Contribution to shock-wave stability theory

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(Submitted 29 July 1984)
Zh. Eksp. Teor. Fiz. **88**, 470–486 (February 1985)

The general theory of decay and branching of arbitrary discontinuities is used as a basis for an investigation of the stability of a shock wave against perturbations of arbitrary intensity. The limits of stability of the shock wave against decay into other "elements," such as stable shock waves, isentropic rarefaction or compression waves, and tangential discontinuities, are determined, and the evolution of a shock discontinuity is examined for those cases where these criteria are not satisfied. The range defined by these limits contains known limits of shock-wave stability against small perturbations of the front surface. This means that a shock wave that cannot be resolved into three "elements" is also stable against small perturbations. The assumption of instantaneous decay of a shock wave that can be expressed in terms of other, including stable, waves propagating with different velocities without overtaking leads to the following conclusion: there are no shock waves that are unstable in the usual sense, i.e., with slowly growing weak perturbations. There are only stable or decaying shock waves. The validity of the above assumption requires further examination. Analysis of the conditions for the existence and development of a three-wave configuration (one wave being weak, i.e., "acoustic") propagating along the original nondecaying shock front has shown that the criteria of stationarity and decay of this configuration are the same as the well-known criteria for the stationarity and decay of weak deformations of the shock front, found by linearizing the gas-dynamic equations for the perturbations.

INTRODUCTION

The fact that the limits of stability of a shock wave against small deformations of its front[1–3] coincide with points on the shock adiabat at which two shock waves propagating in the same direction without approaching one another can exist,[4–6] or the shock discontinuity can be represented by a shock front of lower intensity and a rarefaction wave propagating in the opposite direction,[7] suggests that problems on the stability of shock waves can be solved by analyzing the decay of discontinuities. Below, we investigate the decay of a shock wave into two other component elements, using the general theory of decay of arbitrary discontinuities and of branching of shock waves. We shall determine the criteria for wave stability and evolution for those cases where these criteria are not satisfied.

1. DECAYING SHOCK WAVES

The instability of a shock wave against particular (small or large) perturbations manifests itself either in that the perturbation will slowly grow in time or the perturbed wave will assume a state that can spontaneously decay into other elements (stable shock waves of other amplitudes, rarefaction waves, tangential discontinuities[8–10]) that propagate with different velocities without overtaking. This decay is, of course, irreversible. A wave specified as being in this state will be referred to as a decaying wave in contrast to a wave that is unstable against perturbations. In Sections 1–3, we shall implicitly assume that the front of a decaying wave cannot exist for any finite time and, having been produced instantaneously, it immediately decays into other elements. The generality of this proposition will not be proved. We cannot exclude the possibility that the region of decaying waves actually consists of adjacent subregions and absolute instability and instability against finite perturbations of the structure of the shock discontinuity. This question is examined in Section 4.

Accepting the above assumptions as valid, we shall next show that there are no shock waves that are unstable in the usual sense (i.e., with slowly growing perturbations). There are only stable or decaying shock waves.

The essential mark of a decaying wave is that the "elementary composition" of its front is not unique. To find the conditions for which this absence of uniqueness will occur, we shall apply to a shock discontinuity the general method of the theory of decay of discontinuities.[8–10] Figure 1 illustrates possible nonunique decompositions of a shock discontinuity into the above three elements (ignoring the same variants for secondary waves). The four resulting wave configurations are shown by the thick lines. Two further configurations are shown in Fig. 1a. They are obtained by replacing the shock-wave traveling in the backward direction with a rarefaction wave and, in Fig. 1c, by replacing the rarefaction wave traveling in the backward direction with a shock wave (dashed lines). The decay of a shock wave with the formation of these six configurations is conveniently represented by the following schemes: the four schemes corresponding to the solid curve in Fig. 1 ($\overleftarrow{S}\to\overrightarrow{S}T\overrightarrow{S}\overrightarrow{S}$, $\overrightarrow{S}\to\overleftrightarrow{RT}S$, $\overrightarrow{S}\to\overleftarrow{R}T\overrightarrow{R}\overrightarrow{S}$, $\overrightarrow{S}\to\overleftarrow{S}T\overrightarrow{S}$) and two schemes corresponding to the dashed lines: $\overrightarrow{S}\to\overleftarrow{R}T\overrightarrow{S}\overrightarrow{S}$, $S\to\overleftarrow{S}T\overrightarrow{R}\overrightarrow{S}$ (S is a shock wave, R is an isentropic rarefaction wave, T is a tangential discontinuity, and the arrows show the direction of propagation). Moreover, the front of any compression shock wave can always be formally decomposed into other elements provided they include the isentropic compression wave. This resolution is, of course, meaningful only when the isentropic compression wave actually exists (i.e., is not compressed into a shock wave but is stretched out in time). The consequences of this in relation to

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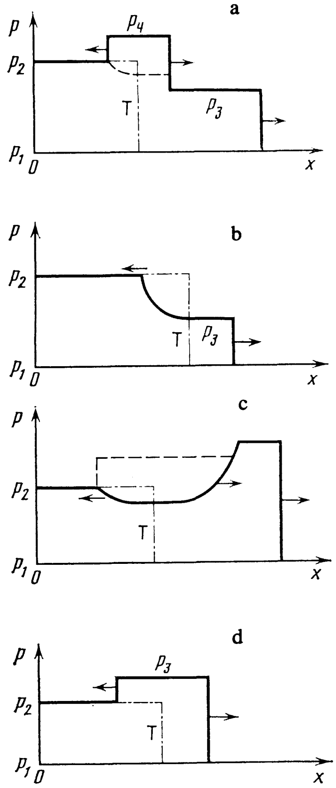

FIG. 1. Possible decay modes of a shock discontinuity. Dot-dash line—initial position of shock discontinuity. T—tangential discontinuity (in Lagrangian coordinates it coincides with the initial discontinuity front). Dashed lines represent two further decay modes in which, instead of the compression (rarefaction) wave, there is a rarefaction (compression) wave propagating in the backward direction.

additional possibilities for the decay of a compression shock wave are examined in Section 2.

The analytic or numerical determination of the conditions for the decay of a shock wave and of the amplitudes of the resulting wave can be carried out in terms of the following coordinates: the pressure p and velocity u of the material (in the laboratory system) or the pressure p and specific volume v. The coordinates (p, u) are convenient in that they are continuous across a tangential discontinuity.

Consider a family of shock adiabats and isentropes on the (p, u) and (p, v) planes, and let us represent these families by the functions

$$u=u_i+f(p,\ p_i,\ v_i), \tag{1}$$

$$u=u_i\pm f_s(p,p_i,v_i),\quad f_s(p,p_i,v_i)\equiv\int\limits_{p_i}^{p}\frac{v}{c}\,dp, \tag{2}$$

$$v=\varphi(p,\ p_i,\ v_i), \tag{3}$$

$$v=\varphi_s(p,\ p_i,\ v_i), \tag{4}$$

where p_i, u_i, and v_i are the values of p, u, and v at the initial point on the adiabat or isentrope, and c is the velocity of sound. The dependence of v and c on p under the integral sign in (2) is determined by the isentrope (4). The positive and negative signs in (2) refer to isentropic waves propagating in the forward and backward directions, respectively. All these functions can be found if we know the equation of state. By definition, $f(p_i, p_i, v_i) = 0$. The initial shock wave corresponds to the adiabat (1), (3) with $u_i = 0, p_i = p_1, v_i = v_1$:

$$u=f(p,\ p_1,\ v_1),\quad v=\varphi(p,\ p_1,\ v_1). \tag{5}$$

To determine the possibility of decay of the initial shock wave with given pressure p_2 and specific volume v_2 into the configuration of Fig. 1a with known state p_3, u_3, v_3 behind the first shock wave [the first kink on the adiabat, see below and Fig. 2; u_3 and v_3 are expressed in terms of p_3 by (1), (3)], we must consider the shock adiabats

$$u=u_3+f(p,p_3,v_3),\quad v=\varphi(p,p_3,v_3), \tag{6}$$

$$u=u_2-f(p,p_2,v_2). \tag{7}$$

The necessary condition for decay into the configuration of Fig. 1a is that the shock adiabats (6) and (7) cross on the (p, u) plane. The crossing point determines the pressure p_4 and the material velocity u_4 in the second shock wave of the configuration of Fig. 1a. The specific volumes on the right (v_r) and left (v_l) sides of the tangential discontinuity (Fig. 1a) are then given by

$$v_r=\varphi(p_4,p_3,v_3),\quad v_l=\varphi(p_4,p_2,v_2).$$

A further condition for the decay of the original wave is that the second wave must lag behind the first, i.e., we must have

$$(p_4-p_3)/(v_3-v_r)<(p_3-p_1)/(v_1-v_3). \tag{8}$$

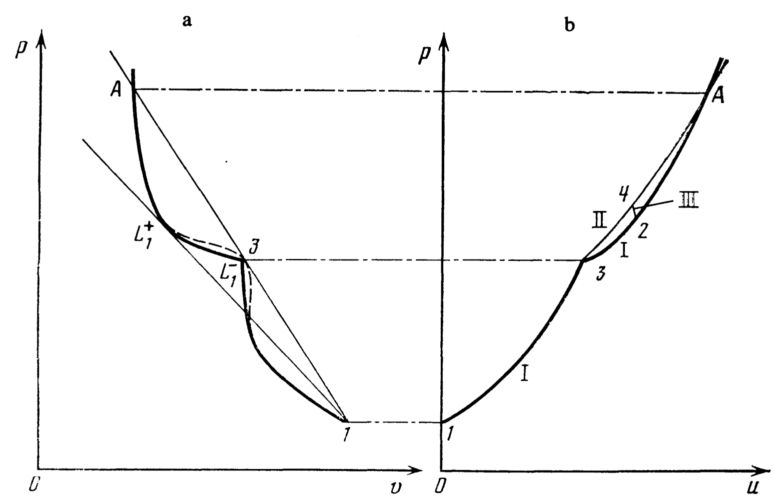

FIG. 2. I—Shock adiabat with a sharp kink or a smooth bend (dashed line) at point 3. $L = -1$ at L_1^- on the adiabat with the smooth bend (which coincides with point 3) and $L = -1$ at L_1^+. Shock adiabats II and III are drawn through points 3 and 2, respectively. Adiabat III corresponds to the shock wave propagating in the backward direction; see Fig. 1a. Pressures p_2, p_3, and p_4 at points 2, 3, and 4, respectively, determine the wave intensities in the configurations of Fig. 1a.

Figure 2 shows an example of a shock adiabatic on the (p, v) and (p, u) plane that admits of the representation of a shock discontinuity in the form of the configuration of Fig. 1a. The adiabat has a kink at the point 3.[1)] At any point on the segment $3A$, the shock wave can be represented by the configuration $\overleftarrow{S}T\overrightarrow{S}\overrightarrow{S}$ (Fig. 1a, solid line). Condition (8) is then satisfied. The disposition of the shock adiabats I and II in Fig. 2 corresponds to the case where $(\partial p/\partial T)_v > 0$. above the kink on the curve. If the kink on the adiabatic is due to a phase transition, this inequality shows that $dp/dT > 0$ on the phase equilibrium line. When the reverse inequality is satisfied, curves I and II above the kink change places. It is then readily shown that the configuration $\overleftarrow{R}T\overrightarrow{S}\overrightarrow{S}$ is formed (Fig. 1a, dashed line). This can be calculated by analogy with the configuration $\overleftarrow{S}T\overrightarrow{S}\overrightarrow{S}$. All that is required is to replace the shock adiabat (7) with the isentrope

$$u=u_2-f_s(p, p_2, v_2). \tag{9}$$

Another variant of wave decay (configuration 1b) can be found and calculated in accordance with the scheme indicated for the $\overleftarrow{R}T\overrightarrow{S}\overrightarrow{S}$ configuration, the adiabat (6) being replaced with the adiabats (1) and (3) of the original shock wave [the adiabat (6) need not be considered at all]. If it turns out that, in addition to the common point (p_2, u_2) the shock adiabat (1) and isentrope (9) have at least one further common point (p_3, u_3) such that $p_3 < p_2$, the original wave will decay into the configuration 1b. This does not require any condition such as (8), since the waves propagate in opposite directions. Figure 3 shows the shock adiabat whose segment $L_2^- \, L_2^+$ corresponds to shock waves decaying into configurations 1b.[7]

As far as decays of the form $\overrightarrow{S}\to\overleftarrow{R}T\overrightarrow{S}\overrightarrow{S}$, $\overleftarrow{S}\to\overrightarrow{S}T\overrightarrow{R}\overrightarrow{S}$ (Fig. 1c) and $\overrightarrow{S}\to\overleftarrow{S}T\overrightarrow{S}$ (Fig. 1d) are concerned, and provided that the decay is not in conflict with any conservation conditions on discontinuties, the shock wave propagating to the right is unstable against decay into configuration 1a in the first case (Fig. 1c) and against decay into the configuration 1b in the second case (Fig. 1d). This is readily verified by analyzing the shape of the shock adiabat on the p, u plane. (See also footnote 3 and Section 4 in this connection.)

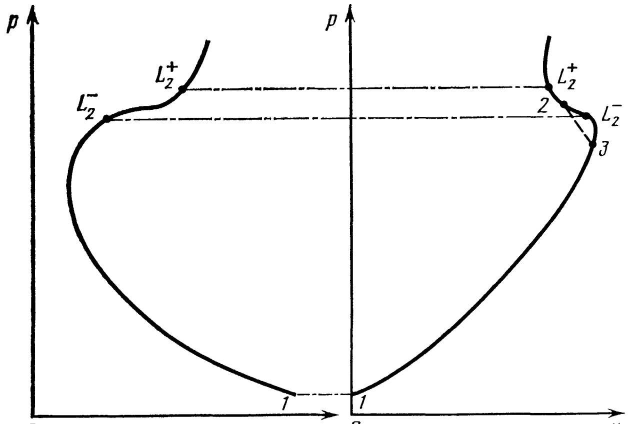

FIG. 3. Shock adiabat on which $L = 1 + 2M$ at L_2^- and L_2^+. The inequality $L > 1 + 2M$ is satisfied on the adiabat between these points. The isentrope (dashed line) drawn between points 2 and 3 corresponds to the rarefaction wave propagating in the backward direction; see Fig. 1c. Pressures p_2 and p_3 at these points determine the wave intensities in the configuration of Fig. 1c.

2. DECAY INTO CONFIGURATIONS THAT INCLUDE THE ISENTROPIC COMPRESSION WAVE

It is always possible to perform a formal decomposition of the compression shock front into an isentropic compression wave and other component elements, which means that the shock front will actually decay only when the leading (trailing) edge of the compression wave propagates more rapidly (slowly) than the shock-wave front. Hence, in order to exclude isentropic compression waves from the discussion, it is assumed in gas-dynamic theory of shock waves that $c_1 < D < c_2 + u_2$, where D is the velocity of the shock waves c_1 and c_2 are the velocities of sound ahead and behind the wave front. It is well known that these inequalities are violated in the case of anomalies in the thermodynamic properties of the medium, which manifest themselves through the fact that the shock adiabat has regions on which

$$(\partial^2 v/\partial p^2)_s < 0, \tag{10}$$

or kinks leading to a discontinuous increase in the isentropic compressibility with increasing pressure.[8,9,4,5,11–13]

When (10) is satisfied, the entropy on the shock adiabat decreases with increasing pressure. Rarefaction shock waves then exist in a particular region of initial and final states.[14,8,5,15] When (10) is satisfied at the intial point (p_1, v_1), a sufficiently weak compression shock will decay in accordance with the scheme of Fig. 4a. As the intensity of the isentropic compression wave increases, we necessarily reach a point $p = p_{s,\max}$ at which the velocity of sound on the isentrope is a minimum. (The region of anomalous thermodynamic properties is always bounded.)

When $p > p_{s,\max}$, the inequality opposite to (10) is satisfied on the isentrope. Hence, for $p_2 > p_{s,\max}$, we have, instead of the configuration 4a, the more complex system of waves shown in Fig. 4b. Further increase in pressure p_2 is accompanied by a reduction in the amplitude of the pressure $p_3 - p_1$ in the isentropic compression wave, reaching zero for $p_2 = p_2^*$, for which $D = c_1$. The corresponding pressure p_2^* is determined by

$$v_1^2(p_2^*-p_1)/(v_1-v_2^*)=c_1^2, \quad v_2^*=\varphi_s(p_2^*, p_1, v_1).$$

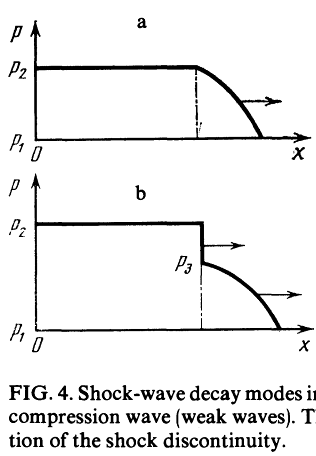

FIG. 4. Shock-wave decay modes involving the formation of an isentropic compression wave (weak waves). The dash-dot line shows the initial position of the shock discontinuity.

The configuration 4b, which appears when the pressure in the original shock front lies in the range $p_{s,\max} < p_2 < p_2^*$, can be calculated in accordance with the above scheme for configuration 1a if the functions (1) and (3) are replaced with the isentropic dependence of u and v on p in the compression wave:

$$u=\int_{p_1}^{p}(v/c)\,dp,\quad v=\varphi_s(p,p_1,v_1).$$

In the approximation in which $\varphi(p,p_1,v_1)=\varphi_s(p,p_1,v_1)$, which is good enough for weak waves, the point $p=p_3$ (Fig. 4b) can readily be found graphically from the condition that the Rayleigh line passing through the point p_2, v_2 must touch the curve $v=\varphi_s(p,p_1,v_1)$.

All that we have said about the possibility of decay of a shock front with the formation of an isentropic compression wave is also valid for the second wave in configuration 1a. When (10) is satisfied behind the first shock wave, configurations[2)] of the form $\overleftarrow{\mathrm{W}}\overrightarrow{\mathrm{TCS}}$ or $\overleftarrow{\mathrm{W}}\overrightarrow{\mathrm{TSCS}}$ will be formed for certain ranges of the pressure p_2 [C is the isentropic compression wave, W represents S or R; W≡R corresponds to the inequality $(\partial p/\partial T)_v < 0$ (see above)] and can be represented by a more involved form of configuration 1a; the role of the second shock wave in configuration 1a is now played by configurations 4a or 4b, where the coordinate of the leading edge of the isentropic compression wave coincides with that of the first shock wave. These configurations correspond to the segment $L_1^- A$ of the adiabat shown in Fig. 2a, with a continuous bend (dashed line) instead of the kink. The point L_1^- is then determined by the point at which the Rayleigh line touches the shock adiabat.

3. LIMITS OF STABILITY OF A SHOCK WAVE AGAINST DECAY

It is not, of course, necessary to investigate each point on the shock adiabat in order to elucidate the possibility of decay of a shock wave. What we need first is to find the limits of instability against decay. Simple analytic expressions can be obtained for instability limits that correspond to the limiting case of two neighboring crossings, i.e., contracting adiabats. The amplitude of the second wave is then infinitesimal and the change in its state is infinitesimal and isentropic. The slope of the adiabat for this wave on the (p,u) plane is determined by the isentropic derivative $(\partial u/\partial p)_s$.

The adiabats of the initial and isentropic waves will come into contact if

$$(\partial u/\partial p)_H=(\partial u/\partial p)_s. \tag{11}$$

The index H indicates that the derivative is evaluated along the Hugoniot adiabat. The derivatives $\partial u/\partial p$ satisfy

$$\left(\frac{\partial u}{\partial p}\right)_H=\frac{1-L}{2J},\quad \left(\frac{\partial u}{\partial p}\right)_s=\pm\frac{v}{c}, \tag{12}$$
$$L\equiv J^2(\partial v/\partial p)_H,\quad J\equiv D/v_1$$

where the plus sign corresponds to the $\overleftarrow{\mathrm{W}}\overrightarrow{\mathrm{TCS}}$ configuration and the minus sign to configuration b (Fig. 1). Substitution of (12) in (11) yields

$$L=1\mp 2\mathrm{M},$$

where M is the Mach number behind the wave front. For the $\overleftarrow{\mathrm{W}}\overrightarrow{\mathrm{TCS}}$ configuration in which the second wave propagates in the same direction as the first, condition (8) must be satisfied and means that $\mathrm{M}=1$ in the required limit (second wave weak, velocities of both waves equal). Thus, the required limits of stability of the shock wave against the decays $\overrightarrow{\mathrm{S}}\to\overleftarrow{\mathrm{W}}\overrightarrow{\mathrm{TCS}}$ and $\overrightarrow{\mathrm{S}}\to\overleftarrow{\mathrm{R}}\overrightarrow{\mathrm{TS}}$ (Fig. 1b) are respectively by the conditions[3)]

$$L=-1\quad\text{and}\quad L=1+2\mathrm{M}. \tag{13}$$

The Rayleigh line touches the shock adiabat at $L=-1$. This limit (L_1^-) occurs on smoothly bending adiabats such as the one shown in Fig. 2a (dashed curve). The limit $L=1+2\mathrm{M}$ occurs on adiabats of the form shown in Fig. 3 (point L_2^-).

If we compare the mutual disposition of the shock adiabat and the isentropes on the (p,u) plane in the neighborhood of these limits, we can readily verify that, for the shock adiabats shown in Figs. 2 and 3, the segments on which[4)]

$$L<-1\quad\text{or}\quad L>1+2\mathrm{M}, \tag{14}$$

correspond to decaying shock waves. Their fronts decay for $L<-1$ into the configuration

$$\overleftarrow{\mathrm{W}}\overrightarrow{\mathrm{TSS}}\ \text{(Fig. 1a)},\quad \overleftarrow{\mathrm{W}}\overrightarrow{\mathrm{TCS}}\ \text{or}\ \overleftarrow{\mathrm{W}}\overrightarrow{\mathrm{TSS}}, \tag{15}$$

which include at least two compression waves propagating in the forward direction, and for $L>1+2\mathrm{M}$ into the configuration $\overleftarrow{\mathrm{R}}\overrightarrow{\mathrm{TS}}$ (Fig. 1b). (We recall that W represents S or R in our notation).

Gardner[7] has shown that the configuration of Fig. 1b can be formed on shock adiabats of the form shown in Fig. 3, provided the condition for a second crossing of the shock adiabat and the isentrope, i.e.,

$$(\partial\bar{u}/\partial p)_H>(\partial\bar{u}/\partial p)_s,$$

is satisfied. (Here, $\bar{u}=-u$. In the scheme of Ref. 7, the positive direction of the velocity of the medium is opposite to that adopted here). In reality, this inequality is sufficient but not necessary. The necessary condition for two (and only two) crossings with the isentrope in the case of adiabats of the type we are considering, which have segments with $L>0$, is the more general inequality

$$(\partial p/\partial\bar{u})_H<(\partial p/\partial\bar{u})_s.$$

This refinement refers to the segment of the adiabat on which the sign of L changes from positive to negative as we run along the shock adiabat in the direction away from its origin (p_1,v_1) (see further text and Fig. 5). The derivative $(\partial p/\partial\bar{u})_s$ is identically equal to c/v [see (12)] and is always positive by virture of the basic thermodynamic inequality, written in the form $c>0$. [In Ref. 7, the positive sign of this derivative is erroneously attributed to the additional assumption that $(\partial^2 p/\partial v^2)_s>0$ on isentropes.]

Equations (13) are identical with the limits of stability, whereas the inequalities (14) are identical with the criteria for the instability of the shock wave against small deformations of its front, obtained by linearizing the equations of gas dynamics for the perturbations, and by solving the characteristic equation for the complex frequency.[1–3] (In the ensu-

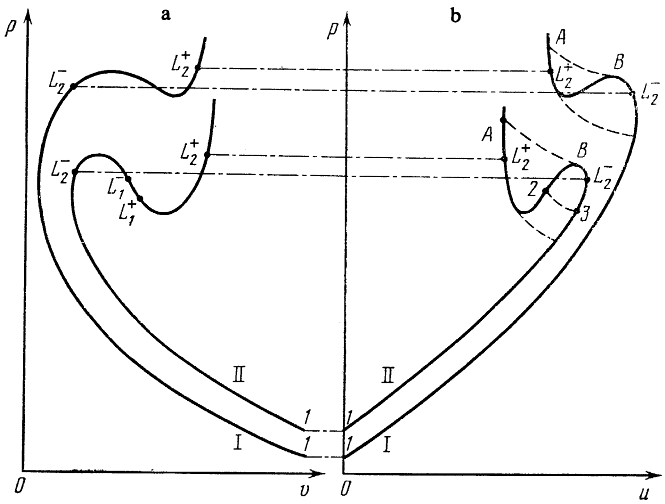

FIG. 5. Shock adiabats I and II. Points L_1^-, L_1^+ and L_2^-, L_2^+ are the same as those in Figs. 2 and 3, respectively. The dashed lines show the isentropes. The isentrope drawn from point 2 to point 3 corresponds to the rarefaction wave propagating in the backward direction; see Fig. 1c. Pressures p_2 and p_3 at these points determine the wave intensity of the configuration of Fig. 1c. The isentrope drawn from point A touches the adiabat on the (p, u) plane of point B.

ing discussion, this approach will be referred to as the linear theory, for brevity.) This means that, for adiabats such as those shown in Figs. 2 and 3, shock waves that are unstable against small deformations of the front surface according to the linear theory are, in fact, unstable against decay as well, i.e., they are decaying shock waves.[5)]

It can be shown that this property of shock waves refers not only to adiabats such as those of Figs. 2 and 3 but, generally, to all shock adiabats having segments satisfying (14). In addition to the anomalies shown in Figs. 2 and 3, there are also two types of anomaly on shock adiabats that are not thermodynamically forbidden and, therefore, at least formally possible; cf. adiabats I and II of Fig. 5 on the (p, v) and (p, u) planes. (An adiabat of type I was examined by D'yakov[1].) These adiabats have segments between the points L_2^- and L_2^+ on which the inequalities (14) are satisfied. Adiabat II is distinguished by a large downward bend, so that it has points L_1^- and L_1^+ for which $L = -1$. If we analyze the position of the shock adiabats and isentropes (broken lines) on the (p, u) plane in Fig. 5b, we can readily verify that the shock discontinuity is unstable against decay into configuration b of Fig. 1 over the entire extent of the segments of adiabats I and II between L_2^- and L_2^+ and further up between L_2^+ and A. The point A lies on the exteme left-hand "branch" of the shock adiabat on the (p, u) plane. The position of this point is determined by the fact that the isentrope drawn from A touches the adiabat on the (p, u) plane at another point (B). The point with pressure p_3 (Fig. 1b) corresponds in Fig. 5 to the crossing of the isentrope drawn from any point 2 on segments $L_2^-L_2^+$ or L_2^+A of the adiabat and the extreme right-hand "branch" (for an example of this isentrope, see the middle dashed line in Fig. 5b).

The segment of adiabat II near L_1^+ (Fig. 5) is qualitatively no different from that of Fig. 2a near L_1^- (adiabat with continuous bending; see dashed line in Fig. 2a) and corresponds to the $\vec{S}\to\overleftrightarrow{W}\overrightarrow{TCS}$ or $\vec{S}\to\overleftrightarrow{W}\overrightarrow{T}\overleftrightarrow{SC}\overrightarrow{S}$ decays. However, in contrast to the adiabats shown in Fig. 2a, such configurations cannot actually appear in the case of adiabat II of Fig. 5 because their first wave would then correspond to some point on the segment $L_2^-L_2^+$ of the adiabat, and should therefore decay in accordance with scheme b of Fig. 1.

It is important to note that the range of values of L for a decaying wave does not only completely include as subregions all values of L for which the wave is unstable in linear theory. Decaying waves also include those for which small deformations of the front do not grow with time in linear theory, i.e., those for which[1–3]

$$-1<L<1+2M. \tag{16}$$

For example, the upper limit (point A) of the region of decaying waves in the case of the adiabats shown in Fig. 2 [the limit of the region of existence of configurations (15)] lies above the point L_1^+, but inequalities (16) are satisfied on the segment L_1^+A. They are also satisfied on the segment $L_1^-L_1^+$ of adiabat II of Fig. 5. The conditions for decay into configuration b of Fig. 1 and inequalities (16) will also be simultaneously satisfied on a certain segment of the adiabat of Fig. 3 lying above the point L_2^+ and segments L_2^+A of the adiabats of Fig. 5.

4. ADDITIONAL NOTES ON SECTIONS 1–3 AND DISCUSSION OF RESULTS

1. So far we have considered the stability and decay scheme of compression shock waves. The results can readily be generalized to the case of rarefaction shock waves. A graphical representation of the possible (nonunique) "elementary composition" of a rarefaction shock wave front (not containing secondary compression shock waves) is obtained by mirror reflection of the configurations examined above (Fig. 1 and others) in the x axis.

2. We must now consider whether waves defined above as decaying cannot exist for any finite time, i.e., whether, having been produced in this somewhat artificial manner, they immediately decay into the component elements. Direct proof (or rejection) of this proposition cannot be obtained within the framework of the differential equations of gas dynamics, or from integral conservation laws, because differential equations are not valid for discontinuities, and integral relationships do not lead to unique results in this case. This difficulty cannot be circumvented by smearing out the discontinuity by artificially introduced dissipative processes because the result may then depend on the width and the unknown initial discontinuity structure which is not nec-

essarily stationary. It is possible that the problem can be solved by numerical simulation of the evolution of a discontinuity within the framework of a microscopic description of the system, using the method of molecular dynamics. Of course, the interaction between the particles must then be specified so as to reproduce the corresponding anomalies of the shock adiabat.

Arguments in favor of the rapid decay of the wave into the component elements may be summarized as follows.

The decay of a wave with the formation of other (stable) waves is irreversible. Once they appear, these waves depart from one another and this, together with their stability, leads to the irreversibility of the process. The problem is thus reduced to finding the probability that they will appear. If the initial structure of the discontinuity is specified in the form of a configuration of such waves, the decay of a discontinuity will occur immediately.

By analyzing all the possible variants of the nonunique representation of a shock discontinuity, we can verify that the wave is of the decaying type, i.e., it exhibits all the marks of a decaying wave noted in Section 1, provided at least one of the following two conditions is satisfied: (a) if the elements into which the discontinuity is decomposed include a wave propagating in the forward direction with a lower pressure and higher velocity than the original wave and (b) if the structure of the resulting configuration attains a high velocity (and, correspondingly, high density and momentum) of matter as compared with the original wave. The increase in the velocity and momentum of the medium is assured by a rarefaction wave propagating in the backward direction. [Condition (a) constitutes a violation of the requirement of ultrasonic propagation of the original wave relative to the medium in front of it, generalized from acoustic to finite perturbations.] The $\overrightarrow{S}\to\overleftarrow{R}T\overrightarrow{S}$ decay is possible only for very strong and essentially nonisentropic shock waves. The development of an initial discontinuity in a strong shock wave is accompanied by very large fluctuations in the state of the medium as compared with regions of steady flow in front of and behind the discontinuity. Under these conditions, the development of the discontinuity involves the likelihood of a practically instantaneous onset of the above maximum velocities and compressions of matter, which correspond to the onset of the irreversible decay of the wave. We note that, if the initial shock discontinuity, represented by the configuration $\overleftarrow{R}T\overrightarrow{S}$ (Fig. 1b), is specified in the form of a structureless and infinitesimally thin discontinuity, at the start of the shock development the velocity of the medium immediately behind the discontinuity rises rapidly to a finite value and a finite-amplitude rarefaction wave traveling in the backward direction appears. It is then found that the original discontinuity decays irreversibly, and probably immediately, into the $\overleftarrow{R}T\overrightarrow{S}$ configuration.

For the anomalies on shock adiabats that are known experimentally (adiabat a in Fig. 1), the decay of the wave has actually been observed.[4,5]

We cannot, however, exclude the possibility that, if the structure of the original discontinuity is specified to be stationary, it may turn out that, for values of L for which the stability criterion (16) of the linear theory is satisfied, the shock wave satisfying the decay conditions will nevertheless not, decay spontaneously. When this is so, segments of adiabats on which both the stability conditions (16) of the linear theory and the decay conditions are simultaneously satisfied will correspond to relatively stable shock waves that decay only upon sufficiently major restructuring under the influence of external sources of perturbation. A relatively stable shock wave will then correspond to definite segments of the extreme "branches" of the adiabats shown in Fig. 5, which are separated from the intermediate "branch" by the points L_2^- and L_2^+. For a sufficiently large perturbation of the shock discontinuity, there will be a transition from one extreme "branch" to another by decay to the $\overleftarrow{R}T\overrightarrow{S}$ configuration (transition with a reduction in pressure in the first shock wave) or the $\overleftarrow{S}T\overrightarrow{S}$ configuration (the reverse transition; on the (p, u) plane this is a transition from the extreme right to the extreme left "branch" of the adiabat).

These questions require further investigation.

We note in conclusion that, in addition to the problem of shock-wave stability that we have discussed, there is the independent problem of the stability of flows containing shock waves (see, for example, Ref. 18). In particular, a wave that is itself stable may lead to the flow instability when it interacts with a piston. This type of instability occurs for $L > 1$ (Refs. 19 and 20).

5. PROPAGATION OF PERTURBATIONS ALONG THE SHOCK-WAVE FRONT

We shall now consider, in terms of plane-wave configurations, small distortions of the front of a nondecaying shock wave and their development in time. We shall first be interested in the development of deformations that are not subject to external perturbations, i.e., perturbations arriving on the wavefront from flows taking place ahead of and behind the front. The basic elements of such deformations are the three-wave configurations containing a weak ("acoustic") compression or rarefaction wave. Figure 6 shows an example of a configuration of this kind. The numbers 1, 2, and 3 in this figure label, respectively, the fronts of the initial (direct, in the laboratory system) wave, an oblique wave, and a weak perturbation. The dashed line shows a tangential discontinuity. The three-wave configuration divides the flow region

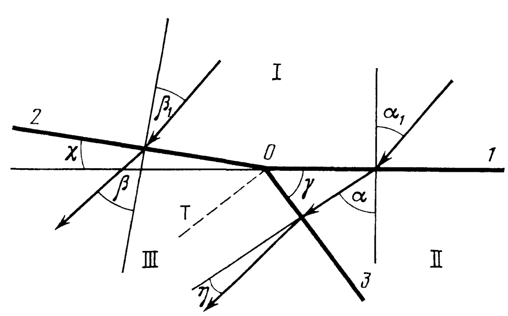

FIG. 6. Configuration consisting of three waves with a peak compression wave 3: 1—unperturbed shock wave, 2—perturbed shock wave, T—tangential discontinuity. Arrows show the directions of current line in the coordinate frame with fixed point O.

into sectors, i.e., the initial state in front of the wave (I), the undisturbed flow behind the wave (II), and the undisturbed flow ahead of the wave (III). Front 3 propagates through the gas with the velocity of sound c. The point O in the laboratory frame moves to the right along front 1 with velocity V_t given by

$$V_t=(c-V\cos\gamma)/\sin\gamma, \tag{17}$$

where V is the velocity of the medium behind the direct wavefront relative to this front and γ is the angle between fronts 1 and 3 of the undisturbed and weak waves. This angle depends on the initial state and the thermodynamic properties of the medium, which determine the shape of the shock adiabat, and on the intensity of the undisturbed shock wave. This dependence is determined by the conditions for the existence of the three-wave configuration, namely, equal pressures and the same directions of flow lines throughout sector III.

It can be shown that there are altogether four qualitatively different configurations of this kind. They can be obtained from the configuration of Fig. 6 by replacing the weak compression wave with a weak rarefaction wave or by interchanging the direct and oblique waves. To find the above dependence on the angle γ, we can consider any of these configurations without loss of generality. Suppose that this is the configuration shown in Fig. 6. In the coordinate frame in which the point O at which the fronts intersect is at rest, the oblique wave is the unperturbed wave. The angles α_1 and α between the flow lines and the normal to front 1 before and after refraction are, respectively, determined by

$$\operatorname{tg}\alpha_1=V_t/V_n=V_t/\theta V, \quad \operatorname{tg}\alpha=V_t/V, \tag{18}$$

where $\theta=\rho_{\rm II}/\rho_{\rm I}$ is the degree of compression in shock wave 1 (ratio of densities in sectors II and I) and V_n is the component of the mass velocity in sector I at right angles to front 1, and its modulus is equal to the velocity D of the initial shock wave in the laboratory frame. Substituting

$$\Gamma=\operatorname{ctg}\gamma, \quad A=\operatorname{tg}\alpha \tag{19}$$

in (17) and (18), we obtain

$$A=(1+\Gamma^2)^{1/2}/\mathrm{M}-\Gamma. \tag{20}$$

Relative to front 2, the normal and tangential components of the velocity of the incident material are respectively given by

$$V_n'=V_n\cos\chi+V_t\sin\chi, \quad V_t'=V_t\cos\chi-V_n\sin\chi, \tag{21}$$

where χ is the angle between fronts 1 and 2 (Fig. 6). The angle χ is small in the case of the weak wave 3 and is related to the difference ΔD between the velocities of shock waves 2 and 1 by

$$\chi=\Delta D/V_t=D'\Delta P/V_t, \tag{22}$$

where ΔP is the increase in pressure in wave 2 relative to wave 1 and D' is the derivative of D with respect to pressure. In the approximation linear in χ, Eq. (21) leads to the following expressions for the angles β_1 and β between the current lines and the normal to front 2 before and after refraction (Fig. 2), respectively:

$$\operatorname{tg}\beta_1=\frac{A}{\theta}\left[1-\chi\left(\frac{\theta}{A}+\frac{A}{\theta}\right)\right], \quad \operatorname{tg}\beta=\theta'\operatorname{tg}\beta_1,$$

where $\theta'=\rho_{\rm III}/\rho_{\rm I}$ is the degree of compression in wave 2. In the linear approximation we have

$$\theta'=\theta(1+\Delta\rho/\rho), \quad \Delta\rho=\rho_{\rm III}-\rho_{\rm II}, \quad \operatorname{tg}\beta=(1-\lambda)A, \tag{23}$$

$$\lambda=\chi(\theta/A+A/\theta)-\Delta\rho/\rho. \tag{24}$$

Expressing $\Delta\rho/\rho$ in terms of Δp and $(\partial v/\partial p)_H$, we can write

$$\lambda=\chi(\theta/A+A/\theta)+\rho(\partial v/\partial p)_H\Delta p. \tag{25}$$

According to (22) and (25), the quantity λ is proportional to Δp and, consequently, in the case of a weak perturbation, $\lambda \ll 1$. When this inequality is satisfied, it follows from (23) that

$$\beta=\alpha-\lambda A/(1+A^2). \tag{26}$$

Analogous calculations, which will not be reproduced here, lead to the following expression for the angle η through which the flow lines rotate as they refract in the weak wave (front 3):

$$\eta=-(\Delta\rho_s/\rho)S/(1+S^2), \tag{27}$$

$$S=\operatorname{tg}(\gamma-\alpha), \tag{28}$$

where $\Delta\rho_s$ is the density discontinuity on front 3, which is related to Δp by

$$\Delta\rho_s=\Delta p/c^2. \tag{29}$$

The angle η is assumed to be positive when the flow lines rotate in the clockwise direction. From (19), (20), and (28), it follows that

$$S=\mathrm{M}(1+\Gamma^2)^{1/2}-\Gamma. \tag{30}$$

All the flow lines in sector III are parallel, provided

$$\beta+\chi=\alpha+\eta. \tag{31}$$

Substituting (26), (27), (22), and (39) in (31), and expressing λ and $\Delta\rho_s$ in accordance with (25) and (29), we obtain

$$\left[\rho\left(\frac{\partial v}{\partial p}\right)_H+\frac{D'}{VA}\left(\frac{\theta}{A}+\frac{A}{\theta}\right)\right]\frac{A}{1+A^2} -\frac{D'}{VA}-\frac{S}{c^2\rho(1+\rho^2)}=0. \tag{32}$$

The derivatives $(\partial v/\partial p)_H$ and D' are related by [see (12)]

$$D'=\frac{\theta}{2}\frac{1+L}{J(\theta-1)}. \tag{33}$$

(This is readily verified by differentiating with respect to p the known expression for the shock-wave velocity as a function of p and v ahead of and behind the front.) Substituting (33) in (32), we obtain the following expression after simple rearrangement:

$$L=\left[1-\frac{\theta}{A^2}+\frac{2\mathrm{M}^2S(A^2+1)}{A(1+S^2)}\right]\Big/\left(1+\frac{\theta}{A^2}\right). \tag{34}$$

This is the relation between the angle characterizing the orientation of front 3 and the quantities L, M, and θ that are uniquely determined by the equation of the shock adiabat and the given point on it. The quantities A and S in (34) can be expressed in terms of cotan γ [see (20) and (30)].

The density jump $\Delta\rho_{\rm I}$ on the tangential discontinuity is due to the fact that the same change in pressure on the left of line T in sector III is achieved by an additional (as compared with wave 1) change in shock-wave intensity, whereas on the right it is achieved by compression in the weak wave 3:

$$\Delta\rho_T = -\left[\rho^2\left(\frac{\partial v}{\partial p}\right)_H + c^{-2}\right]\Delta p.$$

The angle γ can assume values between 0 (fronts 1 and 3 are parallel and both waves propagate in the same direction) and π (fronts 1 and 3 parallel and waves propagate in opposite directions). For angles in the range $0<\gamma<\gamma_0 \equiv \arccos M$, the weak wave 3 is the incident wave, whereas for all other angles ($\gamma_0<\gamma<\pi$) it is the outgoing wave (according to the Landau and Lifshitz classification)[8]. The angle γ_0 corresponds to the propagation of a weak wave 3 along the front 1. We then have $A=(1-M^2)^{1/2}/M$, $S=0$, and the flow lines cross front 3 at right-angles (in the stationary frame).

From (34), (20), and (30) with $\gamma=0$, γ_0 and π, we obtain

$$L(0)=1-2M,\quad L(\gamma_0)=\frac{1-M^2-\theta M^2}{1-M^2+\theta M^2}, \tag{35}$$

$$L(\pi)=1+2M. \tag{36}$$

It may be shown that the quantity L, calculated for fixed values of M and θ, decreases monotonically as the angle γ increases from γ_0 to π. Hence, in accordance with (35) and (36), the conditions for the existence of the triple configurations with a weak departing wave 3 are determined by the inequalities

$$\frac{1-M^2-\theta M^2}{1-M^2+\theta M^2}<L<1+2M. \tag{37}$$

For angles in the range $0<\gamma<\gamma_0$ (incident waves 3), the dependence of L on γ will, in general, be nonmonotonic, and it may be that $L<1-2M$. However, the values of L then always lie in the range[6)]

$$1<L<(1-M^2-\theta M^2)/(1-M^2+\theta M^2), \tag{38}$$

and the lower limit is reached only for $M=1$.

Whatever its dependence on L, the angle γ for the incident wave regarded as an external perturbation can be arbitrary. If it does not satisfy (34), other configurations are formed instead of the three-wave configuration. In particular, in a certain range of values of γ there is a configuration consisting of four waves crossing at a single point[8,9] (configurations with four waves were examined by Fowles[21]; see also the reflection of sound by a shock wave[22]). Within the framework of the approximation employed in Ref. 21, we cannot, however, determine the range of values of γ in which the four-wave configuration can exist. The essential point for our further analysis is that such configurations must necessarily contain an incident wave, quite apart from the unperturbed wave.

The configurations involving the outgoing and incident wave 3 (Fig. 6) evolve in time in qualitatively different manners. Let us suppose that the shock wave satisfies (37) and that there is a random perturbation of the front in the form of a three-wave configuration with a weak outgoing wave. It is readily verified that the front line of this wave lengthens continuously, so that the configuration that we are considering will exist for an infinite time in an infinite medium, and its component waves will retain constant intensities.

Let us now suppose that the shock wave satisfies (38) and a perturbation of its front results in the appearance of the triple configuration (or a configuration consisting of four waves with an incoming weak wave whose front has a limited size).[7)] Any such configuration will exist for a finite time. It will vanish when the outer boundary of front 3 coalesces with the shock-wave front. Thereafter, the deformation of the wave front will decay rapidly as a consequence of the local character of the initial perturbation and because the triple configuration with outgoing weak wave cannot arise when (38) is satisfied. The three-wave configuration with an incoming wave will not decay only when the front of wave 3 extends without limit in the downward direction along the flow right from the beginning. The incoming wave is then a constant external factor, and the stationary perturbation of the shock-wave front is the reaction of a stable system to this factor.

These properties of three-wave configurations, which refer to their stationarity and decay, are in complete agreement with the results deduced in linear theory,[1–3] in which small deformations of the shock-wave front are stationary and decay accordingly when (37) and (38) are satisfied.

In conclusion, let us briefly consider the generation of acoustic disturbances by a shock wave.[1] When conditions (37) are satisfied, weak perturbations of the front, produced by some external causes (surface irregularities on the shock tube wall, inhomogeneities in the original material, and so on) remain on the front surface and propagate downward along the flow. In terms of the three-wave configuration, this constitutes the propagation of an outgoing weak wave 3 (Fig. 6). When (37) is satisfied, one expects to see a sharp enhancement of acoustic noise behind the wave front. The propagation of unattenuated perturbations along the shock wave front and downward along the flow when (37) are satisfied in undoubtedly one of the reasons for the breakdown of uniform flow that is often observed when strong shock waves propagate through gases (see, for example, Refs. 33 and 24).

We emphasize that the generation of sound waves should not be looked upon as the effective loss mechanism responsible for the attenuation of the shock wave. The shock wave constantly transfers kinetic and thermal energy to the material flowing into it. The type-3 outgoing wave (Fig. 6) constitutes only one (and not the most important by far) component of the overall energy flux. The time dependence of the shock-wave intensity is determined by the entire field of subsonic flow and the boundary conditions. In particular, if an infinite region of constant flow is located behind the undisturbed shock-wave front, small deformations of the front, which remain stationary when (37) is satisfied, do not lead to the attenuation of the shock wave in a finite time.

CONCLUSIONS

1. Segments of shock adiabats that correspond to unstable shock waves in linear theory [and satisfy the inequalities (14)] are always component parts of long segments of a shock adiabat on which a shock discontinuity can be represented

by two other stable waves propagating with different velocities without overtaking one another.

2. If the shock discontinuity cannot be resolved into such waves, the shock wave is stable against relatively small perturbations (the perturbations do not grow with time).

3. Further studies are necessary to settle the question as to whether a shock wave that can be represented in the form of the waves specified in item 1 above, but does not satisfy the instability criteria (14) of the linear theory, is also absolutely unstable (decays instantaneously) or is stable only against relatively finite perturbations of the structure of the discontinuity under the influence of the external source.

4. A finite perturbation of the shock-wave front does not destroy its stability if it does not take the wave to a state that can be represented as described in conclusion 1 above.

5. The following three conclusions may be drawn on the assumption that a shock wave that can be represented by the two other waves specified in conclusion 1 above will immediately decay into such waves (the validity of this assumption has not been proved in the general case), at least when criteria (14) are satisfied:

A. Shock waves that are unstable in the usual sense, i.e., those for which an infinitesimal perturbation gradually grows in time, do not exist.[8)]

B. Linearization of the gas-dynamic equations for the perturbations in the form in which it is used to investigate the stability of shock waves will not, in general, yield the correct solution of the problem of the development of a perturbation of a decaying wave even in the initial stage of the process, since the unperturbed flow in this approach is a stationary and not a decaying wave, and the former does not exist in reality. The exponential growth of a small perturbation that is obtained in this method is only an indirect indication of the nonstationary character of the flow, which does not reflect the true picture of its development.

C. In contrast to linear theory, which can be used to determine the development of decaying perturbations of a stable (nondecaying) shock wave, the theory of decay and branching of discontinuities gives a complete solution of the problem of the stability of a shock wave against perturbations of arbitrary amplitude and the evolution of the decaying shock wave. The theory of decay and branching of discontinuities is distinguished by ease of physical interpretation and simplicity: it employs mostly algebraic and trigonometric transformations.

6. Weak deformations of the surface of a plane front of a nondecaying shock wave are found to attenuate when inequalities (38) are satisfied, and retain constant intensity when the inequalities (37) are satisfied. These attenuation and stationarity criteria were obtained by analyzing the propagation of three-wave configurations along the surface of the initial shock-wave front, and are in complete agreement with the corresponding results obtained by linearizing the gas-dynamic equations for perturbed motion and by their "spectral" analysis.[1–3]

7. The above results also lead to the following additional qualitative conclusions: (a) the limits of the region of existence of the three-wave configuration with outgoing wave 3 (Fig. 6) of finite amplitude δp depend not only on L but also on δp; (b) if the length of front 2 (Fig. 6) of this configuration is bounded, the intensity of shock wave 3 decreases with time in a way similar to the attenuation of any plane wave of finite width and amplitude, and (c) all that we have said above about configurations with ingoing waves remains in force in the case of finite-amplitude waves of this kind, and the only change is in the limits of the region in which they exist.

The author is greatly indebted to Ya. B. Zel'dovich and E. M. Lifshitz for their interest in this research and for valuable suggestions.

1)The configuration of Fig. 1a is obtained only when the shock adiabat has the kink which is observed in phase transitions. Continuous bending of the adiabat is accompanied by the formation of configurations in which, instead of the pressure plateau between the two shock discontinuities, we have an isentropic compression wave (see Section 2) which, in the limit of long times, becomes a straight line on the (p, x) plane.

2)The wave adiabat connecting the initial and final states in a system of waves without tangential discontinuities, including the isentropic compression wave propagating in the same direction, was examined by Gallin and Sidorenko.[16]

3)Fowles[17] has derived (13) as the limits of existence of the $\overleftrightarrow{RS}$ and $\overleftrightarrow{SS}$ configurations (which, for the correct resolution of the shock discontinuity, should be of the form $\overleftarrow{W}\overrightarrow{TS}S$ and $\overleftarrow{S}T\overrightarrow{S}$). However, Fowles[17] erroneously assumed that such configurations were stable for $L < -1$ and $L > 1 + 2M$, respectively, and did not consider at all the truly stable configurations a of Fig. 1, a and b of Fig. 4, and b of Fig. 1.

4)It can be shown that $L > 0$ [in particular, the second inequality in (14)] only if $(\partial p/\partial T)_v > 0$.

5)Adiabats containing an inflection point (Fig. 2a) were not considered in the linear theory. Nevertheless, since, in the linear analysis, the only important properties are the local properties of the shock adiabats, and the conditions $c_1 < D < c_2 + u_2$ are satisfied, the results of this analysis are valid for the adiabats shown in Fig. 2a everywhere except for the segment $L_1^- L_1^+$.

6)It can be shown that, whatever the dependence of the specific internal energy E on p and v, we have

$$L = -1 + (1 - M^2)[p + (\partial E/\partial v)_p]/[p + (\partial E/\partial v)_p - (p - p_1)/2].$$

Hence, it is clear that the parameters L and M may be regarded as independent (when varying the thermodynamic properties of the medium, the initial conditions ahead of the wave, and its intensity) only for $M \neq 1$.

7)have in mind here an arbitrary range within which front 3 can be looked upon as rectilinear with sufficient precision.

8)There is a great variety of shock discontinuities in magnetohydrodynamics; see the so-called evolutionary and nonevolutionary discontinuities.[25]

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Translated by S. Chomet