Influence of precession of transverse trajectories of axially channeled electrons on the radiation spectrum

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We have developed the theory of the electromagnetic radiation of ultrarelativistic electrons in motion in unclosed transverse trajectories in axial channeling. We have obtained general formulas for the spectral and angular distributions of the energy of the radiation with allowance for precession of the orbits and nondipole nature of the radiation. We have shown that the precession of the orbits leads to appearance of a multiplet structure of the spectrum for each harmonic investigated. Here the spectral and angular distributions of the radiation do not depend on the azimuthal angle of the radiation. We have made a detailed calculation of the radiation spectra in the dipole approximation with use of a model potential of a crystallographic axis.

INTRODUCTION

The theory of the x radiation and γ radiation arising in motion of electrons at small angles to the axes of a crystal has been discussed previously in a number of studies.[1–7] Kumakhov[1] in the first study of this radiation gave only estimates of the intensity and characteristic frequencies of the radiation, based on a certain analogy of the axial channeling of particles of sufficiently high energy to their helical motion in a magnetic field. The quantum theory of the radiation of electrons in axial channel was developed by Bazylev, Glebov, and Zhevago.[2] It was shown that at relatively low electron energies (1–10 MeV) their motion transverse to the axes turns out to be quantized and calculation of the radiation spectra reduces to calculation of the matrix elements of the dipole moments of transitions between levels of the transverse energy. Concrete calculations of the transverse energy levels and dipole moments for electrons with energies from 1.5 to 4.5 MeV channeled in a silicon crystal have been given in Refs. 8 and 9. At low electron energies the number of levels is small and individual radiation lines corresponding to the different transitions are observed experimentally.[10,11] With increase of the electron energy the number of levels in the transverse-motion potential well increases in proportion to the energy and at $E\approx 100$ MeV can reach several hundred. There is a still more rapid rise of the number of different transitions between levels, which makes it practically impossible to analyze the radiation spectra of high-energy electrons on the basis of purely quantum-mechanical representations of the transverse motion of the particles. In addition to the increase of the number of levels, at sufficiently high electron energies, as was shown in detail in Ref. 12, the parametric dependence of the transverse-motion wave functions on the total energy of the particles can turn out to be important, and this also greatly complicates the theoretical analysis and concrete calculations. As a result it turns out that although the general results of the quantum theory[2,5a] formally retain their validity up to very high electron energies, it becomes more suitable in this case to make a theoretical analysis of the radiation spectrum on the basis of the classical transverse motion of the electrons. This method permits one to discover the general features of the radiation spectra at high electron energies and to obtain comparatively simple expressions for the spectral and angular distributions of the radiation with allowance for various effects (nondipole nature, longitudinal oscillations of the particles in the channel), which are suitable for subsequent numerical calculations.

The theory of radiation by electrons during axial channeling in single crystals, based on a classical description of their transverse motion, was developed by the authors of Ref. 2, by Kumakhov and Trikalinos,[3] and by Baĭer, Katkov, and Strakhovenko.[4]

However, in all of these studies the transverse motion of the electrons was assumed to be periodic, which made it possible to expand the radiation spectra in individual harmonics (with subsequent allowance for the Doppler effect as a consequence of the longitudinal motion of the particles along the axes). It is known, however, (see for example Ref. 13) that purely periodic two-dimensional motion occurs only in potentials of the form $U(\rho)=-\alpha/\rho$ or $U=\beta\rho^2$, where ρ is the distance of the electron from the axis and α and β are certain constants. Therefore the radiation theory developed in Refs. 2–4 is in essence limited to these dependences of the axis potentials. On the other hand, as was shown first by Kreiner *et al.*[14] and subsequently by other authors[15–17] by numerical modeling of the transverse trajectories of the electrons in a more realistic crystal potential, the trajectories of a significant fraction of the electrons have an appreciable precession. This means that nowhere in the accessible region of variation of the distances of the channeled electrons from the axis can the actual potential be represented entirely by one of the dependences given above. In addition, it is clear to begin with that the singular dependence of the potential α/ρ can lead to an appreciable distortion of the actual radiation spectrum if the channeling conditions are such that the region of small ρ is accessible for electrons. For these reasons it is necessary to generalize the existing classical theory of the radiation of electrons in channeling to the case of an arbitrary axially symmetric field. In regard to superbarrier elec-

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trons, departures of the actual potential from the model dependences given above do not lead to an important change of the unlimited nature of their transverse motion. Therefore the general formulas for calculation of the radiation of high-energy superbarrier particles obtained by Avakyan *et al.*[5] and by Akhiezer and Shul'ga[6,7] do not require any modifications.

In the present work we have developed for the first time a classical theory of the radiation during axial channeling of electrons in an arbitrary axially symmetric field, simultaneously taking into account effects of precession of the transverse orbits, the nondipole nature of the radiation, and longitudinal oscillations. The general expressions obtained for the spectral and angular densities of the intensity of radiation with inclusion of these effects can be used directly for concrete numerical calculations of the radiation spectra from high-energy electrons. In addition on the basis of the general formulas we have carried out further analytical calculations of the spectra of the dipole radiation by an electron for a sufficiently real potential. We have shown that a more correct inclusion of the behavior of the potential at small distances from the axis can lead to an important change of the spectrum in comparison with the spectrum which is obtained on the basis of the usually used α/ρ model. We discuss the relation between the classical theory which we have developed and the quantum theory of the radiation in axial channeling of particles, and the limits of applicability of the classical approach to the problem. As a result we have shown that the description of the radiation spectra in terms of classical trajectories is possible only for electrons of sufficiently high energies and with the additional condition of smallness of the energy of the radiated photon in comparison with the electron energy.

1. GENERAL EXPRESSIONS FOR THE INTENSITY OF RADIATION WITH ALLOWANCE FOR PRECESSION

The special and angular distribution of the energy radiated by a charged particle moving along a trajectory $\mathbf{r}(t)$ with velocity $\mathbf{v}(t)$ has the form (see for example Ref. 18)

$$\frac{d^2W}{do\,d\omega}=\left(\frac{e\omega}{2\pi}\right)^2|[\mathbf{n}\times\mathbf{j}_\omega(\mathbf{k},\omega)]|^2,$$

$$\mathbf{j}_\omega=\int_{-\infty}^{\infty}\mathbf{v}(t)\exp[i\mathbf{k}\mathbf{r}(t)-i\omega t]\,dt, \tag{1}$$

where ω and $\mathbf{k}$ are the frequency and wave vector of the radiation wave, $\mathbf{n}=\mathbf{k}/k$, and do is the differential of the solid angle.

In axial channeling in a crystal the electrons move in the field of atomic strings, averaged along the direction of motion. The averaged potential will depend only on the distance ρ to the string. The motion in the plane orthogonal to the string is described by the equation

$$E\ddot{\rho}=-\nabla U(\rho), \tag{2}$$

where E is the relativistic mass of the particle and $c=m=1$.

From the conservation of the longitudinal momentum of the particle it follows[5] that the longitudinal component of the velocity of an ultrarelativistic particle changes with time according to a law

$$v_z(t)=1-{}^1/_2[E^{-2}+(\dot{\rho}(t))^2]. \tag{3}$$

The square of the vector product (indicated by the square brackets) in Eq. (1) can be expressed in terms of the longitudinal component $j^{(z)}$ and the transverse component $\mathbf{j}_\rho$ of the Fourier components of the current $\mathbf{j}_\omega$:

$$|[\mathbf{n}\times\mathbf{j}_\omega]|^2=|[\mathbf{n}_\rho\times\mathbf{j}_\rho]|^2+|\theta j^{(z)}-\mathbf{n}_\rho\mathbf{j}_\rho|^2, \tag{4}$$

where $\mathbf{n}_\rho$ is the transverse component of the unit vector of the wave vector and is expressed in terms of $\cos\varphi_r$ and $\sin\varphi_r$, θ and φ_r are the polar and azimuthal angles of the radiation,

$$\mathbf{j}_\rho=\int_{-\infty}^{\infty}\mathbf{v}_\rho\exp[i\mathbf{k}\mathbf{r}(t)-i\omega t]\,dt,$$

$$j^{(z)}=\int_{-\infty}^{\infty}\exp[i\mathbf{k}\mathbf{r}(t)-i\omega t]\,dt, \tag{5}$$

and $\mathbf{v}_\rho\equiv\dot{\boldsymbol{\rho}}$ is the transverse component of the particle velocity. The phase of the exponential in Eq. (1) can be represented in the form

$$\mathbf{k}\mathbf{r}(t)-\omega t\approx\omega\theta\mathbf{n}_\rho\boldsymbol{\rho}-\frac{\omega}{2}\left[(\theta^2+E^{-2})t+\int_0^t\mathbf{v}_\rho^2(t)\,dt\right].$$

Instead of the Cartesian components of the transverse component of the current we shall introduce the linear combinations

$$j^{(\pm)}=j_x\pm ij_y. \tag{6}$$

Let $\varphi(t)$ be the azimuthal angle of rotation of the electron relative to the string. Then

$$x(t)=\rho(t)\cos\varphi(t),\qquad \dot{x}=\dot{\rho}\cos\varphi-\rho\dot{\varphi}\sin\varphi,$$

$$y(t)=\rho(t)\sin\varphi(t),\qquad \dot{y}=\dot{\rho}\sin\varphi+\rho\dot{\varphi}\cos\varphi.$$

Substituting these relations into Eq. (6), we obtain

$$j^{(\pm)}=\sum_{\nu=-\infty}^{\infty}\exp[i(\nu\mp1)(\pi/2-\varphi_r)]\int_{-\infty}^{\infty}(\dot{\rho}\pm i\rho\dot{\varphi})J_{\nu\mp1}(\theta\rho\omega)\cdot$$

$$\times\exp\left\{i\nu\varphi(t)-\frac{i\omega}{2}\left[(\theta^2+E^{-2})t+\int_0^t\mathbf{v}_\rho^2\,dt\right]\right\}dt,$$

$$j^{(z)}=\sum_{\nu=-\infty}^{\infty}\exp[i\nu(\pi/2-\varphi_r)]\int_{-\infty}^{\infty}J_\nu(\theta\rho\omega)$$

$$\times\exp\left\{i\nu\varphi(t)-\frac{i\omega}{2}\left[(\theta^2+E^{-2})t+\int_0^t\mathbf{v}_\rho^2\,dt\right]\right\}dt. \tag{7}$$

Here we have used the relation

$$\exp i\omega\theta\mathbf{n}_\rho\boldsymbol{\rho}=\sum_{\nu=-\infty}^{\infty}J_\nu(\theta\rho\omega)\exp\{i\nu[\varphi(t)-\varphi_r+\pi/2]\},$$

where J_ν is a Bessel function of order ν.

In motion in a central field $U(\rho)$ the distance of the particle from the axis $\rho(t)$ is a periodic function with a period

T_ρ, i.e., $\rho(t+\mu T_\rho)=\rho(t)$, where μ is an integer. The quantities $\dot\rho$ and $\dot\varphi$ are also periodic functions of time. At the same time as a consequence of the precession of the orbit the rotation angle is, generally speaking, a quasiperiodic function, i.e., it satisfies the equality

$$\varphi(t+\mu T_\rho)=\varphi(t)+(\Delta\varphi+2\pi)\mu; \tag{8}$$

here $\Delta\varphi$ is the precession angle, which for a given transverse energy ε and an angular momentum of the particle with respect to the axis M is calculated according to the formula (see Section 14 of Ref. 9)

$$\Delta\varphi=2\int\limits_{\rho_{min}}^{\rho_{max}}\frac{M/\rho^2\,d\rho}{[2E(\varepsilon-U(\rho))-M^2/\rho^2]^{1/2}}-2\pi, \tag{9}$$

where $\rho_{\min}$ and $\rho_{\max}$ are the perihelion and aphelion of the orbit.

Using the condition (8) and the properties of periodicity of the quantities $\rho(t)$, $\dot\rho(t)$, and $\dot\varphi(t)$, the integration in Eq. (7) over the entire interaction time can be reduced to integration over the period of the radial oscillations. The result has the form

$$j^{(\pm)}=\sum_{n=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}\exp\{i(\nu\mp1)(\pi/2-\varphi_r)\}$$
$$\times\exp\left[-\frac{i\omega}{2}(\theta^2+E^{-2}+\langle v_\rho^2\rangle)T_\rho n+i\nu\Delta\varphi n\right]$$
$$\times\int\limits_0^{T_\rho}(\dot\rho\pm i\rho\dot\varphi)J_{\nu\mp1}(\theta\rho\omega)$$
$$\times\exp\left\{i\nu\varphi-\frac{i\omega}{2}\left[(\theta^2+E^{-2})t+\int\limits_0^t v_\rho^2\,dt\right]\right\}dt, \tag{10}$$
$$j^{(z)}=\sum_{n=-\infty}\sum_{\nu=-\infty}\exp\{i\nu(\pi/2-\varphi_r)\}$$
$$\times\exp\left[-\frac{i\omega}{2}(\theta^2+E^{-2}+\langle v_\rho^2\rangle)T_\rho n+i\nu\Delta\varphi n\right]$$
$$\times\int\limits_0^{T}J_\nu(\theta\rho\omega)\exp\left\{i\nu\varphi-\frac{i\omega}{2}\left[(\theta^2+E^{-2})t+\int\limits_0 v_\rho^2\,dt\right]\right\}dt.$$

Here

$$\langle v_\rho^2\rangle=\frac{1}{T_\rho}\int\limits_0^{T_\rho}v_\rho^2\,dt$$

represents the square of the transverse velocity of the particle, averaged over the period of the radial oscillations.

The Cartesian components of the current (4) are related to the quantities $j^{(\pm)}$ as follows:

$$j_x=\frac{1}{2}(j^{(+)}+j^{(-)}),\qquad j_y=\frac{1}{2i}(j^{(+)}-j^{(-)}).$$

The square of the double summations in Eq. (10) can be represented in the form

$$\left|\sum_{n=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}\exp\left[-\frac{i\omega}{2}(\theta^2+E^{-2}+\langle v_\rho^2\rangle)nT_\rho+i\nu\Delta\varphi n\right]\right|^2$$
$$=\frac{2\pi N}{T_\rho}\sum_{n=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}\delta\left[\frac{\omega}{2}(\theta^2+E^{-2}+\langle v_\rho^2\rangle)-\nu\Omega-n\omega_\rho\right], \tag{11}$$

where $\omega_\rho=2\pi/T_\rho$ is the frequency of radial oscillations, $\Omega=\Delta\varphi/T_\rho$, is the frequency of precession, $N\to\infty$ is the number of radial oscillations during the interaction with a string, and δ is the Dirac delta function.

In substitution of the quantities $j^{(\pm)}$ and $j^{(z)}$ into Eq. (4) we can emit the phase factor $\exp[i\nu(\pi/2-\varphi_r)]$ which is common for $j^{(\pm)}$ and $j^{(z)}$. As a result the spectral and angular density of the energy of radiation in axial channeling of an electron per unit path in the crystal can be represented in the form

$$\frac{d^3W}{do\,d\omega\,dl}$$
$$=\frac{e^2\omega^2}{8\pi}\sum_{n=0}^{\infty}\sum_{\nu=-\infty}^{\infty}[|j_{n\nu}^{(+)}+j_{n\nu}^{(-)}|^2+|2\theta j_{n\nu}^{(z)}+j_{n\nu}^{(+)}-j_{n\nu}^{(-)}|^2]$$
$$\times\delta\left[\frac{\omega}{2}(\theta^2+E^{-2}+\langle v_\rho^2\rangle)-\Omega\nu-n\omega_\rho\right], \tag{12}$$

$$j_{n\nu}^{(\pm)}=\frac{2}{T_\rho}\mathrm{Im}\int\limits_0^{T_\rho/2}$$
$$\times(\dot\rho\pm i\rho\dot\varphi)J_{\nu\mp1}(\theta\rho\omega)\exp[i\nu\varphi-i(\nu\Omega+n\omega_\rho)t+i\omega\delta z]\,dt, \tag{12'}$$

$$j_{n\nu}^{(z)}=\frac{2}{T}\mathrm{Re}\int\limits_0^{T_\rho/2}J_\nu(\theta\rho\omega)\exp[i\nu\varphi-i(\nu\Omega+n\omega_\rho)t+i\omega\delta z]\,dt.$$

Here

$$\delta z(t)=\frac{1}{2}\int\limits_0^t[\langle v_\rho^2\rangle-v_\rho^2]\,dt$$

describes the longitudinal oscillations of the electron in the channel.[2] We have used the relations

$$\rho(T_\rho-t)=\rho(t),\qquad \dot\rho(T_\rho-t)=\dot\rho(t),$$
$$\varphi(T_\rho-t)=2\pi+\Delta\varphi-\varphi(t),\qquad \dot\varphi(T_\rho-t)=-\dot\varphi(t),$$
$$\delta z(T_\rho-t)=-\delta z(t).$$

In a number of cases Eq. (12) can be converted to a form which is more convenient for concrete calculations:

$$\frac{d^3W}{do\,d\omega\,dl}=\frac{e^2\omega^2}{2\pi}\sum_{n=0}^{\infty}\sum_{\nu=-\infty}^{\infty}\left[|j_1|^2+\left(\frac{\nu}{\omega\theta}\right)^2|j_2|^2\right]$$
$$\times\delta\left\{\frac{\omega}{2}(\theta^2+E^{-2}+\langle v_\rho^2\rangle)-\nu\Omega-n\omega_\rho\right\},$$

$$j_1=\frac{1}{T_\rho}\int\limits_0^{T_\rho}(\dot\rho+i\rho\dot\varphi)J_\nu'(\theta\omega\rho)\exp[i\nu\varphi-i(\nu\Omega+n\omega_\rho)t+i\omega\delta z]\,dt,$$

$$j_2=\frac{1}{T_\rho}\int_0^{T_\rho}\left(\frac{\dot\rho}{\rho}-i\dot\varphi+i\frac{\omega\theta^2}{\nu}\right)J_\nu(\theta\omega\rho)\exp[i\nu\varphi-i(\nu\Omega+n\omega_\rho)t+i\omega\delta z]dt, \quad (13)$$

where J'_ν is the derivative of the Bessel function with respect to its argument.

The characteristic features of the radiation of electrons in motion along trajectories with precession are as follows. For a given harmonic with number n a multiplet structure of the spectrum appears, with a splitting of neighboring lines of the multiplet

$$\Delta\omega=2\Omega[\theta^2+E^{-2}+\langle v_\rho^2\rangle]^{-1}.$$

Radiation also appears at frequencies which are multiples of the precession frequency and which are shifted as a consequence of the Doppler effect. These frequencies correspond to the terms in Eq. (12) with $n=0$. The spectral and angular distribution of the radiation (12) does not depend on the azimuthal angle φ_r. This last result is a consequence of neglecting the interference of the radiation from the subharmonics with different values of ν. This neglect is permissible for the condition of smallness of the width of the radiation line $\delta\omega=\omega/N$, where the line arises in a finite path in the crystal (N is the number of radial oscillations in the entire length of the crystal), in comparison with the splitting $\Delta\omega$. In other words, the precession angle must be sufficiently large: $N\Delta\varphi\gg 2\pi$. In the opposite case when the precession angle is zero, the spectral and angular distributions of the radiation, generally speaking, have azimuthal asymmetry.[2-4] It can be shown, however, that these results are applicable also to this case if by the quantity $d^3W/do\,d\omega\,dl$ we understand the spectral-angular density of the energy of radiation per unit path, averaged over the azimuthal angle φ_r of the radiation.

A special case arises in motion with zero orbital angular momentum $M=0$ with respect to the atomic axis. This case corresponds to planar trajectories of the channeled electrons.[19] The expression (9) for the precession angle in this case leads formally to the result $\Delta\varphi=-\pi$ (see Appendix I). Here $\rho_{\min}=0$, and therefore the frequency of the one-dimensional oscillations ω_0 is a factor of two smaller than the frequency of the radial oscillations ω_ρ. Then we find that $\nu\Omega+n\omega_\rho=(2n-\nu)\omega_0$. Further, for one-dimensional motion along the x axis in the formulas (13) we set $\rho=x, \dot\rho=\dot x$, $\dot\varphi=0$, and $\varphi=0$ for $0\leqslant t\leqslant T_\rho/2$ and $\varphi=\pi$ for $T_\rho/2<t\leqslant T_\rho$ (we shall assume that at the initial moment of time the electron is at its aphelion). In this case Eq. (13) takes the form

$$\frac{d^3W}{\theta\,d\theta\,d\omega\,dl}=e^2\omega^2\sum_{n=1}^{\infty}\sum_{\nu=-\infty}^{\infty}\left[|j_{1x}^{(n,\nu)}|^2+\left(\frac{\nu}{\omega\theta}\right)^2|j_{2x}^{(n,\nu)}|^2\right]\delta\left[\frac{\omega}{2}(\theta^2+E^{-2}+\langle v_x^2\rangle)-n\omega_0\right],$$

$$j_{1x}^{(n,\nu)}=\frac{1}{T}\int_0^T\dot x(t)J_\nu'(\omega\theta x)\exp(-in\omega_0t+i\omega\delta z)\,dt,$$

$$j_{2x}^{(n,\nu)}=\frac{1}{T}\int_0^T(\dot x/x+i\omega\theta^2/\nu)J_\nu(\omega\theta x)\times\exp(-in\omega_0t+i\omega\delta z)\,dt,\qquad T=2\pi/\omega_0. \quad (14)$$

The expression (14), can be useful also in calculation of the spectral distribution of radiation in planar channeling of electrons and positrons.

2. CORRESPONDENCE WITH THE QUANTUM THEORY

We shall show that Eq. (12) for the spectral and angular density of the energy of radiation per unit path in axial channeling in the cylindrically symmetric field of a crystal axis can be obtained from the corresponding result of the quantum theory of radiation in axial channeling.[2] With inclusion of the quantum nature of the transverse motion of the channeled particles, the authors of Ref. 2 [see Eqs. (20) and (21) of that article] obtained the following representation for the intensity of radiation of relatively soft[1)] photons ($\omega\ll E$):

$$\frac{d^3W}{d\omega\,do\,dl}=\frac{e^2\omega^2}{2\pi}\sum_f\{|[\mathbf{j}_{if}^{(\rho)}\mathbf{n}_\rho]|^2+|\theta j_{if}^{(z)}-\mathbf{j}_{if}^{(\rho)}\mathbf{n}|^2\}\delta\left[\omega\left(\frac{\theta^2+\gamma^{-2}}{2}-\frac{\partial\varepsilon_f}{\partial E_i}\right)-\tilde\omega_{if}\right]. \quad (15)$$

Here $\tilde\omega_{if}=\varepsilon_i(E)-\varepsilon_f(E)$ is the difference of the energy levels of the transverse motion of a particle with total energy E, and

$$\mathbf{j}_{if}=\int[\psi_f^*(\rho,E-\omega)\hat{\mathbf{v}}\psi_i(\rho,E)]e^{i\mathbf{k}_\rho\rho}\,d^2\rho \quad (16)$$

are the matrix elements of the transition current between transverse-motion states $|i\rangle$ and $|f\rangle$, where $\hat{\mathbf{v}}=(i\nabla_\rho/E;c)$ is the particle velocity operator and $\nabla_\rho=\partial/\partial\boldsymbol{\rho}$.

We see at once that in the classical limit the argument of the δ function in Eq. (15) coincides exactly with the argument of the δ function in Eq. (12). In an axially symmetric field the energy of the states is determined by the combination of the radial quantum number n and the orbital angular momentum l. Thus, $i=\{n,l\}, f=\{n',l'\}$. In the classical limit all quantum numbers are large, and their differences are considered to be relatively small, i.e., $|n-n'|\ll n$, $|l-l'|\ll l$. Therefore the quantum numbers can be considered to be continuous quantities, and for the energy of the final state $\varepsilon_f(E)$ we can use the approximate equality

$$\varepsilon_{n'l'}(E)\approx\varepsilon_{nl}(E)-\frac{\partial\varepsilon_{nl}}{\partial n}(n-n')-\frac{\partial\varepsilon_{nl}}{\partial l}(l-l').$$

The Bohr-Sommerfeld quantization condition for the energy of transverse motion has the form

$$\int_{\rho_{\min}}^{\rho_{\max}}[2E(\varepsilon_{nl}(E)-U)-l^2/\rho^2]^{1/2}\,d\rho=\pi(n+1/2), \quad (17)$$

where $\rho_{\max}$ and $\rho_{\min}$ are the aphelion and perihelion of the classical orbit of the particle in the field of the axis $U(\rho)$. Differentiating the two sides of Eq. (17) with respect to the parameters E, n, and l, we can obtain expressions for the classical analogs of the derivatives of the transverse energy of the particle with respect to the corresponding parameters. For example, as was pointed out in Ref. 2, for the derivative $\partial\varepsilon_f/\partial E$ in the classical limit we obtain

$$\partial\varepsilon_f/\partial E=-\langle\varepsilon_{kin}\rangle/E, \quad (18)$$

where $\langle\varepsilon_{kin}\rangle$ is the kinetic energy of transverse motion of the particle, averaged over the period of radial oscillations. The derivation of this relation is completely analogous to that carried out in Ref. 20 for planar channeling. Differentiating with respect to n, we obtain

$$\int_{\rho_{min}}^{\rho_{max}} \frac{E\,d\rho}{[2E(\varepsilon_{nl}-U)-l^2/\rho^2]^{1/2}}\frac{\partial\varepsilon_{nl}}{\partial n}=\pi. \tag{19}$$

The integral in (19) in the classical limit is equal to the half period $T_\rho/2$ of the radial oscillations, and therefore

$$\partial\varepsilon_{nl}/\partial n=2\pi/T_\rho. \tag{20}$$

Differentiation with respect to l gives the equality

$$\int_{\rho_{min}}^{\rho_{max}} \frac{E\,\partial\varepsilon/dl-l/\rho^2}{[2E(\varepsilon_{nl}-U)-l^2/\rho^2]^{1/2}}\,d\rho=0.$$

From this with use of Eq. (9) we obtain for the precession angle $\Delta\varphi$

$$\partial\varepsilon_{nl}/\partial l=(\Delta\varphi+2\pi)/T_\rho. \tag{21}$$

Thus, with use of the equalities (18)–(21) we obtain complete agreement of the dependences of the frequency and radiation angle in the quantum and classical cases. Here the harmonic number n corresponds to the difference of the radial quantum numbers $n-n'+1$, and the harmonic number ν corresponds to the difference of the orbital quantum numbers $l-l'$.

In a similar manner we can show that in the classical limit the current matrix elements (16) go over to the Fourier components (12). For this purpose we shall take into account that in an axially symmetric field the wave functions of the transverse motion can be represented in the form

$$\begin{aligned}\psi_i(\rho,E)&=(2\pi)^{-1/2}e^{il\varphi}\rho^{-1/2}\chi_{nl}(\rho,E),\\ \psi_f(\rho,E-\omega)&=(2\pi)^{-1/2}e^{il'\varphi}\rho^{-1/2}\chi_{n'l'}(\rho,E-\omega),\end{aligned} \tag{22}$$

where ρ, φ are the cylindrical coordinates of the particle. The radial wave functions in the quasiclassical limit have the form

$$\chi_{nl}(\rho,E)=\left(\frac{2\omega_\rho(E)}{\pi}\right)^{1/2}\times[2E(\varepsilon_{nl}(E)-U_{eff}(\rho))]^{-1/4}\cos\Phi(E,n,l), \tag{23}$$

where

$$U_{eff}(\rho)=U(\rho)+(l^2-{}^1/_4)/2E\rho^2,$$

$$\Phi(E,n,l)=\int_{\rho_{min}}^{\rho}[2E(\varepsilon_{nl}(E)-U_{eff}(\rho))]^{1/2}\,d\rho-\pi/4.$$

Computing, for example, the z-component of the current (16), it is easy to perform the integration over the coordinate φ by means of the relation

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\{-i[\Delta l\varphi+\omega\theta\rho\cos(\varphi-\varphi_r)]\}d\varphi=\exp[-i\Delta l(\pi/2-\varphi_r)]J_{\Delta l}(\omega\theta\rho),$$

where φ_r is the azimuthal angle of radiation of the photon and $\Delta l=l-l'$. Then, taking into account that the phases of the quasiclassical wave functions are large, the integration over the radial coordinate ρ in the matrix element $j_{if}^{(z)}$ can be carried out by the stationary-phase method. The difference of phases $\Delta\Phi\approx\Phi(E,n,l)\Phi(E-\omega,n',l')$ which arises we shall represent in the form

$$\Delta\Phi\approx\frac{\partial\Phi}{\partial E}\omega-\frac{\partial\Phi}{\partial n}(n-n')-\frac{\partial\Phi}{\partial l}(l-l').$$

We shall take into account also the relation

$$d\rho=[2(\varepsilon-U_{eff}(\rho))/E]^{1/2}dt.$$

As a result we obtain

$$\Delta\Phi\approx\omega_\rho(n-n'+1)t+\Omega(l-l')t-\omega\delta z(t),$$

and therefore the matrix element $j_{if}^{(z)}$ in the classical limit coincides with the Fourier component $j_n^{(z)}$. A similar correspondence is obtained for the components $j_{if}^{(\pm)}=j_{if}^{(x)}\pm ij_{if}^{(y)}$.

The foregoing analysis shows that the parametric dependence of the levels of the transverse energy of the electron on its total energy in the classical limit leads to an effect which can be treated as a change of the average longitudinal velocity of the particle under the action of the averaged potential of the axis, which results in a Doppler shift of the frequency of the radiation. In a similar manner the parametric dependence of the wave functions of the transverse motion in the classical limit reduces to the effect of the longitudinal oscillations of the particles on the formation of the frequency and angular distributions of the radiation.[2)] All of these effects are important at sufficiently high energies of the channeled particles, when the condition $EU\gtrsim m^2c^4$ is satisfied.[12] The description of the radiation of channeled particles in terms of trajectories is possible for the condition that all quantum numbers of the states of the transverse motion are large, only transitions with relatively small change of the quantum numbers are effective for the radiation, and the energy of the radiated photon is substantially below the electron energy ($\hbar\omega\ll E$). The latter condition is characteristic for channeled particles just as a consequence of the parametric dependence of the transverse energy levels and the wave functions on the total energy of the particles. If this condition is not satisfied, then the expansions of the final-state energy $\varepsilon_{n'l'}(E_f)$ and of the phase difference of the quasiclassical wave functions $\Delta\Phi$ in powers of ω/E are no longer valid, and this does not permit the characteristics of the radiation of sufficiently energetic photons ($\hbar\omega\sim E$) to be expressed in terms of trajectories. In regard to other similar forms of particle radiation in undulators or in the field of an intense electromagnetic wave, in these cases the parametric dependence which we have noted is not present and the radiation spectra apparently can be expressed in terms of classical trajectories of the particles even for $\hbar\omega\sim E$ (see for example Ref. 4).

3. THE DIPOLE APPROXIMATION

Let us consider the case of relatively low electron energies, when the inequality $\varepsilon E\ll(mc^2)^2$ is satisfied; here $\varepsilon=E\dot\rho^2/2+U(\rho)$ is the energy associated with the transverse motion of the electron. Then we can neglect the dependence of the longitudinal velocity on the time, i.e., the second term in the square brackets of Eq. (3). As a result in the argument of the exponential in Eq. (1) we can set $k_z v_z - \omega t \approx (\omega/2)(\theta^2 + E^{-2})t$, where θ is the angle between the crystallographic axis and the wave vector. Then, for the condition $\varepsilon E \ll (mc^2)^2$, which means also that the angle of deflection of the electron by the field of the string is small in comparison with the effective angle of the radiation $\theta_{\text{eff}} \approx mc^2/E$, we can use the dipole approximation. In the argument of the exponentials in Eq. (12) in this case we can omit the term $i\omega\delta z$, and in the argument of the δ function we can neglect the quantity $\langle \mathbf{v}_\rho^2 \rangle$. In addition, in the case considered the argument of the Bessel functions in Eq. (12) turns out to be small: $\theta\omega\rho(t) \ll 1$, and therefore in the sum over the parameter ν in Eq. (12) the main contribution will be from terms with $\nu = 0$ (for $j_{n\nu}^{(\pm)}$) or with $\nu = 1$ (for $j_{n\nu}^{(z)}$). Then we shall use the approximate formula for Bessel functions

$$J_\nu(x) \approx (x/2)^\nu/\nu!.$$

As a result in the dipole approximation the spectral-angular distribution of the radiation energy per unit length with allowance for precession takes the form

$$\frac{d^3W}{do\,d\omega\,dl}=\frac{e^2\omega^4}{16\pi}(\theta^4+E^{-4})\sum_{n=0}^{\infty}\left\{|\rho_n^{(+)}|^2\delta\left[\frac{\omega}{2}(\theta^2+E^{-2})-\Omega-n\omega_\rho\right]+|\rho_n^{(-)}|^2\delta\left[\frac{\omega}{2}(\theta^2+E^{-2})+\Omega-n\omega_\rho\right]\right\},$$

$$\rho_n^{(\pm)}=\frac{2}{2\pi n\pm\Delta\varphi}\,\mathrm{Im}\int_0^{T_\rho/2}(\dot{x}\pm i\dot{y})\exp\{-i(n\omega_\rho\pm\Omega)t\}\,dt. \qquad (24)$$

After integration of (24) over solid angle we obtain the spectral distribution of dipole radiation

$$\frac{d^2W}{d\omega\,dl}=\frac{e^2\omega^2}{2}\sum_{n=0}\left[\omega_n^{(+)2}|\rho_n^{(+)}|^2 f\left(\frac{\omega}{2\gamma^2\omega_n^{(+)}}\right)+\omega_n^{(-)2}|\rho_n^{(-)}|^2 f\left(\frac{\omega}{2\gamma^2\omega_n^{(-)}}\right)\right] \qquad (25)$$

where

$$\omega_n^{(\pm)}=n\omega_\rho\pm\Omega,\qquad f(x)=(1-2x+2x^2)\eta(1-x),$$

$\eta(x)$ is the Heaviside step function, and γ is the Lorentz factor.

It is evident from the above relations that in the dipole approximation for a given harmonic with a number $n \neq 0$ a doublet structure of the spectrum appears, with a splitting

$$\Delta\omega=4\Omega/(\theta^2+E^{-2}). \qquad (26)$$

Let us consider in more detail the case of planar trajectories of the channeled electrons with zero orbital angular momentum $M = 0$, when $\Delta\varphi = -\pi$. In this case

$$\omega_n^{(\pm)}=\omega_0(2n\mp 1),$$

in which $\omega_0 = \omega_\rho/2$ is the frequency of the one-dimensional oscillations. Dipole radiation occurs in the odd harmonics $k = 2n \pm 1$ ($n = 0, 1, 2, \ldots$), as must be the case in one-dimensional transverse oscillations in a symmetric potential well.

4. TRAJECTORIES OF THE CHANNELED ELECTRONS

The averaged potential of an atomic string $U(\rho)$ can be represented with sufficient accuracy in the form

$$U(\rho)=\begin{cases}-\alpha/\rho+U_0 & \rho_1<\rho<D/2,\\ -U_1+\beta\rho^2, & \rho\leqslant\rho_1,\end{cases} \qquad (27)$$

where the parameters α, U_0, β, and U_1 are chosen from the condition of best agreement of the model (27) with the more exact potential. Values of α for various axes and crystals are given, for example, in Ref. 5a. The value of U_0 is chosen in such a way that the potential (27) vanishes at half the distance to the nearest axis $D/2$. Here the depth of the potential well, which is equal to $U_1 + U_0$, must coincide with the depth of the real potential well (see Ref. 5a). The remaining parameters β and ρ_1 are determined from the conditions of continuity of the potential (27) and its derivative at $\rho = \rho_1$ and have the form

$$\rho_1=3\alpha/2(U_0+U_1),\qquad \beta=\alpha/2\rho_1^3.$$

In Fig. 1 the solid curve shows a computer-calculated potential for the $\langle 111\rangle$ axis of a tungsten crystal, averaged over the thermal vibrations of the atoms. Here the Moliere approximation was used for the potential of an individual atom, and the thermal vibrations of the atoms were taken into account in the framework of the Debye model. The dashed curve shows the model dependence (27) with the parameters $\alpha = 65$ eV·Å, $U_0 = 50$ eV, $\beta = 28460$ eV/Å^2, and $U_1 = 885$ eV.

In the general case the trajectory of a particle in the potential (27) can be broken up into two portions, one of which lies in the region $\rho_1 < \rho < D/2$, and the other in the region $\rho < \rho_1$. For $\rho > \rho_1$ the transverse motion of a channeled electron is determined by the equations (see for example Refs. 2 and 3)

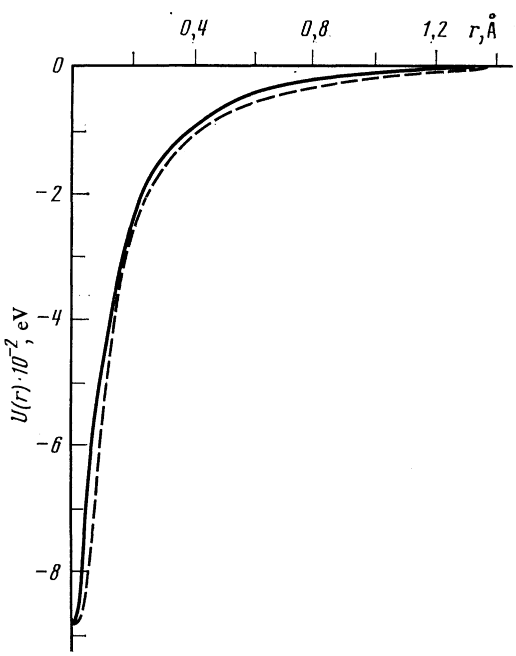

FIG. 1. Potential of an atomic string in tungsten $\langle 111\rangle$. The solid curve is a Moliere potential averaged over the thermal vibrations and the azimuthal angle. The dashed curve is the potential (27).

$$x=a(e+\cos\delta),\quad y=a(1-e^2)^{1/2}\sin\delta,\quad t=\frac{T_1}{2\pi}(\delta+e\sin\delta),\tag{28}$$

where

$$a=\frac{\alpha}{2(U_0-\varepsilon)},\quad T_1=\frac{2\pi\alpha}{c}E^{1/2}[2(U_0-\varepsilon)]^{-3/2},$$

$$e=\left[1-\frac{2M^2c^2}{\alpha^2E}(U_0-\varepsilon)\right]^{1/2}.$$

Here ε is the transverse energy of the channeled electron, E is the total energy, and M is the orbital angular momentum with respect to the axis. The initial conditions in Eq. (28) are chosen in such a way that at the initial moment $t=0$ the electron is at the aphelion $\rho_{\max}=a(1+e)$.

In the region $\rho<\rho_1$ the transverse motion of the electron is determined by harmonic dependences of the coordinates on the time of the form (see Ref. 13, Section 23)

$$x(t)=x_m\cos[\tilde\omega(t-t_1)+\varphi_1],$$
$$y(t)=y_m\cos[\tilde\omega(t-t_1)+\varphi_2],\tag{29}$$

where $\tilde\omega=(2\beta/E)^{1/2}c$ and the parameters x_m, y_m, φ_1, and φ_2 are found from the conditions of continuity of the coordinates $x(t)$, $y(t)$ and the velocities $\dot x(t)$, $\dot y(t)$ at the moment of transition t_1 from the region $\rho>\rho_1$ to the region $\rho<\rho_1$. By means of Eq. (28) we find

$$t_1=\frac{a}{c}\left[\frac{E}{2(U_0-\varepsilon)}\right]^{1/2}\left[\delta_1+\frac{1}{a}(\rho_{max}-\rho_1)^{1/2}(\rho_1-\rho')^{1/2}\right],\tag{30}$$

where

$$\rho'=a(1-e),\quad \delta_1=\arccos\left[\frac{1}{e}(\rho_1/a-1)\right].$$

For the parameters of the orbit (29) we obtain

$$x_m=(x_1^2+v_{x1}^2/\tilde\omega^2)^{1/2},\quad y_m=(y_1^2+v_{y1}^2/\tilde\omega^2)^{1/2},$$

$$\varphi_1=-\arcsin(v_{x1}/x_m\tilde\omega),\quad \varphi_2=-\arcsin(v_{y1}/y_m\tilde\omega),\tag{31}$$

$$\tilde\omega=(2\beta/E)^{1/2}c.$$

Here we have used the following notation for the coordinates and velocities at the moment of time t_1:

$$x_1=a\left[e+\frac{1}{e}\left(\frac{\rho_1}{a}-1\right)\right],$$

$$y_1=a(1-e^2)^{1/2}\left[1-\frac{1}{e^2}\left(\frac{\rho_1}{a}-1\right)^2\right]^{1/2},$$

$$v_{x1}=-\frac{2\pi a^2}{\rho_1T_1}\left[1-\frac{1}{e^2}\left(\frac{\rho_1}{a}-1\right)^2\right]^{1/2},$$

$$v_{y1}=\frac{2\pi a^2}{\rho_1T_1e}(1-e^2)^{1/2}\left(\frac{\rho_1}{a}-1\right).$$

Equations (30) and (31) are valid for those energies ε and angular momenta M for which the transverse trajectories of the electrons pass through the region $\rho<\rho_1$ near the axis, i.e., when the inequality $\rho'<\rho_1<\rho_{\max}$ is satisfied.

If the quantity $\rho'=a(1-e)$ turns out to be larger than ρ_1, then the trajectory of the channeled electron lies entirely in the region $\rho>\rho_1$ and consequently at all times has the form (28), i.e., it turns out to be closed. On the other hand, if the quantity $\rho_{\max}=a(1+e)$ turns out to be less than ρ_1, then the trajectory lies entirely in the region $\rho<\rho_1$ and at all values of t it is determined by Eqs. (29) and (31), where the quantities x_1, y_1, v_{x1}, and v_{y1} now have the meaning of initial transverse coordinates and velocities. In this case the transverse component of the trajectory also turns out to be closed.

The period of the radial oscillations in a field of the form (27) can be represented as

$$T_\rho=2[t_1\eta(\rho_{max}-\rho_1)+t_2\eta(\rho_1-\rho')],\tag{32}$$

where t_2 is the time of motion over the portion of the trajectory from the point ρ_1 to the perihelion:

$$t_2=\frac{1}{c}\left(\frac{E}{8\beta}\right)^{1/2}\arccos\left[\frac{1}{\tilde e}\left(1-\frac{2\rho_1^2\beta}{U_1+\varepsilon}\right)\right],$$
$$\tilde e=\left[1-\frac{2M^2c^2\beta}{E(U_1+\varepsilon)^2}\right]^{1/2}.\tag{33}$$

The formula (9) leads to the following expression for the precession angle $\Delta\varphi$ in the field (27):

$$\Delta\varphi=\left\{\arcsin\left[\frac{1}{\tilde e}-\frac{\rho_{min}^2}{\tilde e\rho_1^2}(1+\tilde e)\right]\right.$$
$$\left.-2\arcsin\left[\frac{1}{e}-\frac{\rho_{max}}{e\rho_1}(1-e)\right]-\frac{\pi}{2}\right\}\eta(\rho_{max}-\rho_1)\eta(\rho_1-\rho');\tag{34}$$

here the perihelion of the orbit is

$$\rho_{min}\equiv\left[\frac{U_1+\varepsilon}{2\beta}(1-\tilde e)\right]^{1/2}.$$

For closed trajectories the precession angle vanishes. In the general case the angle $\Delta\varphi$ (34) turns out to be negative. In particular, for zero orbital angular momentum the precession angle approaches the value $\Delta\varphi=-\pi$, which agrees with the general derivation (see Appendix I).

The frequency of the radial oscillations, the precession frequency, and the parameters of the orbits of the channeled electrons are determined in the last analysis by two integrals of motion in the field of the axis: the transverse energy ε and the angular momentum M with respect to the axis. For a specified energy $-U_1<\varepsilon<0$ the region of allowable angular momenta is bounded above by the value $M_{\max}$ which is determined by equality of the energy of the transverse motion to the minimal sum of the potential energy $U(\rho)$ and the centrifugal energy $M^2c^2/(2E\rho^2)$ (see Ref. 9, Section 14). For the potential (27) we obtain

$$M_{max}(\varepsilon)=\begin{cases}\dfrac{\alpha}{c}[E/2(U_0-\varepsilon)]^{1/2}, & |\varepsilon|<{}^1/_3(U_1-U_0),\\ c^{-1}(E/2\beta)^{1/2}(U_1+\varepsilon), & |\varepsilon|\geqslant{}^1/_3(U_1-U_0).\end{cases}\tag{35}$$

The transverse trajectories of electrons with angular momentum $M_{\max}(\varepsilon)$ are circles. For other angular momenta with the same value of ε the trajectories are shown in Fig. 2. The dependence of the precession angle (34) on the angular momentum M for various transverse energies is illustrated in Fig. 3, in which

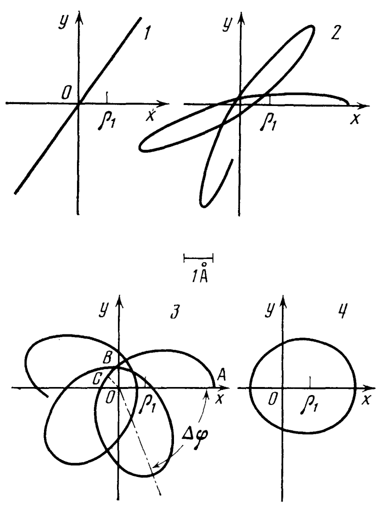

FIG. 2. Most characteristic trajectories of electrons in an axial channel of tungsten in the transverse plane.

$$M_1=(\rho_1/c)(2E)^{1/2}(\alpha/\rho_1-U_0+\varepsilon)^{1/2}. \tag{36}$$

5. SPECTRA OF THE RADIATION OF CHANNELED ELECTRONS AS A FUNCTION OF THE TRANSVERSE ENERGY AND THE ORBITAL ANGULAR MOMENTUM

From the known form of the trajectories of electrons in the field (27) found in the preceding section, we can compute the quantities $\rho_n^{(\pm)}$ which determine the frequency and angular distribution (24) and the frequency distribution (25) for the dipole radiation of channeled electrons. The result has the form (see Appendix II)

$$\rho_n^{(\pm)}=\frac{-2}{2\pi n\pm\Delta\varphi}\left[K_n^{(\pm)}\eta(\rho_{max}-\rho_1)+G_n^{(\pm)}\eta(\rho_1-\rho')\right], \tag{37}$$

$$K_n^{(\pm)}=-a\sum_{m=-\infty}^{\infty}(g_n^{(\pm)}+m)^{-1}\sin[(g_n^{(\pm)}+m)\delta_1]\times\left[-J_m'(g_n^{(\pm)}e)\pm(1-e^2)^{1/2}\frac{m}{g_n^{(\pm)}e}J_m(g_n^{(\pm)}e)\right], \tag{38}$$

$$G_n^{(\pm)}=x_m\left[S_1\cos\left(\frac{1+\tilde g_n^{(\pm)}}{2}\xi_1+\varphi_1+\varphi_3\right)-S_2\cos\left(\frac{1-\tilde g_n^{(\pm)}}{2}\xi_1+\varphi_1-\varphi_3\right)\right]$$

$$\pm y_m\left[S_1\sin\left(\frac{1+\tilde g_n^{(\pm)}}{2}\xi_1+\varphi_2+\varphi_3\right)+S_2\sin\left(\frac{1-\tilde g_n^{(\pm)}}{2}\xi_1+\varphi_2-\varphi_3\right)\right], \tag{39}$$

$$S_1=(1+\tilde g_n^{(\pm)})^{-1}\sin\left(\frac{1+\tilde g_n^{(\pm)}}{2}\xi_1\right),$$

$$S_2=(1-\tilde g_n^{(\pm)})^{-1}\sin\left(\frac{1-\tilde g_n^{(\pm)}}{2}\xi_1\right),$$

$$g_n^{(\pm)}=\frac{T_1}{2\pi T_\rho}(2\pi n\pm\Delta\varphi),\quad \xi_1=\tilde\omega t_2,$$

$$\tilde g_n^{(\pm)}=\frac{1}{T_\rho\tilde\omega}(2\pi n\pm\Delta\varphi),\quad \varphi_3=\frac{t_1}{T_\rho}(2\pi n\pm\Delta\varphi).$$

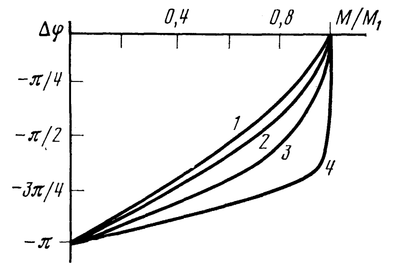

FIG. 3. Precession angle $\Delta\varphi$ as a function of the angular momentum M for various transverse energies ε which are greater than $-(1/3)(U_1-U_0)$: 1—$\varepsilon=0$, 2—$\varepsilon=-0.1U_1$, 3—$\varepsilon=-0.23U_1$, 4—$\varepsilon=-0.32U_1$. The value of M_1 is given by Eq. (36).

Here the values of a, T_1, and e are determined by the formulas (28); t_1, ρ', and δ_1 are determined by the formulas (30); x_m, y_m, φ_1, φ_2, and $\tilde\omega$ are given by the formulas (31); t_2 and $\tilde e$ are given by (33); the period of radial oscillations T_ρ and the precession angle $\Delta\varphi$ are determined by Eqs. (32) and (34).

In the limiting case of motion along ellipses of the form (28) in the region $\rho>\rho_1$ we have

$$\Delta\varphi=0,\quad t_2=0,\quad T_\rho=T_1,\quad \delta_1=\pi,$$
$$g_n^{(+)}=g_n^{(-)}=n,\quad \omega_n^{(+)}=\omega_n^{(-)}=2\pi n/T_1.$$

In the sum over m in Eq. (38) we are left with terms having $m=-n$. The second term in the square brackets of (37) disappears as a consequence of the inequality $\rho_1<\rho'$. As a result the spectral density of the energy radiated by an electron per unit path in a crystal (25) takes the form

$$\frac{d^2W}{d\omega dl}=\frac{e^2\omega\bar\omega^2a^2}{c^4}\sum_{n=1}^{\infty}\left[J_n'^2(ne)+\frac{1-e^2}{e^2}J_n^2(ne)\right]f\left(\frac{\omega}{2\gamma^2\bar\omega n}\right), \tag{40}$$

where $\bar\omega=2\pi/T_1$ is the period of motion in the elliptical orbit. This result[3)] coincides with Eq. (36) of Ref. 2.

In the other limiting case of motion along the elliptical trajectories of the form (29) entirely in the region $\rho<\rho_1$ we obtain

$$\Delta\varphi=0,\quad t_1=0,\quad \omega_n^{(+)}=\omega_n^{(-)}=n\tilde\omega,\quad \xi_1=\pi/2,$$
$$\varphi_3=0,\quad \tilde g_n^{(+)}=\tilde g_n^{(-)}=n.$$

Without loss of generality we can assume $\varphi_1=0$, $\varphi_2=\pi/2$ (which corresponds to the aphelion of the orbit at the initial moment of time). The first term in square brackets in (37) disappears, since $\rho_1>\rho_{max}$. The second term turns out to be nonzero only for the first harmonic ($n=1$), in which case we obtain

$$\frac{d^2W}{d\omega dl}=\frac{e^2\omega\tilde\omega^2}{4c^4}(x_m^2+y_m^2)f\left(\frac{\omega}{2\gamma^2\tilde\omega}\right).$$

This formula agrees with the result obtained previously for radiation in axial channeling of positrons [see Eq. (30) of Ref.

2], which was derived for the case of channeling in a parabolic potential.

In the general case of precessing orbits, the results of the calculation of the spectral distribution of the energy of radiation by an electron per unit path during channeling in the field (27) are given in Fig. 4. Here the spectra 1, 2, 3, and 4 correspond to the trajectories shown in Fig. 2. The spectral density of the radiated energy per unit pathlength is measured in units of

$$W_0=\frac{(2\pi)^3e^2\gamma^2}{ca_F}\left(\frac{2Ze^2}{Ed}\right)^{3/2},$$

where d is the distance between atoms in the string, Z is the charge of the nucleus, and a_F is the Thomas-Fermi radius.

In one-dimensional motion (Fig. 4) $M=0$, $\Delta\varphi=-\pi$, and the maximum intensity of the radiation occurs at the frequency $\Omega=\omega_p/2$ of the zeroth harmonic. With increase of the angular momentum M the intensity of the radiation in the zeroth harmonic decreases. The contribution to the radiation from the Fourier components $\rho_n^{(+)}$ also decreases [see Eq. (37)].

6. DISCUSSION OF RESULTS

Comparison of the radiation spectra of channeled electrons (24) and (37) obtained with allowance for precession of transverse orbits with the similar spectra (40) in which precession is not taken into account shows a number of substantial differences between them. These differences are most noticeable at relatively small orbital angular momenta

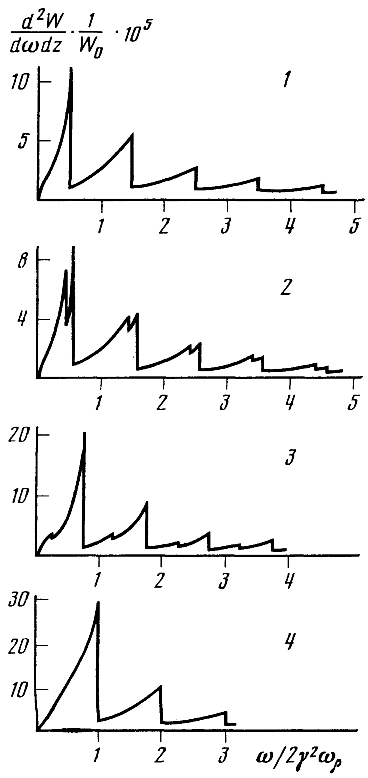

FIG. 4. Spectra of radiation of electrons in tungsten ⟨111⟩ corresponding to the trajectories shown in Fig. 2.

M. In particular, analysis of the asymptotic behavior of the quantities $\rho_n^{(\pm)}$ at large n and $M=0$ shows that the spectral density of the energy radiated falls off inversely in proportion to the number of the radiated harmonic. At the same time according to (40) without precession this falloff would occur much more slowly as a result of the corresponding behavior of the Bessel function $J_n(n)$. Thus, a more accurate inclusion of the behavior of the potential of the axis at small distances lead to a substantial change of the spectrum in the region of relatively high frequencies. In the region of the low harmonics, taking into account precession leads as a whole to a broadening of the spectrum as the result of splitting of harmonics and a corresponding decrease of the spectral density of the radiated energy, which must be taken into account in analysis of the measured radiation spectra in axial channeling of electrons.

In calculation of the spectral and angular characteristics of the radiation from a beam of channeled electrons, the spectra obtained above must be averaged over all possible trajectories of electrons in the field of the crystal axes. This averaging is carried out with a distribution function $f(\varepsilon, M, l)$ over the transverse energies ε and the orbital angular momentum M at depth l from the crystal surface; this function satisfies a kinetic equation of the Fokker-Planck type.[21] Here in general it is necessary to take into account the contribution to the radiation from superbarrier electrons ($\varepsilon>0$).[5–7] This averaging leads as a rule to a significant smoothing of the resulting spectrum, even when the spectrum of radiation from individual electrons has a nonmonotonic nature. In spite of this, in an experiment[22] carried out with good resolution in photon energy a fine structure of the spectrum was observed in the region of the intensity maximum. The appearance of this structure can be explained by the contribution from channeled electrons which have an appreciable precession of the transverse orbits. These effects should increase on introduction of additional collimation of the γ-ray beam which is radiated.

APPENDIX I

For the case $M\to 0$ we have $\rho_{\min}\to 0$ and the main contribution to the integral is from the region near $\rho=\rho_{\min}\approx 0$. The perihelion $\rho_{\min}$ of the orbit is found from the condition

$$M^2/\rho_{min}^2=2E[\varepsilon-U(\rho_{min})].$$

Hence at small $\rho_{\min}$ we have the approximate equality

$$\rho_{min}\approx M/2E(\varepsilon-U(0)).$$

Here the denominator in Eq. (9) can be represented in the form $M(\rho_{\min}^{-2}-\rho^{-2})^{1/2}$, and the upper limit can be extended to infinity. Then, after elementary integration we obtain $\Delta\varphi=-\pi$.

APPENDIX II

The integral over a half period in Eq. (26) can be broken up into two integrals with integration limits $(0, t_1)$ and $(t_1, T_p/2)$. The first integral corresponds to the contribution to the radiation from the portion of the trajectory with $\rho>\rho_1$. This is shown by the arc AB in Fig. 3. The second integral corresponds to the contribution to the radiation from the part of the trajectory with $\rho < \rho_1$. This contribution is shown by BC in Fig. 3. The expression (39) for the contribution to the radiation for motion in the region $\rho < \rho_1$ is obtained directly after substitution into (26) of the equations of motion (29) and simple integration over the region $(t_1, T_\rho/2)$.

Let us consider the calculation of the contribution to radiation from the portion of the trajectory with $\rho > \rho_1$. In the integrand of Eq. (26) we transform to integration over δ, by means of the relations (28); then the integral over the segment $(0, t_1)$ goes over into an integral over the segment $(0, \delta_1)$:

$$K_n^{(\pm)} = a \operatorname{Im} \left\{ \int_0^{\delta_1} (\sin\delta \mp i(1-e^2)^{1/2} \cos\delta) \times \exp[-ig_n^{(\pm)} (\delta + e \sin\delta)] d\delta \right\}, \quad \text{(A1)}$$

where $g_n^{(\pm)} = \omega_n^{(\pm)}/\bar{\omega}$, $\bar{\omega} = 2\pi/T_1$.

In the first term of the integrand in Eq. (A1) we convert from $\sin\delta$ and $\cos\theta$ to their exponential representation. In addition, we represent the second exponential in the integrand in the form

$$\exp(-ig_n^{(\pm)} e \sin\delta) = \sum_{m=-\infty}^{\infty} J_m(g_n^{(\pm)} e) \exp(-im\delta).$$

After a simple integration and straightforward manipulations in Eq. (A1) we obtain, separating the imaginary part, $K_n^{(\pm)}$

$$= -\frac{a}{2} \sum_{m=-\infty}^{\infty} J_m(g_n^{(\pm)} e) \left\{ \frac{-1 \pm (1-e^2)^{1/2}}{g_n^{(\pm)} + m + 1} \sin[(g_n + m + 1)\delta_1] + \frac{1 \pm (1-e^2)^{1/2}}{g_n^{(\pm)} + m - 1} \sin[(g_n^{(\pm)} + m - 1)\delta_1] \right\}. \quad \text{(A2)}$$

This formula is most convenient for calculations by computer. By means of the recurrence relations for Bessel functions and trigonometric conversions, Eq. (A2) goes over into Eq. (38).

1) Generalization of the results to the case in which the photon energy ω can be comparable with the particle energy E is carried out in Ref. 5a

2) In the work of Kumakhov and Trikalinos[2] this effect was not taken into account, which makes their results inapplicable at high energies of the electrons.

3) Equation (3.12) of Ref. 3 as the result of a misprint gives a result which is a factor of two too large.

[1] M. A. Kumakhov, Zh. Eksp. Teor. Fiz. **72**, 1489 (1977) [Sov. Phys. JETP **45**, 781 (1977)].

[2] V. A. Bazylev, V. I. Glebov, and N. K. Zhevago, Zh. Eksp. Teor. Fiz. **78**, 62 (1980) [Sov. Phys. JETP **51**, 31 (1980)].

[3] M. A. Kumakhov and K. Trikalinos, Phys. Stat. Sol. (b) **99**, 449 (1980).

[4] V. N. Baĭer, V. M. Katkov, and V. M. Strakhovenko, Zh. Eksp. Teor. Fiz. **80**, 1348 (1981) [Sov. Phys. JETP **53**, 688 (1981)].

[5] A. L. Avakyan, N. K. Zhevago, and Yan Shi, a) Zh. Eksp. Teor. Fiz. **82**, 584 (1982) [Sov. Phys. JETP **55**, 341 (1982)]. b) Radiat. Eff. **56**, 39 (1981).

[6] N. F. Shul'ga, Pis'ma Zh. Eksp. Teor. Fiz. **32**, 179 (1980) [JETP Lett. **32**, 166 (1980)].

[7] A. I. Akhiezer and N. F. Shul'ga, Usp. Fiz. Nauk **137**, 561 (1982) [Sov. Phys. Uspekhi **25**, 541 (1982)].

[8] J. U. Andersen, E. Bonderup, E. Laegsgaard, B. B. Marsh, and A. H. Sorensen, Nucl. Instr. Meth. **194**, 209 (1982).

[9] V. V. Ivanov and A. V. Tulupov, Fiz. Tverd. Tela (Leningrad) **25**, 1357 (1983) [Sov. Phys. Solid State **25**, 780 (1983)].

[10] J. U. Andersen and E. Laegsgaard, Phys. Rev. Lett. **44**, 1079 (1980).

[11] N. Cue, E. Bonderup, B. B. Marsh, *et al.*, Phys. Lett. **80A**, 26 (1980).

[12] V. A. Bazylev and N. K. Zhevago, Usp. Fiz. Nauk **137**, 605 (1982) [Sov. Phys. Uspekhi **25**, 565 (1982)].

[13] L. D. Landau and E. M. Lifshits, Mekhanika (Mechanics), Nauka, Moscow, 1965, p. 46.

[14] H. J. Kreiner, F. Bell, R. Sizmann, *et al.*, Phys. Lett. **33A** 133 (1970).

[15] Y. Yamamura and Y. H. Ohtsuki, Phys. Rev. **B24**, 3430 (1981).

[16] A. R. Avakyan, V. I. Glebov, V. V. Goloviznin, N. K. Zhevago, and Yan Shi, Preprint Erevan Physics Institute No. 595(82)-82, Erevan, 1982; Rad. Eff., 1984) (in press).

[17] E. G. Vyatkin and Yu. L. Pivovarov, Izv. vuzov, ser. fizika **8**, 78 (1983).

[18] L. D. Landau and E. M. Lifshits, Teoriya polya (Field Theory), Nauka, Moscow, 1973.

[19] H. C. H. Nip, M. J. Hollis, and F. C. Kelly, Phys. Lett. **28A**, 324 (1968).

[20] V. A. Bazylev, V. V. Beloshitskiĭ, V. I. Glebov, *et al.*, Zh. Eksp. Teor. Fiz. **80**, 608 (1981) [Sov. Phys. JETP **53**, 306 (1981)].

[21] M. Kitagawa and Y. H. Ohtsuki, Phys. Rev. **B8**, 3117 (1973).

[22] V. B. Ganenko, L. É. Gendenshteĭn, I. I. Miroshnichenko, *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **32**, 397 (1980) [JETP Lett. **32**, 373 (1980)].

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