Self-focusing and two-dimensional collapse of whistlers

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We study the non-linear self-compression of a whistler wave (whistler) in a collisionless plasma under the following conditions: a) A wave beam with a time-independent amplitude (stationary self-focusing is incident on the boundary of the plasma ($z = 0$); b) at $t = 0$ there is in the plasma a wave beam with a z-independent amplitude (two-dimensional collapse). We obtain the basic equations which are convenient for a numerical study of these processes and we give such a study which allows us to trace the dynamics of the self-compression up to its final stage. We show that in this case the leakage of energy caused by the transformation of a collapsing wave into a diverging one plays the deciding role. As a result of the de-excitation the self-compression stops and after this the wave structures smear out.

I. INTRODUCTION

The self-focusing and collapse of whistlers is of great interest in connection with the possibility of the corresponding active experiments in the Earth's magnetosphere and also because of the role which these processes may play in the strong turbulence of a magnetized plasma (e.g., in the magnetospheres of the Earth and the planets, the solar corona, laboratory plasmas, and so on).

The theoretical study in Refs. 1, 2 showed that the self-focusing of whistlers should be highly diversified. It will become clear from what follows that this also applied to the collapse. The analysis given in Refs. 1, 2 was based on a study of plane-parallel stationary uniform wave beams which are described by the solutions of a Schrödinger equation and the equations of magneto-hydrodynamics (MHD) supplemented by taking into account the ponderomotive force produced by the wave. In Ref. 3 similar results were obtained also for cylindrically symmetric beams. The solutions describing such beams (uniform and stationary) are looked for in the form of functions depending on r and the variable

$$\xi=\varkappa(z-\sigma c_S t), \tag{1.1}$$

where c_S is the sound speed and $\varkappa$ and σ are constants. After such solutions are found we let $\varkappa\to 0$ for fixed σ, as the result of which we obtained a family of solutions containing the parameter $\sigma (0<\sigma<\infty)$, where all quantitites depended only on r. It then turned out that when

$$0<\sigma<1 \tag{1.2}$$

waveguide channels were obtained with a density $N(r)$ larger than in the surrounding plasma ($N(r)>N_0$, where N_0 is the density as $r\to\infty$). It is important that the relative change in the constant (in time) magnetic field $\mathbf{B}(r)$ is then very small in comparison with the relative change in the density, i.e., if we define the quantitites

$$\nu=(N-N_0)/N_0, \quad \mathbf{b}=(\mathbf{B}-\mathbf{B}_0)/B_0, \tag{1.3}$$

where $\mathbf{B}_0$ is the external magnetic field as $r\to\infty$, $b\ll\nu$ then in the case (1.2).

If

$$1<\sigma\ll c_A/c_S, \tag{1.4}$$

where c_A is the Alfvén velocity (one assumes the plasma to be strongly magnetized so that $c_A/c_S\gg 1$) we obtain channels with a density smaller than in the surrounding plasma, i.e., $\nu<0$, with $b\ll\nu$ also in that case.

Finally, when

$$c_A/c_S\lesssim\sigma<\infty \tag{1.5}$$

one obtains again channels with $\nu<0$, but now already $b\sim\nu$ while

$$b=\nu \quad (\sigma\gg c_A/c_S). \tag{1.6}$$

However, the analysis[1,2] shows that if one uses the Schrödinger equation one loses a very important effect, the leakage of the whistler from the waveguide, owing to the transformation of the trapped wave into an untrapped mode which is not described by the Schrödinger equation. This effect, which is a kind of tunnel effect (we call it a tunnel transformation), can be described only on the basis of the complete set of Maxwell equations. The theory[4] leads to the conclusion that when $b\ll\nu$ it occurs when $\nu>0$, i.e., in the case (1.2). As to waveguides with $\nu<0$, this effect occurs only when $b\gtrsim\nu$ and, in particular, when $b=\nu$,[3] i.e., in the case (1.6). The leakage effect, which becomes very strong for narrow beams, makes the whistler self-focusing process unique. Thanks to it uniform beams with σ lying in the ranges (1.2) and (1.5) cannot be formed at all. Under conditions which might lead to such beams from the point of view of the Schrödinger equation the beams are, indeed, "de-excited" before they become rather narrow.

The present paper is devoted to numerical studies of the processes of self-compression of whistler waves under such circumstances that the leakage plays a decisive role leading to a discontinuation of the compression due to de-excitation. Taking into account what was said above we can at once indicate two such regimes.

Firstly, the case when an axially symmetric right-handedly polarized wave is incident on the boundary of the plasma with a frequency corresponding to whistlers, and with an amplitude which is constant in time. In that case we can drop

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all time derivatives in the initial equations. If the boundary condition were to correspond to a solution depending solely on r and ξ [see (1.1)] we would be led to the case (1.2) with $\sigma=0$ for which leakage occurs; the latter must, clearly, occur also for any other stationary boundary condition. One may thus expect that the self-compression of a beam with an amplitude which is constant in time (we call this process below stationary self-focusing) must be accompanied by leakage.

The second case when leakage must occur corresponds to the situation when intially there is a beam with an amplitude which is independent of z. If such a beam were stationary, we should assume in (1.1) $\varkappa\to 0$, but $\varkappa\sigma=$ const, so that $\sigma\to\infty$. According to (1.6) in that case $b=\nu<0$. As we noted already above in that case leakage should also occur which, of course, must also happen for the compression of any uniform beam. The self-compression of a beam, uniform with respect to z, can be considered to be a two-dimensional collapse.

One may thus expect that for the two simplest self-compression regimes of whistler wave beams—stationary self-focusing and two-dimensional collapse—leakage occurs and must lead to the discontinuation of self-compression due to the de-excitation of the beams when the latter become sufficiently narrow. We shall show below that this is, indeed, the case. More complicated regimes will be considered in other papers.

The paper is constructed as follows. We write down in section 2 the basic equations which are the starting point for the two above-mentioned problems of stationary self-focusing and two-dimensional collapse. Then in section 3 we consider self-focusing and in section 4 collapse.

2. BASIC EQUATIONS

We shall assume that the electromagnetic field is quasi-monochromatic:

$$\frac{1}{2}[\tilde{\mathbf{E}}(\mathbf{r},t)\exp(-i\omega t)+\text{c.c.}], \tag{2.1}$$

where the amplitude $\tilde{\mathbf{E}}(r,t)$ depends rather slowly on t, but is an arbitrary function of r. In that case the Maxwell equations give

$$\operatorname{rot}\operatorname{rot}\tilde{\mathbf{E}}=\frac{\omega^2}{c^2}(\hat{\varepsilon}\tilde{\mathbf{E}})+\frac{i}{c^2}\frac{\partial(\omega^2\hat{\varepsilon})}{\partial\omega}\frac{\partial\tilde{\mathbf{E}}}{\partial t}-\frac{1}{2c^2}\frac{\partial^2(\omega^2\hat{\varepsilon})}{\partial\omega^2}\frac{\partial^2\tilde{\mathbf{E}}}{\partial t^2}. \tag{2.2}$$

Here $\hat{\varepsilon}$ is the permittivity tensor. We shall also assume that the external magnetic field $\mathbf{B}_0$ is directed along the z-axis and that the plasma is "cold."

In the whistler frequency range

$$\omega_{hl}\ll\omega<\omega_c\ll\omega_p, \tag{2.3}$$

where ω_p, ω_c, and ω_{hl} are the electron plasma, electron cyclotron, and the lower hybrid frequencies, respectively, the non-vanishing components of $\hat{\varepsilon}$ have the form

$$\varepsilon_{xx}=\varepsilon_{yy}=\varepsilon,\quad \varepsilon_{yx}=-\varepsilon_{xy}=ig,\quad \varepsilon_{zz}=\eta,$$
$$\varepsilon=\frac{\gamma^2(N,B)}{1-u^2},\quad g=\frac{\gamma^2(N,B)}{u(1-u^2)},\quad \eta=-\frac{\gamma^2(N,B)}{u^2}, \tag{2.4}$$

$$\gamma(N,B)=\omega_p(N)/\omega_c(B),\quad u=\omega/\omega_c(B). \tag{2.5}$$

We consider in what follows cylindrically symmetric whistler beams propagating along the external magnetic field $\boldsymbol{\beta}_0$. We introduce the quantities

$$F=\tilde{E}_r-i\tilde{E}_\varphi,\quad G=\tilde{E}_r+i\tilde{E}_\varphi, \tag{2.6}$$

where $\tilde{E}_r$, $\tilde{E}_\varphi$ are the components of $\tilde{\mathbf{E}}$ in a cylindrical system of coordinates with axis of symmetry along z. For the beam considered all quantities in (2.6) depend on r, z, t, but are independent of φ. Using (2.6) we get from Eq. (2.2) the following set of equations, which is equivalent to (2.2):

$$\frac{\omega^2}{c^2}(\varepsilon+g)F+\frac{\partial^2F}{\partial z^2}+\frac{1}{2}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}r(F-G)\right]-\frac{\partial^2E_z}{\partial r\,\partial z}+\frac{i}{c^2}\frac{\partial\omega^2(\varepsilon+g)}{\partial\omega}\frac{\partial F}{\partial t}-\frac{\partial^2\omega^2(\varepsilon+g)}{2c^2\partial\omega^2}\frac{\partial^2F}{\partial t^2}=0, \tag{2.7}$$

$$\frac{\omega^2}{c^2}(\varepsilon-g)G+\frac{\partial^2G}{\partial z^2}-\frac{1}{2}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}r(F-G)\right]-\frac{\partial^2E_z}{\partial r\,\partial z}+\frac{i}{c^2}\frac{\partial\omega^2(\varepsilon-g)}{\partial\omega}\frac{\partial G}{\partial t}-\frac{\partial^2\omega^2(\varepsilon-g)}{2c^2\partial\omega^2}\frac{\partial^2G}{\partial t^2}=0, \tag{2.8}$$

$$\frac{\omega^2}{c^2}\eta E_z+\frac{1}{r}\frac{\partial}{\partial r}r\left[\frac{\partial E_z}{\partial r}-\frac{1}{2}\frac{\partial}{\partial z}(F+G)\right]=0. \tag{2.9}$$

In the stationary self-focusing problem which is considered in the next section is is convenient to use instead of Eq. (2.9), which is the z component of Eq. (2.2), the equation $\operatorname{div}(\hat{\varepsilon}\tilde{\mathbf{E}})=0$ which follows from (2.2) when $\partial\tilde{\mathbf{E}}/\partial t=0$. In the notation (2.6) it has the form

$$\frac{\partial}{\partial z}(\eta E_z)+\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}r[(\varepsilon+g)F+(\varepsilon-g)G]=0. \tag{2.10}$$

In (2.4) ω_p and ω_c are local frequencies; their spatial and temporal variations are caused by the action of the averaged ponderomotive forces of the hf field on the plasma. As in Refs. 1 to 3, we shall describe the "slow" motions of the plasma by the linearized MHD equations supplemented by the ponderomotive force of the wave field:

$$\frac{\partial\mathbf{b}}{\partial t}-[\nabla\times[\mathbf{v}\times\mathbf{B}_0]]/B_0=0, \tag{2.11}$$

$$\frac{\partial\nu}{\partial t}+\nabla\mathbf{v}=0, \tag{2.12}$$

$$\frac{\partial\mathbf{v}}{\partial t}=c_A^2[[\nabla\times\mathbf{b}]\times\mathbf{B}_0]/B_0-c_S^2\nabla\nu+\mathbf{f}/\rho_0, \tag{2.13}$$

where $\rho_0=(Zm_e+m_i)N_0\simeq m_iN_0$ is the unperturbed mass density, ν and $\mathbf{b}$ are defined in (1.3) and $\mathbf{f}$ is the ponderomotive force density of the hf field which in our notation has the form

$$f_r=-\frac{\gamma^2}{32\pi(1-u)^2}\frac{\partial}{\partial r}\left\{|F|^2+\left(\frac{1-u}{1+u}\right)^2|G|^2+\frac{2(1-u)^2}{u^2}|E_z|^2\right\}, \tag{2.14}$$

$$f_z=\frac{\gamma^2}{32\pi u(1-u)}\frac{\partial}{\partial z}\left\{|F|^2-\frac{1-u}{1+u}|G|^2-\frac{2(1-u)}{u}|E_z|^2\right\}. \tag{2.15}$$

These expressions are obtained from the general formula (see, e.g., Ref. 5) without the terms with $\partial/\partial t$. For whistlers those are unimportant when

$$u \gg (m_e/m_i)^{1/3}, \tag{2.16}$$

which will be assumed to be satisfied.

Numerical calculations show that if the initial perturbation is sufficiently small the perturbations of the plasma remain small during the whole process (i.e., $\nu \ll 1$, $|\mathbf{b}| \ll 1$) because of the de-excitation of the collapsing beam; this justifies the linearization of the MHD Eqs. (2.11) to (2.13).

3. STATIONARY SELF-FOCUSING

We state the problem as follows. Parallel to the external magnetic field there is incident on the plasma boundary ($z = 0$) an axially symmetric right-handedly polarized wave. The amplitude of the wave at the plasma boundary $z = 0$ is constant in time. The evolution of the wave in the plasma for $z > 0$ is then described by Eqs. (2.7), (2.8), (2.10), (2.11) to (2.15) in which we must put $\partial/\partial t = 0$.

We write the field amplitude in the form

$$\tilde{F}=F(r,z)e^{ikz},\quad \tilde{G}=G(r,z)e^{ikz},\quad \tilde{E}_z=E_z(r,z)e^{ikz}, \tag{3.1}$$

where we do not assume that the dependence of F, G, and E_z on r and z is slow,

$$k=\omega_p\omega^{1/2}/c(\omega_c-\omega)^{1/2} \tag{3.2}$$

is the wave number of the whistler propagating along the z-axis. We also introduce dimensionless variables

$$\rho=kr,\quad \zeta=kz, \tag{3.3}$$

$$V=F/E_0,\quad U=[F-(1-u)G/(1+u)(1-2u)]/E_0,$$
$$E_0^2=32\pi u(1-u)c_s^2\rho_0/\gamma_0^2, \tag{3.4}$$

where $\gamma_0 = \gamma(N_0, B_0)$ (see (2.5)). The reason for introducing the quantity U will be explained later. Using (2.10) to eliminate E_z from (2.7), (2.8) and using the fact that when $\nu \ll 1$, $|\mathbf{b}| \ll 1$

$$\frac{\partial \ln \varepsilon_{\alpha\beta}}{\partial z} \ll k,\quad \frac{\partial \ln \varepsilon_{\alpha\beta}}{\partial r} \ll k \tag{3.5}$$

($\alpha, \beta = x, y, z$) we get instead of (2.7), (2.8)

$$\left(i\frac{\partial}{\partial\zeta}+\frac{1}{2}\frac{\partial^2}{\partial\zeta^2}\right)(b_{11}V+b_{12}U)+\hat{L}V=B_{11}V+B_{12}U, \tag{3.6}$$

$$b_{21}\left(i\frac{\partial}{\partial\zeta}+\frac{1}{2}\frac{\partial^2}{\partial\zeta^2}\right)V+\hat{L}U=B_{21}V, \tag{3.7}$$

where

$$\hat{L}=\partial^2/\partial\rho^2+(1/\rho)\partial/\partial\rho-1/\rho^2,$$
$$b_{11}=-2(1-2u^2)/u^2,\quad b_{12}=(1+u)(1-2u)/u^2,$$
$$b_{21}=4(1-u)/(1-2u),\quad B_{21}=-2(1-u)\nu/(1-2u), \tag{3.8}$$
$$B_{11}=-(1-2u)/u^2+2(1-u)\nu/u,$$
$$B_{12}=(1-2u)/u^2+(1-u)(1-2u)\nu/2u^2.$$

We now turn to the MHD Eqs. (2.11) to (2.13) which in the stationary case take the form

$$\frac{\partial \mathbf{v}}{\partial z}=0,\quad \frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{\partial v_z}{\partial z}=0, \tag{3.9}$$

$$c_s^2\nabla\nu+c_A^2(\nabla b_z-\partial\mathbf{b}/\partial z)=\mathbf{f}/\rho_0. \tag{3.10}$$

It follows from (3.9) that $\mathbf{v}\equiv 0$. From the z-component of Eq. (3.10) and from Eq. (2.15) we get after simple transformations

$$\nu=|V|^2-\frac{1+u}{1-u}(1-2u)^2|V-U|^2-\frac{2(1-u)}{u}|W|^2, \tag{3.11}$$

where $W = E_z/E_0$ is given by Eq. (2.10) which can be rewritten in the form

$$\frac{\partial W}{\partial\zeta}+iW=\frac{u^2}{1-u}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(V+\frac{1-2u}{2u}U\right). \tag{3.12}$$

Substituting (3.11) into the r-component of Eq. (3.10) and combining this with the equation div $\mathbf{b} = 0$ we can determine b_r, b_z. We shall not do this here, noting solely that one then obtains

$$|\mathbf{b}|\sim(c_s/c_A)^2\nu, \tag{3.13}$$

i.e., $|\mathbf{b}| \ll \nu$. We can thus neglect the variations of the magnetic field which is constant in time.

The problem thus reduces to solving the set of Eqs. (3.6), (3.7), (3.11), and (3.12). Beforehand we elucidate some important properties of the quantity U defined in (3.4). To do this we note that the wave emerging from the waveguide which is formed as a result of the tunnel transformation of the self-focusing wave has, as $r \to \infty$, the form[2]

$$r^{-1/2}\mathbf{E}^0\exp\{i(k_\perp r+kz-\omega t)\}, \tag{3.14}$$

where ω and k are the same as in (2.1), (3.1), while

$$k_\perp=\frac{k(1-2u)^{1/2}}{u},\quad E_r^0+iE_\varphi^0=\frac{(1+u)(1-2u)(E_r^0-iE_\varphi^0)}{1-u}. \tag{3.15}$$

Using (3.15) one checks easily that the quantity U in contrast to V [see (3.4)] must vanish rapidly as $r \to \infty$. Moreover, one must expect that U is a smoother function of r, z than V, since V is the superposition of two waves—the self-focusing wave and the wave formed as the result of the tunnel transformation, while in the quantity U the field of the latter to an appreciable extent is subtracted. This assumption is confirmed by numerical results (see below). Taking what we have said into account we can in Eq. (3.6) neglect the term with $\partial^2U/\partial\zeta^2$ as a result of which the set (3.6), (3.7) simplifies considerably and becomes convenient for a numerical solution. The solution then is uniquely determined by V, U, and $\partial V/\partial\zeta$ for $\zeta = 0$.

We note also the following. If the transverse size of the beam is much larger than k^{-1} and $\nu \ll 1$, we have $U \simeq V$ and from (3.6), (3.7), (3.11), and (3.12) there follows a Schrödinger equation[2]

$$i\frac{\partial V}{\partial\zeta}+\frac{1-2u}{4(1-u)}\hat{L}V+\frac{1}{2}|V|^2V=0, \tag{3.16}$$

where the operator $\hat{L}$ is given in (3.8). Equation (3.16) describes the initial stage of the evolution of the beam if the

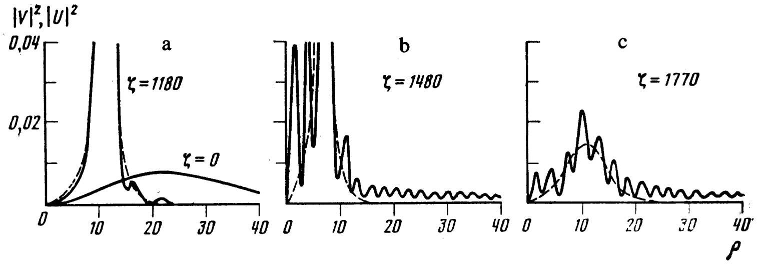

FIG. 1. Self-focusing of a whistler beam for $u = 0.3$, $V_0^2 = 0.008$, $s = 21.5$. Solid lines give $|V|^2$, dashed lines give $|U|^2$.

conditions for the applicability of this equation are satisfied for small z.

The set (3.6), (3.7), (3.11), and (3.12), and also Eq. (3.16) were solved numerically. The initial field of the beam was given in the form

$$V|_{\zeta=0} = V_0 \rho s^{-1} \exp(-\rho^2/2s^2 + 1/2). \tag{3.17}$$

For the set (3.6), (3.7), (3.12) moreover we gave

$$U|_{\zeta=0} = V|_{\zeta=0}, \quad W|_{\zeta=0} = 0, \tag{3.18}$$

$$\frac{\partial V}{\partial \zeta}\Big|_{\zeta=0} = i\left[\frac{1-2u}{4(1-u)}\hat{L}V + \frac{1}{2}|V|^2 V\right]_{\zeta=0}. \tag{3.19}$$

The last condition is connected with the fact that for small ζ the Schrödinger Eq. (3.16) must be satisfied. The range of the calculation was $\zeta \geqslant 0$, $0 \leqslant \rho \leqslant \rho_{\max} = 4s$. For $\rho = 0$ we put $V = U = 0$. At the outer boundary $\rho = \rho_{\max}$ the condition was that the energy leaking out the range covered by the calculation was absorbed.

We now give the results (part of these results for a simplified self-focusing model were given in a short communication[6]). Self-focusing occurs when $V_0^2 > V_{cr}^2$, where V_{cr}^2 is a critical value (equal to 0.005 for $s = 21.5$, $u = 0.3$). One can split the process into two stages. In the first stage of the self-focusing a tubular wave structure is formed collapsing towards the beam axis, with the quantity V smoothly changing. In that case $U \approx V$ and $v > 0$. In that stage the solution of the set (3.6), (3.7) is close to the solution of the Schrödinger Eq. (3.16). The end of the first stage approximately corresponds to $\zeta = 1180$ (see Fig. 1a) where $|V|^2_{\max} \approx 0.09$, $|U|^2_{\max} \approx 0.07$, and $v_{\max} \approx 0.09$. When the transverse size of the beam becomes comparable to the longitudinal whistler wavelength, the second stage starts. The collapse stops due to the strong tunnel transformation of the wave trapped in the waveguide into a diverging wave of asymptotic form (3.14) as is clear from Fig. 1b where $|V|^2_{\max} \approx 0.11$, $|U|^2_{\max} \approx 0.08$, and $v_{\max} \approx 0.1$. After that the leakage of energy becomes so large that $|V|^2_{\max}$ starts to decrease and the beam gradually gets de-excited (Fig. 1c). It is clear from Fig. 1 that the function $U(\rho, \zeta)$ is appreciably smoother than $V(\rho, \zeta)$ and its width is of the order of the width of the waveguide. It follows from the definition of U that this fact is the basic evidence for the fact that the oscillations in Fig. 1 are connected with the energy leakage. It is natural that in the second stage the solution of the set (3.6), (3.7) differs appreciably for the solution of the Schrödinger Eq. (3.16) as can be clearly seen from Fig. 2.

We turn attention to the following fact. In Ref. 6 we solved the set (3.6), (3.7) under conditions which differed from (3.17) to (3.19) in that $\partial V/\partial \zeta = 0$ for $\zeta = 0$. It then turned out that the corresponding solution differs from the one considered here only for small ζ. One can conclude from this that the value of $\partial V/\partial \zeta$ for $\zeta = 0$ (the other conditions being fixed) very little affects the shape of the solution for rather large ζ.

The energy characteristics of the self-focusing process are shown in Fig. 3, where we show graphs of the integrals

$$I_1 = \int_0^{\rho_{\max}} \rho |V|^2 d\rho, \quad I_2 = \int_0^{\rho_{\max}} \rho |U|^2 d\rho \tag{3.20}$$

as functions of ζ. In the first stage, of course $I_1 \approx I_2$. In the second stage I_2 initially decreases fast; this is connected with the loss of energy into tunneling and after that enters a plateau when the leakage stops due to the spreading out of the beam. At the end of the process we have again $I_1 \approx I_2$.

It is important to note also that if in the set (3.6), (3.7) we drop the terms with $\partial^2 V/\partial \zeta^2$ we get results which are close to those given above. This indicates that the ζ-dependence of the field $E(\rho, \zeta)$ may be taken to be rather slow. This explains, in particular, the fact that the results given here are close to the results obtained by assuming that $\partial \tilde{E}_z/\partial z \approx ik\tilde{E}_z$, as was done in Ref. 6.

When the dimensionless frequency $u = \omega/\omega_c$ approaches $\frac{1}{2}$ the leakage becomes stronger and the self-focusing is stopped earlier for otherwise the same conditions. This is noteworthy, when one compares the result given above with the numerical results, already when $u = 0.4$ and agrees with the qualitative theory.[2]

When $u = 0.3$ and the value of V_0^2 [see (3.17)] is twice that of Fig. 1 one obtains a two-focus structure.[7] We note merely that a consecutive de-excitation then takes place, first of the first and then of the second focus.

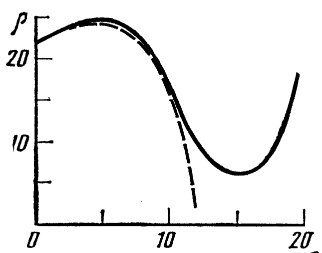

FIG. 2. Position of the maximum of the field $|U|^2$ (solid line) and maximum of the field in the Schrödinger Eq. (3.16) (dashed line) for the parameter values of Fig. 1.

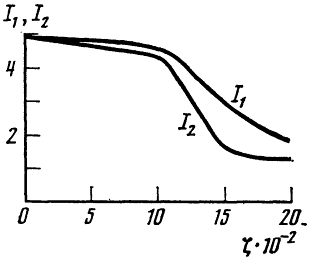

FIG. 3. The integrals I_1/I_2 as functions of ζ for the parameter values of Fig. 1.

4. COLLAPSE

We now consider the collapse of a whistler beam which is uniform with respect to z. Its temporal evolution ($t>0$) is described by Eqs. (2.7) to (2.9) and (2.11) to (2.15) in which we shall put

$$\tilde{F}=F(r,t)e^{ikz},\quad \tilde{G}=G(r,t)e^{ikz},\quad \tilde{E}_z=E_z(r,t)e^{ikz}, \tag{4.1}$$

where $\mathbf{k}$ is given by Eq. (3.2).

We introduce the dimensionless variables

$$\begin{gathered}\rho=kr,\quad \tau=\omega t,\\ V=F/E_1,\quad U=(F+\mu G)/E_1,\quad W=E_z/E_1,\\ E_1^2=32\pi(1-u)^2c_A^2\rho_0/\gamma_0^2,\quad \mu=-(1-u)/(1+u)(1-2u).\end{gathered} \tag{4.2}$$

The quantities V, U, and W differ from those introduced in section 3 only in their normalization.

Using (3.5) Eqs. (2.7) to (2.9) take in the dimensionless variables the form

$$\begin{gathered}\left(i\frac{\partial}{\partial\tau}+\frac{u}{1+u}\frac{\partial^2}{\partial\tau^2}\right)(b_{11}'V+b_{12}'U)-(1+\mu)\hat{L}V+\hat{L}U\\ =B_{11}'V+B_{12}'U+2i\mu\frac{\partial W}{\partial\rho},\\ b_{21}'\left(i\frac{\partial}{\partial\tau}-\frac{u}{1-u}\frac{\partial^2}{\partial\tau^2}\right)V+(1+\mu)\hat{L}V-\hat{L}U\\ =B_{21}'V+2i\mu\frac{\partial W}{\partial\rho},\\ \frac{\partial^2W}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial\rho}-\frac{1-u}{u}W=\frac{i}{2\mu}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho[(\mu-1)V+U],\end{gathered} \tag{4.3}$$

where

$$\begin{gathered}b_{11}'=-b_{12}'=2(1-u)/(1+u)^2,\\ B_{12}'=-B_{11}'=4/(1+u)+2u(1-u)\nu/(1+u)^2,\\ b_{21}'=-2/(1+u)(1-2u),\quad B_{21}'=u\nu b_{21}.\end{gathered}$$

The operator $\hat{L}$ is defined in (3.8). We shall in what follows neglect in Eqs. (4.3) the term with $\partial^2U/\partial\tau^2$ (similar to the neglect of the term with $\partial^2U/\partial\zeta^2$ in the self-focusing problem).

We can write the MHD equations in the case considered in the form

$$\frac{\partial}{\partial t}\left(\nu\frac{\mathbf{B}_0}{B_0}-\mathbf{b}\right)=0,\quad \operatorname{div}\mathbf{b}=0, \tag{4.4}$$

$$\frac{\partial\nu}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(rv_r)=0, \tag{4.5}$$

$$\frac{\partial v_z}{\partial t}=0,\quad \frac{\partial v_r}{\partial t}+c_S^2\frac{\partial\nu}{\partial r}+c_A^2\frac{\partial b_z}{\partial r}=\frac{f_r}{\rho_0}, \tag{4.6}$$

where f_r is given by Eq. (2.14). It follows from (4.4) that $b_r=0$, $\nu=b_z+\varphi_1(r)$, where $\varphi_1(r)$ is an arbitrary function. The beam spreads out due to de-excitation (which will be confirmed below) as $t\to\infty$. We must thus assume that $\varphi_1(r)=0$. From (4.6) follows that $v_z=\varphi_2(r)$. Assuming that $v_z=0$ at $t=0$, we get $v_z(r,t)=0$.

We thus get to the set of Eqs. (4.3), (4.5), and (4.6) with $v_z=0$, $b_r=0$, $\nu=b_z=b$, which agrees with (1.6). We can then replace (4.5) and (4.6) by a single equation

$$\frac{\partial^2\nu}{\partial t^2}-\frac{c_S^2+c_A^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\nu}{\partial r}\right)=-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{f_r}{\rho_0}\right). \tag{4.7}$$

Substituting here f_r from (2.14) and changing to the dimensionless variables (4.2) we get

$$\frac{\partial^2\nu}{\partial\tau^2}=\beta\frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{\rho\frac{\partial}{\partial\rho}\left[\nu+|V|^2+(1-2u)^2|V-U|^2+\frac{2(1-u)^2}{u^2}|W|^2\right]\right\}, \tag{4.8}$$

where $\beta=(c_A^2+c_S^2)k^2/\omega^2$.

Of the transverse size of the beam is much larger than k^{-1} and $\nu\ll1$, we have $U\approx V$, $|W|^2\ll|V|^2$ and from (4.3), (4.8) there follow the equations

$$i\frac{\partial V}{\partial\tau}+\frac{1-2u}{2}\hat{L}V-u\nu V=0, \tag{4.9a}$$

$$\frac{\partial^2\nu}{\partial\tau^2}=\beta\frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{\rho\frac{\partial}{\partial\rho}(\nu+|V|^2)\right\}. \tag{4.9b}$$

The set of Eqs. (4.3), (4.8), and also Eqs. (4.9) were solved numerically. The initial conditions for the set (4.3), (4.8) were written in the form

$$V|_{\tau=0}=V_0\rho s^{-1}\exp\{-\rho^2/2s^2+{}^1/_2\},\quad U|_{\tau=0}=V|_{\tau=0}, \tag{4.10}$$

$$\left.\frac{\partial V}{\partial\tau}\right|_{\tau=0}=i\left[\frac{1-2u}{2}\hat{L}V-u\nu V\right]_{\tau=0}, \tag{4.11}$$

$$\nu|_{\tau=0}=-\{|V|^2+2(1-u)^2|W|^2/u^2\}_{\tau=0}. \tag{4.12}$$

It is clear from (4.6) that condition (4.12) is the same as the assumption $\partial v_r/\partial\tau=0$ when $\tau=0$.

For Eqs. (4.9) the quantity $V_{|\tau=0}$ was taken in the form (4.10) and $\nu_{|\tau=0}=-|V|^2_{\tau=0}$. The range of the calculation was $0\leqslant\rho\leqslant\rho_{\max}=4s$. At the outer boundary $\rho=\rho_{\max}$ we put the condition of absorption of the energy flowing from the range of the calculation. For $\rho=0$ we put

$$V=U=0,\quad \partial W/\partial\rho=0, \tag{4.13}$$

$$\nu=-2(1-u)^2|W|^2/u^2. \tag{4.14}$$

Equation (4.14) follows from (4.5), (4.6) and the condition that the solution be finite as $\rho\to U$. For Eqs. (4.9) we put $\nu=0$ for $\rho=0$.

We give now the results. The collapse of the initial distribution (4.10) to (4.12) occurs when $V_0^2>V_{cr}^2$ where V_{cr} is a critical value (equal to 0.014 for $s=20$, $U=0.3$). As in the

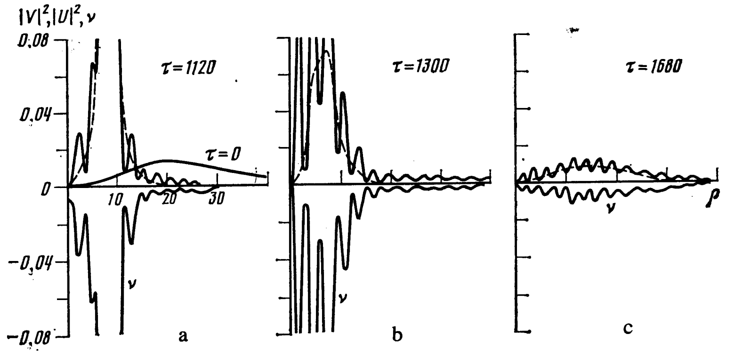

FIG. 4. Collapse of whistlers for $u = 0.3$, $V_0^2 = 0.016$, $s = 20$. Full drawn line gives $|V|^2$, the dashed line gives $|U|^2$.

case of self-focusing, the process can be split into two stages. In the first stage of the collapse a tubular wave structure is formed which collapses towards the beam axis. With a smoothly changing magnitude of $V \approx U$ and $\nu < 0$. The fact that ν is negative in the collapse distinguishes this process significantly from the self-focusing considered above where $\nu > 0$. This corresponds with the fact that $\nu < 0$ under condition (1.6). The end of the first stage roughly corresponds to $\tau = 1120$ (see Fig. 4a). In that case $|V|^2_{\rm max} \approx 0.2$, $|U|^2_{\rm max} \approx 0.15$, , $\varphi_{\rm max} \approx -0.21$. When the transverse size of the beam becomes comparable with the longitudinal whistler wavelength the second stage starts. The collapse ceases due to the strong tunnel transformation of the wave trapped in the waveguide into a diverging wave with asymptotic shape (3.14); this is clear from Fig. 4b where $|V|^2_{\rm max} \approx 0.12$, $\nu_{\rm max} \approx -0.11$. After that the leakage becomes very large so that $|V|^2_{\rm max}$ decreases and the beam gets smeared out (Fig. 4c).

In the first stage the structure of the beam is very close to the one obtained from Eqs. (4.9). In the second stage the solutions of the set (4.3), (4.8) differ considerably from the solutions of Eqs. (4.9). We then have graphs similar to Fig. 2 where we have τ instead of ζ. The integrals I_1 and I_2 [see (3.20)] as functions of τ also show similar behavior as that shown in Fig. 3.

Finally we draw attention to the following. When studying the collapse we retained in Eqs. (2.2) only second derivatives of the field amplitude with respect to time. Indeed, due to the complicated dependence of the operator $\hat{\varepsilon}$ on $\hat{\omega} = i\partial/\partial t$ the Maxwell equations contain, strictly speaking, time derivatives of arbitrarily high order. However, one can neglect them. This follows from the fact that if we neglect the term with $\partial^2\widetilde{E}/\partial t^2$ in Eq. (2.2) we get results close to those given above [obtained using (2.2)]. This enables us to assume that the higher time-derivatives which are dropped in Eq. (2.2) are unimportant.

It follows from the results of the present paper that the collapse of whistlers differs appreciably from the collapse of Langmuir waves[8,9] due to the fact that the leakage of energy due to the tunnel transformation of the collapsing wave leads to a de-excitation of the wave with subsequent smearing out of the wave structure. And this can occur long before one reaches intensities sufficient to transfer an appreciable energy to the plasma particles.

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