{jk}\exp[-i(p_{zj}z - p_{zk}z')] + B^{jk}\exp[i(p_{zj}z + p_{zk}z')] + \bar{B}^{jk}\exp[-i(p_{zj}z + p_{zk}z')], \quad zz' < 0, \tag{11}$$

where we should set the subscripts j, $k = 1(2)$ in the region $z < -\delta (z > \delta)$, and the index $+(-)$ corresponds to $z > z' (z < z')$. As follows from (5) and (11), the matrices $A_\pm^j$, $\bar{A}_\pm^j$, B^j, and $\bar{B}^j$, which do not depend on z and z' (but, generally speaking, depend on $\boldsymbol{\rho}$), give the values of the functions $\check{g}$ and $\check{\mathcal{G}}$ in the immediate neighborhood of the boundary. On the other hand, we can write down the solutions to (10), using the two linearly independent solutions to the Schrödinger equation $H(z)\psi_{1,2}(z) = 0$, which have, in the region $|z| > \delta$, the following form:

$$\psi_1(z)=\begin{cases} e^{ip_{z1}z}+re^{-ip_{z1}z}, & z<-\delta \\ de^{ip_{z2}z}, & z>\delta, \end{cases}$$

$$\psi_2(z)=\begin{cases} e^{-ip_{z2}z}+\tilde{r}e^{ip_{z2}z}, & z>\delta, \\ \tilde{d}e^{-ip_{z1}z}, & z<-\delta. \end{cases} \quad (12)$$

Here, as in (11), $p_{zj}=(p_{Fj}^2-p_{\|}^2)^{1/2}$ is assumed to be a positive quantity. Using the properties of the solutions to the equation $H\psi=0$, we easily obtain the relations[2)]

$$\tilde{d}=dp_{z2}/p_{z1}, \quad \tilde{r}=-r^*d/d^* \quad (r=|r|e^{i\vartheta_r}, \quad d=|d|e^{i\vartheta_d}). \quad (13)$$

The transmission and reflection coefficients D and R are given by the following relations:

$$D=|d|^2(p_{z2}/p_{z1})=\tilde{D}=|\tilde{d}|^2(p_{z1}/p_{z2}),$$
$$R=|r|^2=1-D=\tilde{R}=|\tilde{r}|^2. \quad (14)$$

It follows from (10) that the solution for $|z|, |z'|\ll z^*$ can be represented in the form

$$\check{G}(z,z')=\begin{cases} e^{ip_{z1}z'}F_1(z)+e^{-ip_{z1}z'}\bar{F}_1(z), & z'<-\delta, & z>z', \\ e^{ip_{z2}z'}F_2(z)+e^{-ip_{z2}z'}\bar{F}_2(z), & z'>\delta, & z<z', \\ e^{ip_{z1}z}P_1(z')+e^{-ip_{z1}z}\bar{P}_1(z'), & z<-\delta, & z<z', \\ e^{ip_{z2}z}P_2(z')+e^{-ip_{z2}z}\bar{P}_2(z'), & z>\delta, & z'<z. \end{cases} \quad (15)$$

The matrices F_i, P_i, etc. satisfy the equation $H(z)\check{\psi}(z)=0$, where $\check{\psi}=F_i$ or P_i, etc.; therefore, they can be expressed in terms of $\psi_{1(2)}$:

$$F_i(z)=f_i^{(1)}\psi_1(z)+f_i^{(2)}\psi_2(z), \qquad P_i(z)=p_i^{(1)}\psi_1(z)+p_i^{(2)}\psi_2(z), \quad (16)$$

and similarly for $\bar{F}_i, \bar{P}_i$; here $f_i^{(1),(2)}, p_i^{(1,2)}$ are constants. By substituting the expressions (16) into (15), and taking (12) into account, we can, after comparing the resulting expressions with (11), arrive at the following set of relations:

$$v_{z1}A_\pm{}^1-v_{z2}A_\pm{}^2=v_{z1}\bar{A}_\pm{}^1-v_{z2}\bar{A}_\pm{}^2=|r|(v_{z2}B^2e^{i\vartheta_2}+v_{z1}\bar{B}^1e^{-i\vartheta_1})=|r|(v_{z1}B^1e^{i\vartheta_1}+v_{z2}\bar{B}^2e^{-i\vartheta_2}),$$

$$v_{z1}\bar{A}_\pm{}^1+v_{z2}A_\pm{}^2=v_{z1}A_\pm{}^1+v_{z2}\bar{A}_\pm{}^2=-|r|^{-1}(v_{z2}B^2e^{i\vartheta_2}-v_{z1}B^1e^{i\vartheta_1})=-|r|^{-1}(v_{z2}\bar{B}^2e^{-i\vartheta_2}-v_{z1}\bar{B}^1e^{-i\vartheta_1}), \quad \vartheta_{1(2)}=\vartheta_r-\vartheta_d\pm\vartheta_d, \quad (17)$$

which furnish the sought boundary equations. Before writing these conditions down, it is expedient, as can be seen from (17), to introduce the following notation:

$$\tilde{\mathscr{G}}=\check{\mathscr{G}}\exp(i\vartheta_{1(2)}\operatorname{sign}p_{z1(2)}), \quad z<-\delta \quad (z>\delta).$$

Notice that the function $\tilde{\mathscr{G}}$ also satisfies Eq. (9). Let us also introduce the symmetric (s) and antisymmetric (a)—in the variable p_{zj}—matrices

$$\check{g}_{\binom{s}{a}}=\tfrac{1}{2}[\check{g}(p_{zj})\pm\check{g}(-p_{zj})], \quad (18)$$

where $j=1(2)$ for $z<-\delta$ $(z>\delta)$; similarly, we define the functions $\tilde{\mathscr{G}}_{s(a)}$. Then the boundary conditions, which follow from (17) with allowance made for (8), (11), and (5), assume the following form:

$$\check{g}_a(-)=\check{g}_a(+)\equiv\check{g}_a(0), \quad \tilde{\mathscr{G}}_a(-)=\tilde{\mathscr{G}}_a(+)\equiv\tilde{\mathscr{G}}_a(0), \quad (19a)$$
$$\check{g}_s(+)-\check{g}_s(-)=-R^{1/2}[\tilde{\mathscr{G}}_s(+)+\tilde{\mathscr{G}}_s(-)], \quad (19b)$$
$$R^{1/2}[\check{g}_s(+)+\check{g}_s(-)]=\tilde{\mathscr{G}}_s(-)-\tilde{\mathscr{G}}_s(+),$$

where $\check{g}(\pm)\equiv\check{g}(z=\pm\delta, \boldsymbol{\rho}), \tilde{\mathscr{G}}(\pm)=\tilde{\mathscr{G}}(z=\pm\delta, \boldsymbol{\rho})$. Let us recall that, in deriving (19), we assumed that $p_{\|}<p_{F1,2}$. If we assume, for definiteness, that $p_{F1}<p_{F2}$, then we should, in the case when $z>\delta$, derive some more boundary conditions that encompass the interval $p_{F1}<p_{\|}<p_{F2}$. They are derived in much the same way as the preceding ones. It should only be taken into account here that the expression for the function $\check{G}(z,z')$ in the region $z<-\delta$ $(z'<-\delta)$ should be determined by that function in (12) which falls off with distance from the boundary into the interior of the first metal. This condition leads automatically to a situation in which the reflection coefficient R for electrons incident on the boundary from the right is equal to unity. As a result, we obtain the following boundary conditions:

$$\check{g}_a(+)=\check{0}, \quad p_{F1}<p_{\|}<p_{F2},$$
$$\check{\mathscr{G}}(+)\exp(iv_r\operatorname{sign}p_{z2})=\check{g}(+)+\check{1}\operatorname{sign}p_{z2}\delta(t-t'). \quad (20)$$

Similar relations will be satisfied for $z=-\delta$ in the case in which $p_{F1}>p_{F2}$.

To the system of equations and boundary conditions obtained must be added expressions for the macroscopic quantities. These expressions should, generally speaking, be found with the aid of the Green function $\check{G}(z,z')$. In doing this, however, it should be borne in mind that, after the evaluation of the integral $\int d^2p_{\|}\check{G}(z,z')$, the contribution from the functions $\check{G}_{12}$ and $\check{G}_{21}$ will, because of the presence of the rapidly oscillating factors $\exp[\pm ip_{zj}(z+z')]$, be small in the case when $|z+z'|\gg p_{Fj}^{-1}$, at least to the extent that the parameter $1/p_F|z+z'|$ is small. For this reason, it turns out that at distances from the boundary much greater than the atomic distances all the macroscopic quantities will be expressed by the usual relations[3,4] in terms of the quasiclassical function $\check{g}$. In particular, for the electric potential and the current density we have the expressions

$$\Phi=-\frac{\pi}{4}\int\frac{d\Omega}{4\pi}\operatorname{Sp}\hat{g}(t,t,\mathbf{p}_{Fj}),$$
$$\mathbf{j}=-\frac{p_{Fj}}{4\pi}\int\frac{d\Omega}{4\pi}\mathbf{p}_{Fj}\operatorname{Sp}\hat{\tau}_z\hat{g}(t,t,\mathbf{p}_{Fj}). \quad (21)$$

For the above-indicated reason, the matrix $\check{\Sigma}$ will also be determined by the function $\check{g}$ only. The expressions for $\check{\Sigma}$ can be found in Refs. 3 and 4. Thus, the first of the equations of the system (9) does not explicitly depend on the function $\check{\mathscr{G}}$. We shall now show that if the interference between the waves (described by the functions $\check{G}_{12}$ and $\check{G}_{21}$) emanating from neighboring boundaries can be neglected (the distance between the boundaries is large compared to the mean free path), then we can derive a boundary condition that contains only the function $\check{g}$. Let us, for the purpose of deriving it, show that the following relation obtains in the indicated case:

$$\check{g}\check{\mathscr{G}}=(-1)^j \operatorname{sign} p_{zj}\check{\mathscr{G}}. \qquad (22)$$

To prove (22), let us consider the function

$$\check{\mathscr{P}}_{nk}(z', \mathbf{R})=\check{G}_{nn}(z', z, \boldsymbol{\rho})\check{G}_{nk}(z, z', \boldsymbol{\rho}),$$

where

$$\check{G}_{nn}\check{G}_{nk}\equiv\int\check{G}_{nn}(t, t_1)\check{G}_{nk}(t_1, t')\,dt_1.$$

It is easy to verify, using (6) and (6′), that for $z\neq z'$

$$\left(v_z\frac{\partial}{\partial z}+\frac{1}{2}\mathbf{v}_\parallel\frac{\partial}{\partial\boldsymbol{\rho}}\right)\check{\mathscr{P}}_{nk}(z', \mathbf{R})=\check{0},$$

from which it follows that this function does not depend[3] on $\mathbf{R}$. Since the function $\check{\mathscr{P}}_{nk}$ is equal to zero at infinity (i.e., for $|z|\gg l_j$), we arrive at the result that for $z>\delta$ $(z<-\delta)$

$$\check{\mathscr{P}}_{nk}(z, \mathbf{R})\,|_{\mathbf{R}=(z\pm 0,\,\boldsymbol{\rho})}=\check{0},$$

and this, with allowance for (8), leads to (22) in the case when $n\neq k$ and to the following well-known normalization relation[4] in the case when $n=k$:

$$\check{g}\check{g}\equiv\check{g}^2=\check{1}\delta(t-t')\equiv\check{1}. \qquad (23)$$

From (22) and (23) we obtain

$$\check{g}_s\check{\mathscr{G}}_s+\check{g}_a\check{\mathscr{G}}_a=(-)^j\check{\mathscr{G}}_a, \qquad (24)$$

$$\check{g}_s\check{\mathscr{G}}_a+\check{g}_a\check{\mathscr{G}}_s=(-1)^j\check{\mathscr{G}}_s,$$

$$\check{g}_s^2+\check{g}_a^2=\check{1}, \quad [\check{g}_s, \check{g}_a]_+=\check{0}; \qquad (25)$$

here and below we assume that $p_{zj}>0$. With allowance for (19), we find from (24) that

$$2\check{\mathscr{G}}_a(0)=(\check{g}_s\check{\mathscr{G}}_s)(+)-(\check{g}_s\check{\mathscr{G}}_s)(-), \qquad (26)$$

$$-2(\check{g}_a\check{\mathscr{G}}_a)(0)=(\check{g}_s\check{\mathscr{G}}_s)(+)+(\check{g}_s\check{\mathscr{G}}_s)(-),$$
$$\check{\mathscr{G}}_s(\pm)=\mp R^{1/2}\check{g}_s^+(0)-R^{-1/2}\check{g}_s^-(0), \qquad (27)$$

where we have introduced the matrices

$$\check{g}_s^\pm(0)={}^1/_2[\check{g}_s(+)\pm\check{g}_s(-)]. \qquad (28)$$

Substituting (27) into (26), and taking into consideration the relations

$$[\check{g}_s^+(0), \check{g}_s^-(0)]_+=\check{0}, \quad [\check{g}_s^\pm(0), \check{g}_a(0)]_+=\check{0},$$

which follow from (25) and (19), we easily obtain the sought boundary condition:

$$\check{g}_a(0)\,[R(\check{g}_s^+(0))^2+(\check{g}_s^-(0))^2]=D\check{g}_c^-(0)\check{g}_s^+(0). \qquad (29)$$

For $p_{F1}<p_\parallel<p_{F2}$ the boundary condition for $\check{g}$ is given by the relation (20).[4)] Notice that the coefficients R and D in (29) are functions of p_{zj} and $\boldsymbol{\rho}$. As a result of our assumption that the potential varies smoothly in the contact plane, we have figuring in the boundary conditions (19), (20), and (29) functions with the same values of $p_\parallel$. As is well known, a boundary that is such that the component $p_\parallel$ is preserved in reflections from it is commonly called a specularly reflecting plane.

For $R=0$ it follows from (29), as it should, that $\check{g}_s^-(0)=\check{0}$, which, together with (19a), implies continuity of the function $\check{g}$, while in the case $D\to 0$ we again arrive at the result that at an impenetrable boundary $\check{g}_a(0)=\check{0}$ (this case has been considered before by Kulik and Omel'yanchuk[11]). For $R\neq 0$ we find from (29) that $\check{g}_s^-(0)\neq\check{0}$, i.e., that the function $\check{g}$ experiences at the boundary a jump, which will also occur when we have metals with different parameters in direct contact. As has already been noted, as a result of the unjustified neglect of part of the waves reflected from the boundary, the opposite conclusion is arrived at in Ref. 5, in which the boundary conditions for the Eilenberger equations are analyzed.

In the case of a junction made up of normal metals the boundary conditions (29) are significantly simpler. Indeed, as is easy to verify, the solutions to the equations for $\hat{g}^{R(A)}$ that satisfy (29), (20), and the boundary conditions at infinity are the constants $\hat{g}^{R(A)}(\varepsilon, \varepsilon')=\pm 2\pi\hat{\tau}_z\delta(\varepsilon-\varepsilon')$. Taking account of the foregoing, as well as the fact that $\hat{g}$ can be expressed in terms of the distribution function f:

$$\hat{g}=2\begin{pmatrix} f & 0\\ 0 & -\bar{f}\end{pmatrix},$$

we easily obtain the boundary conditions, which reduce to the following form:

$$f(p_{z2}, \mathbf{p}_\parallel; +)=Df(p_{z1}, \mathbf{p}_\parallel; -)+Rf(-p_{z2}, \mathbf{p}_\parallel; +), \quad p_\parallel<p_{F1,2},$$

$$f(p_{z1}, \mathbf{p}_\parallel; -)-f(-p_{z1}, \mathbf{p}_\parallel; -)$$
$$=f(p_{z2}, \mathbf{p}_\parallel; +)-f(-p_{z2}, \mathbf{p}_\parallel; -), \quad p_\parallel<p_{F1,2},$$
$$f(p_{z2}, \mathbf{p}_\parallel; +)-f(-p_{z2}, \mathbf{p}_\parallel; +)=0, \quad p_{F1}<p_\parallel<p_{F2}.$$

It is precisely such relations that are usually written down for the distribution function on the basis of nonrigorous, but obvious arguments.

Notice that the continuity of the current automatically follows, on the basis of (21), from the continuity of the function $\check{g}_a$ at the boundary. At the same time, such quantities as Φ and Δ (determined by the function $\check{g}_s$) will, generally speaking, undergo a jump at the boundary.

Let us now consider the consequences that follow from (29) and (20) for tunnel junctions ($D\ll 1$). It can be seen from (29) that $\check{g}_a\sim D$; therefore, if we limit ourselves in the determination of $\check{g}_a$ and the current to terms of the order of D, then we should substitute into (29) expressions for $\check{g}_s^+$ and $\check{g}_s^-$ in zeroth order in D, in which approximation we have

$$\check{g}_s\approx\check{g}, \quad R(\check{g}_s^+)^2+(\check{g}_s^-)^2\approx\check{1}.$$

As a result, we find $\check{g}_a$ from (29) and (20), and, using (21), we arrive at the following expression for the current:

$$I=-\frac{p_{F1}^2}{8\pi}\int_S d^2\rho\,\operatorname{Sp}\hat{\tau}_z\langle\alpha_1 D(\alpha_1)$$
$$\times(\hat{g}_1^R\hat{g}_2+\hat{g}_1\hat{g}_2^A-\hat{g}_1^R\hat{g}_1-\hat{g}_2\hat{g}_1^A)_{\varepsilon}(t,t)\rangle. \qquad (30)$$

In (30) and in what follows we shall, for definiteness, assume that $p_{F1}<p_{F2}$, and angle brackets denote integration over $\alpha_1=\cos\vartheta_1=p_{z1}/p_{F1}$:

$$\langle(\ldots)\rangle=\int_0^1 (\ldots)\,d\alpha_1.$$

The index zero on the brackets indicates, firstly, that all the matrices should be evaluated in the immediate neighborhood of the boundary. For example,

$$(\hat{g}_1{}^R\hat{g}_2)_0(t,t)=\int dt_1\hat{g}_1{}^R(-;t,t_1)\hat{g}_2(+;t_1,t).$$

Furthermore, the index zero indicates that all the functions should be taken in the zeroth order in D; consequently, we should, in finding them from Eq. (9a), use the boundary condition $\check{g}_a(\pm)=\check{0}$. In the case when the junction is made up of homogeneous metals, such solutions are obvious: $\check{g}=\check{g}_{1(2)}$, $z<-\delta$ $(z>\delta)$, where the $\check{g}_j$ are the equilibrium values corresponding to their values of the phase χ_j (which we shall assume to be independent of ρ, assuming that the area of the contact is sufficiently small):

$$\check{g}(t,t')=\check{S}(t)\check{g}(t-t')\check{S}^+(t'),\qquad \check{g}(t)=\int\check{g}(\varepsilon)e^{-i\varepsilon t}\frac{d\varepsilon}{2\pi},$$

$$\check{g}(\varepsilon)=\begin{pmatrix}\hat{g}^R(\varepsilon) & [\hat{g}^R(\varepsilon)-\hat{g}^A(\varepsilon)]\,\mathrm{th}(\varepsilon/2T)\\ \hat{0} & \hat{g}^A(\varepsilon)\end{pmatrix},$$

$$\hat{g}^{R(A)}=g^{R(A)}\hat{\tau}_z+f^{R(A)}i\hat{\tau}_y, \tag{31}$$

$$\check{S}=\begin{pmatrix}\hat{S} & \hat{0}\\ \hat{0} & \hat{S}\end{pmatrix},\quad \hat{S}=\begin{pmatrix}e^{i\chi/2} & 0\\ 0 & e^{-i\chi/2}\end{pmatrix},\quad \chi(t)=\chi-2\int^t\Phi(t')\,dt',$$

where the subscript j has been dropped and

$$g^{R(A)}=\left(\frac{\varepsilon}{\Delta}\right)f^{R(A)}=\varepsilon[(\varepsilon\pm i0)^2-\Delta^2]^{-1/2}.$$

In this case we find from (30) and (31) that

$$I=\frac{\pi}{8R_N}\,\mathrm{Sp}\,\hat{\tau}_z(\hat{g}_1{}^R\hat{g}_2+\hat{g}_1\hat{g}_2{}^A-\hat{g}_2{}^R\hat{g}_1-\hat{g}_2\hat{g}_1{}^A)(t,t), \tag{30'}$$

where the resistance (in the normal state) R_N is given by the relation

$$R_N{}^{-1}=\frac{p_{F1}{}^2}{\pi}\int_S d^2\rho\langle\alpha_1 D(\alpha_1)\rangle. \tag{30''}$$

Notice that the tunneling-Hamiltonian method leads to the same expression for the current in tunnel junctions.[6–8] The formula (30) is not based on any assumptions about the shape of the potential barrier, nor does its validity depend on what metals form the junction, or what voltage potential is applied across the junction (in deriving it we only assumed that $V(t)$, $\omega\ll\varepsilon_{Fj}$).

Let us note that, in Ref. 22, boundary conditions are given without derivation for the interface between metals with identical parameters, at which there exists a potential barrier $U_0\delta(z)$. In our notation these boundary conditions have the form

$$\check{g}_a(+)=\check{g}_a(-)=(iv_z/U_0)[\check{g}_s(+)-\check{g}_s(-)]. \tag{32}$$

It is clear that (32) differs from (29); thus, for example, for $D\ll 1$ it follows from (32) that $\check{g}_a\sim D^{1/2}$, whereas we have from (29) that $\check{g}_a\sim D$.

§2. PROPERTIES OF CONSTRICTED MICROCONTACTS

Let us now use the results obtained to study the properties of constricted microcontacts. As a model for the constrictions, we shall consider an aperture of radius a in a thin impenetrable screen,[11,23] assuming that, for $\rho<a$ (the point $\rho=z=0$ corresponds to the center of the aperture), the reflection coefficient does not depend on ρ (for $\rho>a$ we have $R=1$). We shall first investigate in detail the case of pure constrictions, whose dimension satisfies the condition

$$a\ll l_j,\quad v_{Fj}/T_{cj}. \tag{33}$$

The Green function $\check{g}$ in the problem under consideration can be found by two methods, one of which (using the boundary conditions (29)) we shall now consider. Let us, as in Ref. 13, take account of the fact that the condition (33) allows us to set the quantities $\check{\Phi}$, $\check{\Delta}$, and $\check{\Sigma}$ in (9) equal to their values at infinity (this fact was first used by Kulik and Omel'yanchuk[11] in their solution of the stationary problem). The solution to the resulting equation can be represented in the form[13]

$$\check{g}=\exp(-\check{K}_j\tau_j)\check{C}_j(\mathbf{R},\mathbf{p}_{Fj})\exp(\check{K}_j\tau_j)+\check{g}_j,$$

where $\tau_j=\mathbf{v}_{Fj}\cdot\mathbf{R}/v_{Fj}^2$, the function $\check{g}_j$ is given by (31), $\check{K}_j$ is given by the formula (11) in Ref. 13, and $\check{C}_j(\mathbf{R},\mathbf{p}_{Fj})$ satisfies the equation $\mathbf{v}_{Fj}\partial\check{C}_j/\partial\mathbf{R}=\check{0}$, from which it follows that, for $\mathbf{R}$ belonging to the trajectories crossing the aperture, i.e., for straight lines parallel to the vector $\mathbf{v}_{Fj}$,

$$\check{C}_j(\mathbf{R},\mathbf{p}_{Fj})=\check{C}_j(\mathbf{p}_{Fj})=\check{g}_j(\mathbf{R}=0,\mathbf{p}_{Fj})-\check{g}_j.$$

For $\mathbf{R}$ lying on the trajectories crossing the screen, as follows from the boundary condition $\check{g}_a(\pm)=\check{0}$, we have $\check{C}_j=\check{0}$. It is shown in Ref. 13 that the matrices $\check{C}_j$ satisfy the relations

$$\check{g}_j\check{C}_j=-\check{C}_j\check{g}_j=\mathrm{sign}\,p_{zj}(-1)^j\check{C}_j,$$

Using these relations and the equalities $\check{C}_{a1}=\check{C}_{a2}\equiv\check{C}_a$, which follow from (19), we obtain

$$\check{C}_{c1(2)}=\check{g}_c(z=\mp0,\ \rho=0)-\check{g}_{1(2)}=\mp\check{g}_{1(2)}\check{C}_a, \tag{34}$$

where we further assume that $p_{zj}>0$. Finding $\check{g}_s^{\pm}$ from (34), and substituting them into (29), we obtain, after simple transformations, the expression[5)]

$$\check{C}_a=\check{g}_-\check{g}_+[\check{1}D^{-1}-(\check{g}_-)^2]^{-1}, \tag{35}$$

where $\check{g}_\pm=(\check{g}_2\pm\check{g}_1)/2$. In the absence of reflections at the boundary (i.e., for $D=1$), the expression (35) coincides with the one found in Ref. 13. From (35) we obtain

$$\begin{aligned}\hat{C}_a=\hat{C}_a^R\hat{n}_+-\hat{n}_+\hat{C}_a^A &+[\hat{1}D^{-1}-(\hat{g}_-^R)^2]^{-1}\{RD^{-1}[(\hat{g}_-^R)^2\hat{n}_--\hat{g}_-^R\hat{n}_-\hat{g}_-^A\\ &-\hat{n}_-(\hat{g}_+^A)^2]+D^{-1}\hat{g}_+^R\hat{n}_-\hat{g}_+^A\\ &-(\hat{g}_+^R)^2\hat{n}_-(\hat{g}_+^A)^2\}[\hat{1}D^{-1}-(\hat{g}_-^A)^2]^{-1}\\ &+\hat{C}_a^R\hat{n}_-\hat{C}_a^A,\quad \hat{C}_a^{R(A)}=(\hat{g}_-^R\hat{g}_+)^{R(A)}[\hat{1}D^{-1}-(\hat{g}_-^{R(A)})^2]^{-1},\end{aligned} \tag{36}$$

where

$$\hat{n}_\pm = (\hat{n}_2 \pm \hat{n}_1)/2,$$

$$\hat{n}_j(t,t') = \hat{S}_j(t)\left(\int \mathrm{th}\frac{\varepsilon}{2T}\exp[-i\varepsilon(t-t')]\frac{d\varepsilon}{2\pi}\right)\hat{S}_j^+(t').$$

The expression for the current has the form

$$I = -\frac{\pi}{2\tilde{R}_N}\mathrm{Sp}\,\hat{\tau}_z\langle\alpha_1 \hat{C}_a(t,t)\rangle, \tag{36'}$$

where we have introduced the quantity $\tilde{R}_N = (p_{F1}{}^2 S/2\pi^2)^{-1}$ ($S = \pi a^2$), which, as we shall see later, has the meaning of the resistance of the constriction in the normal state in the case of a smooth boundary and direct contact between the metals.

Specific expressions for the transmission coefficient and, hence, the current can be found only for certain models for the potential barrier. In particular, the model

$$U_M(z, \rho<a) = U_0\delta(z) + \theta(-z)(\varepsilon_{F2}-\varepsilon_{F1}) \quad (\mu = \varepsilon_{F2} > \varepsilon_{F1}),$$

for which

$$D = v_{z1}v_{z2}[U_0^2 + (v_{z1}+v_{z2})^2/4]^{-1}, \tag{37}$$

is a useful one. Using (37), we can make a qualitative judgment about the dependence of the CVC on the parameters of the metals (and about the height of the potential barrier) in contacts with sharp boundaries. As a more general parameter for the CVC, we can choose the quantity D_n (or R_n), the transmission coefficient for electrons incident normally on the surface. Since the coefficients $D = D_n F(\alpha_1)$ for different boundaries with the same D_n value differ from each other only in the form of the function $F \leqslant 1$, it is clear that the dependence $I(V, D_n)$ obtained with the use of (37) will qualitatively describe the shape of the CVC of contacts with boundaries of arbitrary shape.

Let us now proceed to study the properties of specific types of contacts.

1. The N_1cN_2 junction

Let us note that the resistance of the NcN junction in the pure limit being considered by us now has been computed before by Omel'yanchuk *et al.*[23] For the N_1cN_2 junction with a boundary of arbitrary transmissivity we find from (36) and (36') that

$$I = V/R_N, \quad R_N^{-1} = \tilde{R}_N^{-1}\cdot 2\langle\alpha_1 D(\alpha_1)\rangle. \tag{38}$$

Notice that, formally, the expression (38) for R_N coincides in form with the expression obtained for tunnel junctions (see (30')), whose dimensions were not assumed to be small compared to the mean free path. For the case in which the metals are in direct contact we have

$$R_N^{-1} = \tilde{R}_N^{-1}\begin{cases} 1, & R_n = 1 - D_n \ll 1, \\ \frac{4}{3}\frac{b(2+b)}{(1+b)^2}, & b = \frac{p_{F1}}{p_{F2}}. \end{cases} \tag{38'}$$

The second of the relations (38') is a model relation, since, in deriving it, we used the expression (37); as to the first relation, which does not depend (in the leading approximation in R_n) on the shape of the potential barrier, it corresponds to the case of a smooth boundary. The formula (38') illustrates the fact that the resistance of metals connected in series is, even when there is direct contact between the metals, not equal to the sum of their resistances (if $p_{F1} \neq p_{F2}$). This circumstance, which is obvious in the case when there is an insulating layer between the metals, stems from the fact that a boundary that reflects electrons has a resistance R_b: there occurs when current flows across it in electric-potential jump[6)] $V_b = IR_b$ (see Fig. 1), for which, using (35), (34), and (21), we obtain

$$V_b = \Phi(z=+0, \rho<a) - \Phi(z=-0, \rho<a) = qV,$$

$$q = \frac{V_b}{V} = \frac{R_b}{R_N} = \frac{1}{2}(1-b^2)^{1/2} + \frac{1}{2}\langle R[1+b^2\alpha_1(1-b^2(1-\alpha_1^2))^{-1/2}]\rangle$$
$$= 1 - \frac{1}{2}\langle D[1+b^2\alpha_1(1-b^2(1-\alpha_1^2))^{-1/2}]\rangle. \tag{39}$$

It can be seen from (39) that the quantity q, which is equal, as it should, to zero for $R = 0$ and $b = 1$, is close to unity in the case of strong reflection from the boundary. In particular, for direct contact in the case of a sharp boundary we find from (37) and (39) that $q = 1 - b/4$, $b \ll 1$; if, on the other hand, the boundary is a smooth one, then, irrespective of the shape of the potential barrier, we have for q in the leading approximation in R_n the expression

$$q = \tfrac{1}{2}(1-b^2)^{1/2}.$$

Let us now proceed to analyze a more complicated system.

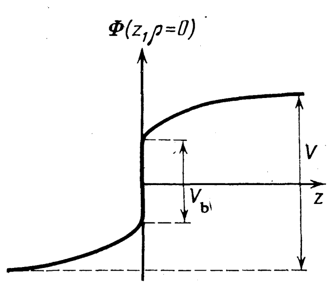

FIG. 1.

2. The ScN junction

Let us, in computing the current, choose as the zero potential the potential of the superconductor at points far from the constriction; then in the normal metal $\Phi(\infty) = V$. We find from (35) and (36) after simple computations that

$$I = \frac{1}{4\tilde{R}_N}\int_{-\infty}^{\infty} d\varepsilon B(\varepsilon)\left(\mathrm{th}\frac{\varepsilon+V}{2T} - \mathrm{th}\frac{\varepsilon-V}{2T}\right), \tag{40}$$

where

$$B(\varepsilon) = \left\langle \frac{4\alpha_1}{(\varepsilon/\Delta)^2 + [1-(\varepsilon/\Delta)^2](2D^{-1}-1)^2}\right\rangle, \quad |\varepsilon| < \Delta,$$

$$B(\varepsilon) = \left\langle \frac{4|\varepsilon|\alpha_1}{|\varepsilon| + (\varepsilon^2-\Delta^2)^{1/2}(2D^{-1}-1)}\right\rangle, \quad |\varepsilon| > \Delta.$$

In the case of a direct contact with a smooth boundary, we obtain from (40) results that, in the leading approximation in $R_n \ll 1$, differ (for $b \neq 1$) from those obtained in Ref. 13 only by a change in the quantity R_N. At $T=0$ the differential conductivity is given by the expression

$$\sigma(V)=dI/dV=B(V)/\tilde{R}_N, \tag{41}$$

from which it follows, in particular, that

$$\tilde{R}_N\sigma(0)=\left\langle\frac{4\alpha_1 D^2}{(1+R)^2}\right\rangle,\quad \tilde{R}_N\sigma(\Delta)=2,\quad \frac{\sigma_N}{\sigma(\Delta)}=\langle\alpha_1 D\rangle. \tag{41'}$$

Thus, for contacts with boundaries of low transmissivity we find that at low temperatures

$$\sigma(0)\sim D_n\sigma_N\sim D_n^2\sigma(\Delta).$$

Notice that the last relation in (41′) allows us to estimate the boundary transmissivity with the aid of experimental data.

In the region of high voltage potentials $V \gg \Delta$, we obtain from (40) for arbitrary temperatures the well-known relation:[12,13,19]

$$I=V/R_N+I_0\,\mathrm{th}\,(V/2T), \tag{42}$$

in which the excess current I_0 is equal to

$$I_0=\frac{\Delta}{R_N}J,\quad J=\frac{1}{2\langle\alpha_1 D\rangle}\left\langle\alpha_1\frac{D^2}{R}\left[1-\frac{D^2}{R^{1/2}}\frac{\mathrm{arth}\,(R)^{1/2}}{(1+R)}\right]\right\rangle$$
$$=\begin{cases} 4/3, & R_n\ll 1,\\ \langle\alpha_1 D^2\rangle/2\langle\alpha_1 D\rangle, & D_n\ll 1.\end{cases} \tag{42'}$$

It can be seen that the product $V_0=I_0R_N$ decreases with decreasing boundary transmissivity; for $D \ll 1$ we obtain $V_0 \sim \Delta D_n$; thus, for example, in the case of a direct contact, we find, using (37) for $U_0=0$, that $V_0 \approx 3/2b\Delta$, $b \ll 1$.

To illustrate the dependence of the CVC on the parameters of the metals in the case of a direct contact with a sharp boundary, we use the model expression (37) with $U_0=0$. Substituting it into (40), we find that

$$B(\varepsilon)=\left[2-\frac{1-b^2}{b^2}\frac{\Delta^2-\varepsilon^2}{|\varepsilon|\Delta}\,\mathrm{arth}\,\frac{2b^2|\varepsilon|\Delta}{\Delta^2-\varepsilon^2+b^2(\varepsilon^2+\Delta^2)}\right]$$
$$\times\theta(\Delta-|\varepsilon|)+2\frac{|\varepsilon|}{\Delta}\left[\frac{|\varepsilon|}{\Delta}-\frac{1}{b}\frac{(\varepsilon^2-\Delta^2)^{1/2}}{\Delta}\right.$$
$$\left.+\frac{\varepsilon^2-\Delta^2}{\Delta^2}\frac{1-b^2}{b^2}\,\mathrm{arth}\,\frac{b\Delta}{(\varepsilon^2-\Delta^2)^{1/2}+b|\varepsilon|}\right]\theta(|\varepsilon|-\Delta). \tag{43}$$

Figure 2a shows the dependence $\sigma(V, T=0)$ following from (41) and (43) for three values of the parameter b, while Fig. 2b shows the CVC; in this model $R_n=(1-b)^2/(1+b)^2$, and therefore the chosen b values correspond to $R_n=0; 0.05; 4/9$. It can be seen that, as b and, hence, D_n decrease, the CVC and $\sigma(V)$ approach the dependences that are characteristic of tunnel junctions. A similar change in the shape of $\sigma(V)$ and $I(V)$ will, as has already been noted, occur as D decreases in the case of an arbitrary potential barrier and, in particular, in the case of the barrier $U=U_0\delta(z)$, which is considered within the framework of the one-dimensional contact model in

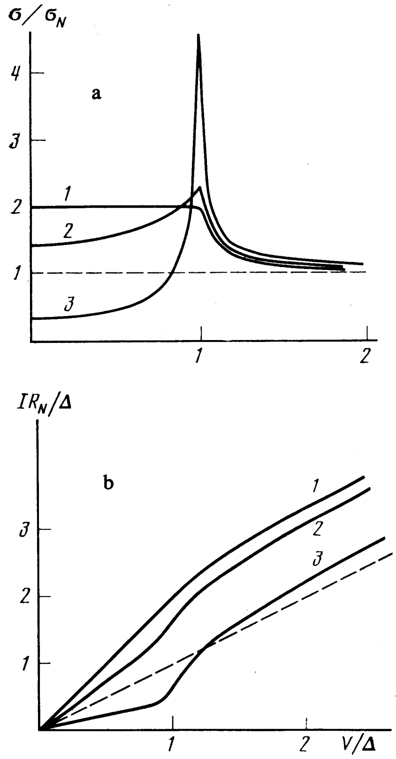

FIG. 2. a) The dependence $\sigma(V, T=0)$ constructed with the use of D given in the form (37) with $U_0=0$ for: 1) $b=1$, $R_n=0$; 2) $b=2/3$, $R_n=0.05$; 3) $b=1/5$, $R_n=4/9=0.44\ldots$. b) The CVC for the same parameters.

Ref. 20. Thus, reflection from the boundary leads at low temperatures to a qualitative change in the shape of the function $\sigma(V)$, which becomes nonmonotonic. The dependence $\sigma_0(T)=(dI/dV)_{V=0}$ also becomes nonmonotonic in the case when $R_n \neq 0$. Let us give the expressions for $\sigma_0(T)$ in some limiting cases. From the general formula

$$R_N\sigma_0(T)=\frac{1}{2\langle\alpha_1 D\rangle}\int_0 B(\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{th}\frac{\varepsilon}{2T}\,d\varepsilon \tag{44}$$

we obtain

$$R_N\sigma_0(T)=\int_\Delta^\infty\frac{\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}\frac{\partial}{\partial\varepsilon}\mathrm{th}\frac{\varepsilon}{2T}\,d\varepsilon,\quad D_n\ll e^{-\Delta/T},$$
$$R_N\sigma_0(T)=\frac{1}{\langle\alpha_1 D\rangle}\left\langle\frac{2\alpha_1 D^2}{(1+R)^2}\right\rangle,\quad D_n\gg e^{-\Delta/T},\quad T\ll\Delta,$$
$$R_N\sigma_0(T)=1+\frac{\Delta}{2T_c}J,\quad T\gg\Delta.$$

It follows from (44), as well as from an analysis of the dependence $\sigma_0(T)$ for arbitrary transmissivity, that the maximum of this function is attained at some point $T^*=cT$, $c\sim 1$, with the quantity $\sigma_0(T^*)/\sigma_N-1\sim 1$ if $R_n\sim 1$. Let us note that the shape of the dependences $I(V)$, $\sigma(V)$, and $\sigma_0(T)$ has been experimentally found to be similar to the shape considered above (see, for example, Refs. 16 and 18).

When current flows through the contact, there occurs a voltage-potential jump V_b at the boundary. We shall not analyze the dependence $V_b(V)$ in detail. Let us only note that at high voltage potentials $V \gg \Delta$ this functions, which is nonlinear in the region $V \sim \Delta$, is given by the relation

$$V_{\cdot}(V) = qV + \tilde{q}\Delta \,\mathrm{th}\,(V/2T),$$

where q (given by (39)) and $\tilde{q}$ are V-independent quantities.

Let us, in conclusion, consider the case of superconducting constrictions.

3. The S_1cS_2 junction

It follows from (35) and (36) that, in the case of zero voltage potential, the current flowing through the contact is connected with the phase difference φ by the following relation:

$$I = \frac{1}{\tilde{R}_N}\int d\varepsilon [Q^R(\varepsilon) - Q^A(\varepsilon)]\,\mathrm{th}\frac{\varepsilon}{2T} = \frac{8\pi Ti}{R_N}\sum_{\omega_n>0} Q^R(i\omega_n), \quad (45)$$

where

$$Q^R = if_1^R f_2^R \langle \alpha_1 (2D^{-1} - 1 + g_1^R g_2^R - f_1^R f_2^R \cos\varphi)^{-1}\rangle \sin\varphi.$$

For $R=0$ and $b=1$ we have from (45) the expression obtained earlier by Kulik and Omel'yanchuk.[11] For $b \neq 1$, in the case of a direct contact and a smooth boundary, the difference between our results and the results obtained by Kulik and Omel'yanchuk[11] consists only in a change in the formula for the normal resistance: $R_N = \tilde{R}_N$. For constrictions of the type ScS we obtain the relation

$$I = \frac{p_F^2 \Delta S \sin\varphi}{2\pi}\left\langle \alpha \frac{D(\alpha)}{[1 - D(\alpha)\sin^2(\varphi/2)]^{1/2}} \times \mathrm{th}\left\{\frac{\Delta}{2T}[1 - D(\alpha)\sin^2(\varphi/2)]^{1/2}\right\}\right\rangle,$$

which was obtained earlier by Haberkorn *et al.*[24] for the case of the model potential barrier $U = U_0\delta(z)$.

For $D \ll 1$ we obtain from (35) and (36) for constrictions with boundaries of low transmissivity the well-known expressions obtained for tunnel junctions (see Refs. 6–8), in which R_N is given by the formula (38). For an arbitrary boundary transmissivity in the region of high voltage potentials we arrive at a relation of the type (42) in which

$$I_0 = \frac{\Delta_1 + \Delta_2}{R_N} J,$$

where J is given by the expression (42′). Sometimes in experiments the excess current I_0 is compared with the critical current I_c. For their ratio we have ($V_c = I_c R_N$)

$$\frac{I_0}{I_c} = \frac{\Delta_1 + \Delta_2}{V_c} J = \begin{cases} 8/3\pi, & \Delta_1 = \Delta_2 \gg T, \quad R_n \ll 1, \\ \dfrac{(\Delta_1+\Delta_2)^2}{4\Delta_1\Delta_2}\dfrac{\langle\alpha_1 D^2\rangle}{\langle\alpha_1 D\rangle}\left[K\left(\dfrac{|\Delta_1-\Delta_2|}{\Delta_1+\Delta_2}\right)\right]^{-1}, \\ \dfrac{4}{\pi}\dfrac{(\Delta_1+\Delta_2)T}{\Delta_1\Delta_2} J, & T \gg \Delta_{1,2}, \end{cases}$$

$$D_n \ll 1, \quad T \ll \Delta_j,$$

where $K(x)$ is the complete elliptic integral of the first kind. Thus, the ratio I_0/I_c decreases with decreasing boundary transmissivity. Notice that, even in the case when $D \ll 1$, the excess current, which is determined by a higher (second) power of the transmission coefficient than the critical current, is not necessarily small in comparison with I_c.

Thus far, we have studied pure constrictions. Let us now briefly discuss the case corresponding to the dirty limit $l_j \ll a$. It turns out that, when a certain condition (see (47)) is fulfilled, the Green functions can be approximately considered to be continuous at the boundary. In this case the computations will be similar to those carried out in Refs. 12 and 19, and we obtain as a result in the leading approximation in the parameter $\varkappa$ the formulas obtained in the indicated papers, in which we should change only the expression for R_N:

$$R_N = R_{N1} + R_{N2}, \quad R_{Nj} = \sigma_j a. \quad (46)$$

For $\varkappa \ll 1$ the discrepancy between our results and those of Refs. 12 and 19 and the departure of the expressions for R_N from (46) will manifest themselves in the next orders in the parameter $\varkappa$.

We can, by comparing the results obtained for constrictions with direct contact between the metals in the pure and dirty limits, see that in the first case, in contrast to the latter: 1) the resistance in the normal state cannot be represented in the form of the sum $R_{N1} + R_{N2}$, where R_{Nj} is determined by the parameters of the j-th metal; 2) in ScN junctions the CVC can lie either above or below the ohmic curve; 3) the quantity $V_0 = I_0 R_N$ can be significantly smaller than max Δ_1, Δ_2 (when $b \ll 1$).

We obtain the condition upon the fulfillment of which the function $\check{g}$ can be considered to be continuous at the boundary from (29) by taking account of the estimate $\check{g}_a \sim \min(l_j/a)$, $\check{g}_s^+ \sim \langle\check{g}\rangle$. As a result we come to the conclusion that $\check{g}_s^- \ll \check{g}$ if

$$\varkappa = \min\,(l_j/aD_n) \ll 1. \quad (47)$$

For $\varkappa \ll 1$ the reflection of the electrons from the boundary is barely noticeable in the background of the more intense impurity scattering. We shall not consider in detail constrictions in which the parameter $\varkappa$ is not small: We only note that in the $\varkappa \gg 1$ case the well-known expressions obtained for tunnel junctions[6–8] will again be valid if for R_N we use the formula (38).

In conclusion, I wish to express my gratitude to K. K. Likharev and M. Yu. Kupriyanov for a useful discussion.

[1)] We use the model of isotropic metals (we neglect the difference in their effective electronic masses). The vector potential is included in the phase of the order parameter with the aid of the change of variables $\chi \to \chi + \tilde{\chi}$, $\nabla\tilde{\chi} = \mathbf{A}$ (we neglect the effects connected with the magnetic field).

[2)] We easily arrive at (13) by taking the nondependence on z of the quantities $(\psi_1 d\psi_1^*/dz - \psi_1^* d\psi_1/dz)$ and $(\psi_2 d\psi_1^*/dz - \psi_1^* d\psi_2/dz)$ into consideration.

[3)] If $\partial\check{G}/\partial\boldsymbol{\rho} \neq \check{0}$, then we must, more accurately, speak of the nondependence on $\mathbf{R}$ lying on the straight lines parallel to the vector $(v_z, \mathbf{v}_\parallel/2)$.

[4)] Relations similar to (19), (20), and (29) will be satisfied for the temperature Green functions $\check{g}_\omega$ and $\hat{\mathscr{F}}_\omega$.

[5)] We could also have proceeded in a second manner, taking into account the fact that the solution (9b) has the form

$$\tilde{\mathscr{F}} = \exp(-\check{K}_j z/v_{zj})\tilde{C}_j \exp(-K_j z/v_{zj}), \quad \check{C}_{2(1)} = \tilde{\mathscr{F}}(z = \pm 0, \boldsymbol{\rho}),$$

where the matrices $\tilde{C}_j$ should satisfy the relations $\check{g}_j\tilde{C}_j = \tilde{C}_j\check{g}_j = \mathrm{sign}p_{zj}(-1)^j\tilde{C}_j$. Taking these relations, as well as (19), into account, we again arrive, after simple computations, at (35).

[6)]More precisely, we should speak of an electrochemical-potential jump, since the electric-potential jump (which cannot be measured by a voltmeter), equal to $\varepsilon_{F2} - \varepsilon_{F1}$, occurs even in the absence of a current.

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