Negative differential resistance and current stratification in metallic plates

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A dropping segment of the static current-voltage characteristic (CVC) is predicted for a thin metallic plate carrying a strong dc current. The nonlinearity mechanism is magnetodynamic and is due to the influence of the current's own magnetic field on the dynamics of the conduction electrons. A unified analytic description of the nonlinear CVC in the entire region of the onset of this nonlinearity is obtained on the basis of an asymptotically accurate solution of the magnetostatics problem. The change of the CVC shape as a function of an external (constant and uniform) magnetic field parallel to the current is investigated and the critical values of the fields at which negative differential resistance (NDR) sets in are obtained. It is proved that a current distribution uniform over the plate width is unstable in the NDR region. Stratification of the current (static domain wall) and hysteresis of the CVC are predicted. The necessary numerical estimates are made.

1. INTRODUCTION

A nonlinear current-voltage characteristic (CVC) of a thin metallic plate was investigated by us theoretically in Refs. 1 and 2. The nonlinearity is due to a magnetodynamic mechanism connected with the influence on the dynamics of the conduction electrons exerted by the magnetic field produced by the current I. A special role is played here by a group of so-called "trapped" electrons that move, without colliding with the boundaries, along trajectories that meander near the plane where the magnetic field reverses sign. The features of the nonlinear CVC are determined by competition between the contributions made to the current by the untrapped and trapped electrons. In an external magnetic field parallel to the plane of the plate and perpendicular to the current direction, the voltage drop V is a monotonic function of the current I. The sample resistance decreases then with increase of the current. This phenomenon was observed experimentally in single-crystal wires of pure zinc[3] and gallium.[4]

Interest attaches to more nontrivial manifestations of the magnetodynamic nonlinearity, particularly to the possible onset of negative differential resistance (NDR). We demonstrate here this possibility, using as an example a plate in an external magnetic field parallel to the current direction. We show that at such a longitudinal orientation of the external field the CVC is an S-shaped curve.

We advance simple physical considerations that explain the onset of the NDR in the situation considered. We have a metallic film of thickness $d \ll l$ (l is the electron mean free path) through which a current I flows (Fig. 1). The magnetic field $\mathbf{H}(x)$ produced by the current and directed along the z axis is equal to zero at the middle of the plate $x = 0$, and has on the opposing faces the equal and opposite values H and $-H$:

$$H = 2\pi I / cD. \tag{1}$$

Here D is the horizontal dimension (width of plate) in a direction perpendicular to the current, and c is the speed of light. The external constant and uniform magnetic field $\mathbf{h}_0$ is directed along the y axis.

Let us estimate the conductivity of the electrons trapped by the combined nonuniform magnetic field $\mathbf{H}(x) + \mathbf{h}_0$, under conditions when the following inequalities hold:

$$d \ll (R_0 d)^{1/2} \ll l. \tag{1.2}$$

Here R_0 is the Larmor radius in the field h_0. We choose a current whose magnetic field H is much less than h_0. The necessary condition for electron trapping is the vanishing of the x component of the Lorentz force at some point inside the plate, i.e.,

$$[\mathbf{v} \times \mathbf{H}(x) + \mathbf{h}_0]_x = 0. \tag{1.3}$$

From this we easily obtain the characteristic value of the z-projection of the trapped-electron velocity. Assuming v_y to be of the same order of magnitude as the Fermi velocity, v_F, we get

$$v_z / v_F \sim H / h_0 \ll 1. \tag{1.4}$$

The x component of the trapped-electron velocity is estimated from the equation

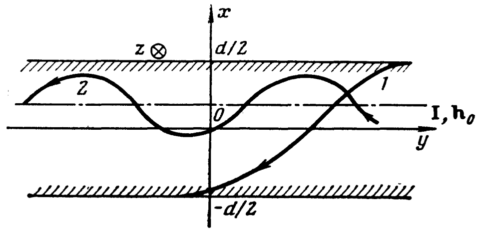

FIG. 1. Coordinate system; trajectories of untrapped (1) and trapped (2) electrons in a metal plate.

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$$. v_x/v_F \sim (d/R)^{1/2} \ll 1, \tag{1.5}$$

where $R \propto H^{-1}$ is the trajectory curvature radius in the field H. According to (1.4) and (1.5), the relative number of trapped electrons is given by

$$\Delta N/N \sim (H/h_0)(d/R)^{1/2}. \tag{1.6}$$

Since the trapped electrons do not collide with the sample boundaries during the entire free-path time, their conductivity can be described by the formula

$$\sigma_{\text{захв}} \sim \sigma_0 (d/R_0)^{1/2} (H/h_0)^{1/2} \propto I^{3/2}, \tag{1.7}$$

σ_0 is the static conductivity of a bulky sample. Obviously, if the mean free path l is long enough the electric field in a sample with diffuse boundaries is completely determined by the conductivity (2.7), so that the voltage drop V is inversely proportional to the square root of the current:

$$V \propto I^{-1/2}. \tag{1.8}$$

In other words, the CVC should have a segment with a negative differential conductivity.

In the second section of this paper we study the dynamics of electrons in the combined magnetic field and present asymptotically accurate expressions, obtained by solving the Boltzmann kinetic equation, for the currents of the untrapped and trapped electrons. In the third section is derived an equation that describes the nonlinear CVC in a wide range of the current I and of the external field h_0; the onset of the S-shaped CVC with increasing h_0 and the change of its shape are demonstrated. In the fourth section we investigate the instability of a current uniformly distributed over the plate width on the dropping segment of the CVC; the inhomogeneous current distributions due to the development of this instability are considered. The corresponding effects (current shutoff, hysteresis) are similar to those observed in semiconductors with S-shaped CVC. The singularities of the CVC under conditions of current stratification are also discussed, numerical estimates are presented, and certain questions that need further study are raised.

2. ELECTRON DYNAMICS. CURRENT DENSITY

Let us study the character of the electron motion in the combined nonuniform magnetic field $\mathbf{H}(x) + \mathbf{h}_0$, which is assumed to be relatively weak:

$$d \ll R_0,\ R. \tag{2.1}$$

We introduce the vector potential

$$\mathbf{A}(x) = \{0, A_y(x), A_z(x)\};$$
$$A_y(x) = \int_{d/2}^{x} dx' H(x'), \qquad A_z(x) = -h_0 x. \tag{2.2}$$

Note that $A_y(x)$ is a positive even function of x inside the plate $[H(d/2) = -H < 0]$ and vanishes on its boundaries, $A_y(d/2) = -A_y(-d/2) = 0$.

The integrals of the electron motion are the total energy (equal to the Fermi energy ε_F) and the generalized momenta

$$p_y = mv_y - \frac{e}{c} A_y(x),$$
$$p_z = mv_z - \frac{e}{c} A_z(x), \tag{2.3}$$

m is the electron mass and e the absolute value of its charge. For simplicity, we assume the electron dispersion law to be quadratic and isotropic. Under conditions (2.1), the second terms of (2.3) are small compared with the characteristic values mv_y and mv_z. The electron moves therefore along the x axis with an energy $\varepsilon_x = \varepsilon_F - (p_y^2 + p_z^2)/2m$ in an effective potential field.

$$U(x) = \frac{e}{mc} [p_y A_y(x) + p_z A_z(x)]. \tag{2.4}$$

Figure 2 shows schematically the regions of electron motion in the (x, p_y) phase plane at $p_z > 0$ (Fig. 2a) and $p_z < 0$ (Fig. 2b). The upper and lower curves that determine the electron turning points ($v_z = 0$) inside the sample are given by the equation $\varepsilon_x = U(x)$ and are described by the equations

$$p_y = p_y^{\pm}(x) \equiv \pm p_y^0 - \frac{e}{c}\left[A_y(x) \mp \frac{p_z}{p_y^0} h_0 x\right], \tag{2.5}$$
$$p_y^0 = (p_F^2 - p_z^2)^{1/2}, \qquad p_F = (2m\varepsilon_F)^{1/2}.$$

The upper and lower signs correspond respectively to the upper and lower curves. The shaded regions in Fig. 2 correspond to the trapped electrons. At a fixed coordinate, the region of existence of trapped electrons in the (p_y, p_z) momentum space is bounded both in p_y and in p_z. Indeed, it can be seen from Fig. 2 that the integral of motion p_y is bounded by

$$p_y^-(x) < p_y < p_y^-(d/2), \qquad p_y^-(x) < p_y < p_y^-(-d/2) \tag{2.6}$$

at $p_z > 0$ and $p_z < 0$, respectively. It is easy to verify that the p_y variation intervals in (2.6) exist only under the condition

$$-\left[1 + \frac{h_0^2 (x+d/2)^2}{A_y^2(x)}\right]^{-1/2} < \frac{p_z}{p_F} < \left[1 + \frac{h_0^2 (x-d/2)^2}{A_y^2(x)}\right]^{-1/2}. \tag{2.7}$$

Note that Fig. 2 describes a situation wherein the effective potential $U(x)$ has a minimum inside the plate $(-d/$

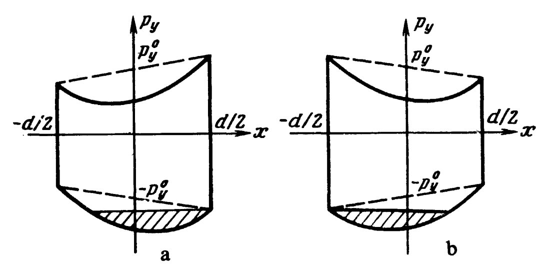

FIG. 2. The (x, p_y) phase plane and the regions where untrapped and trapped electrons exist in a nonuniform magnetic field; a) $p_z > 0$, b) $p_z < 0$. The shaded regions correspond to the trapped electrons.

$2<x<d/2$). It follows from inequalities (2.7) that the nonuniform magnetic field can trap only those electrons having

$$|p_z|<p_F[1+h_0^2/H^2]^{-1/2}. \quad (2.8)$$

At $H\ll h_0$ this result corresponds to Eq. (1.4) which was used in the Introduction to estimate the conductivity of the trapped electrons.

The moving untrapped electrons collide with the sample boundaries and occupy under conditions (2.1) the greater part of the (x, p_y) phase plane (see Fig. 2). The principal role in the formation of the conductivity is played by those carriers that move almost parallel to the plate surface. If

$$d\ll l\ll (R_+d)^{1/2}, \quad R_+=cp_F/e(H+h_0), \quad (2.9)$$

the influence of the magnetic field can be neglected, and the density of the untrapped-electron current in a plate with diffuse boundaries is given by the known formula[5]

$$j_{\text{untr}}=\frac{3}{4}\sigma_0E\frac{d}{l}\ln\frac{l}{d}. \quad (2.10)$$

In a sufficiently strong magnetic field, when

$$d\ll (R_+d)^{1/2}\ll l, \quad (2.11)$$

calculations perfectly analogous to those of Ref. 2 lead to the following asymptotic expression:

$$j_{\text{untr}}=\frac{3}{8}\sigma_0E\frac{d}{l}\ln\frac{R_+}{d}. \quad (2.12)$$

Expression (2.12) is obtained from (2.10) by replacing the mean free path l by the effective range $(R_+d)^{1/2}$ of the untrapped electrons. The current density (2.12) is independent of l and is determined by electron scattering from the sample boundaries.

To find the trapped-electron current we must solve the Boltzmann kinetic equation (linearized in terms of the electric field E) for the nonequilibrium increment δf to the fermi distribution function $f_F(\varepsilon)$. If the current I is strong enough and the inequalities

$$d\ll (Rd)^{1/2}\ll l, \quad (2.13)$$

are satisfied, we obtain

$$\delta f=-eEl\frac{p_y^0}{p_F}\frac{\partial f_F}{\partial \varepsilon}. \quad (2.14)$$

Expression (2.14) does not depend on the coordinate x in view of the uniformity of the electric field E and of the conditions (2.13). In addition, it does not contain the current's own field, since the kinetic-equation term with the Lorentz force, according to (2.13), is much smaller than the collision term $\nu\delta f$. The magnetodynamic-nonlinearity mechanism manifests itself in the density of the trapped-electron current in view of the integration over the phase-space region (2.6), (3.7), whose dimensions depend strongly on $H(x)\propto I$. Simple transformations yield

$$j_{\text{trap}}(x)=\frac{3\sigma_0}{\pi(2R_0)^{1/2}}E\left[\left(\frac{d}{2}-x\right)^{1/2}F\left(\frac{A_y(x)}{h_0(d/2-x)}\right)\right.$$

$$\left.+\left(\frac{d}{2}+x\right)^{1/2}F\left(\frac{A_y(x)}{h_0(d/2+x)}\right)\right]. \quad (2.15)$$

We have introduced here the function

$$F(a)=a^{3/2}\int_0^1\frac{dt\,(1-t)^{1/2}}{(1+a^2t^2)^{5/4}}, \quad (2.16)$$

which increases monotonically with increase of its argument and has simple asymptotic terms at small and large value of x. The argument a is of the order of the ratio of the current's own field to the external magnetic field, $a\sim H/h_0$. We obtain accordingly the following asymptotic equations for the trapped-electron current density:

$$j_{\text{trap}}(x)=\frac{4\sqrt{2}}{\pi}\frac{\sigma_0E}{1-(2x/d)^2}\left(\frac{d}{R_0}\right)^{1/2}\left(\frac{A_y(x)}{h_0d}\right)^{3/2} \quad (2.17)$$

at $H\ll h_0$, and

$$j_{\text{trap}}(x)=\frac{36\sqrt{\pi}}{5\Gamma^2(1/4)}\sigma_0E\left[\frac{e}{cp_F}A_y(x)\right]^{1/2} \quad (2.18)$$

at $h_0\ll H$. The current density (2.18) is independent of the external field h_0 and agrees with the density obtained in Ref. 2 at $h_0=0$.

3. NONLINEAR CVC. CONDITIONS FOR REALIZATION OF NDR

We now use the equations derived in the preceding section to describe the nonlinear CVC $V(I)$ [or, equivalently, $E(H)$] and to investigate its form as a function of h_0. Rigorously formulated, the problem is to solve the magnetostatics equation

$$A_y''(x)=-\frac{4\pi}{c}j(A_y(x),x) \quad (3.1)$$

with boundary conditions

$$A_y'(0)=0, \quad A_y'(\pm d/2)=\mp H. \quad (3.2)$$

In this manner, however, asymptotically exact results can be obtained only for individual segments of the CVC, where the current density is specified by one of the expressions (2.10), (2.12), (2.17), or (2.18). A unified analytic description of the CVC can be obtained by using a simple and physically clear model based on the use of the effective conductivity, which takes into account the untrapped as well as the trapped electrons.

According to (2.10) and (2.12), the conductivity of the untrapped electrons can be expressed by

$$\sigma_{\text{untr}}=\frac{3}{8}\sigma_0\frac{d}{l}\ln\left[\left(\frac{d}{l}\right)^2+\frac{d}{R_+}\right]^{-1}. \quad (3.3)$$

The conductivity of the trapped particles is determined by the following equations [see (2.17), (2.18)]:

$$\sigma_{\text{trap}}=\sigma_0\left(\frac{d}{R_0}\right)^{1/2}\frac{(\bar{A}/h_0d)^{3/2}}{1+\bar{A}/h_0d}. \quad (3.4)$$

we have left out here for simplicity the numerical coefficients contained in (3.17) and (2.18), and replaced the vector potential $A_y(x)$ by its characteristic value $\bar{A} = A_y(0)$. As a result we have for the total conductivity $\sigma = \sigma_{\text{untr}} + \sigma_{\text{trap}}$

$$\sigma(\bar{A}) = \frac{3}{8}\sigma_0 \frac{d}{l} \ln\left[\left(\frac{d}{l}\right)^2 + \frac{d}{R_+}\right]^{-1} + \sigma\left(\frac{d}{R_0}\right)^{1/2} \frac{(\bar{A}/h_0 d)^{3/2}}{1+\bar{A}/h_0 d} \tag{3.5}$$

Putting $\bar{A} = Hd$ and replacing the left-hand side of (3.1) by $-2H/d$, we obtain the CVC

$$E = cH/2\pi d\sigma(Hd). \tag{3.6}$$

Following Ref. 2, we introduce the natural scales, under the conditions of magnetodynamic nonlinearity, for the magnetic and electric fields:

$$h = \gamma \frac{cp_F d}{el^2} \quad \mathscr{E} = \frac{4chl}{3\pi\sigma_0 d^2} \quad \gamma = \left(\frac{3}{8}\right)^2. \tag{3.7}$$

In terms of these scales, the current-voltage characteristic takes the explicit form

$$\frac{E}{\mathscr{E}} = \frac{H}{h}\left\{\ln\left[\left(\frac{l}{d}\right)^2\left(1+\gamma\frac{H+h_0}{h}\right)^{-1}\right] + \left(\frac{H}{h}\right)^{1/2}\frac{h}{H+h_0}\right\}^{-1}. \tag{3.8}$$

Equation (3.8) is the sought interpolation formula that provides a unified analytic description of the nonlinear CVC. The agreement between the asymptotically exact calculation results of the Appendix with the various limiting formulas that follow from (3.8) corroborate the model proposed here.

We consider now the behavior of the $E(H)$ plots with change of h_0. So long as the external field is weak enough and does not exceed a certain critical value h_{cr}, i.e., $h_0 < h_{cr}$, the function $E(H)$ increases monotonically. At $h = h_{cr}$ an inflection point with a horizontal tangent appears on the $E(H)$ curve. The "critical" magnetic field is determined from the equation

$$\frac{h_{cr}}{h} = 625\left\{\ln\left[\left(\frac{l}{d}\right)^2 \frac{h}{h_{cr}}\right]\right\}^2. \tag{3.9}$$

The coordinates of the critical point in the (H, E) plane are

$$\frac{H_{cr}}{h} = 0{,}464\frac{h_{cr}}{h}, \quad \frac{E_{cr}}{\mathscr{E}} = 1{,}92\left(\frac{h_{cr}}{h}\right)^{1/2}. \tag{3.10}$$

When $h_0 > h_{cr}$ the CVC are S-shaped curves (in the coordinates V and I). The dropping segment of the CVC is quite pronounced at $h_0 \gg h_{cr}$. In this range of the parameter h_0 the coordinates of the extremal points of the function $E(H)$ can be obtained explicitly:

$$\frac{H_{max}}{h} = \left(2\ln\frac{R_0}{d}\right)^{2/3}\left(\frac{h_0}{h}\right)^{2/3},$$

$$\frac{E_{max}}{\mathscr{E}} = \frac{2}{3}\left(2\ln\frac{R_0}{d}\right)^{-1/3}\left(\frac{h_0}{h}\right)^{2/3}, \tag{3.11}$$

$$\frac{H_{min}}{h} = \frac{h_0}{h}, \quad \frac{E_{min}}{\mathscr{E}} = 2\left(\frac{h_0}{h}\right)^{1/2}.$$

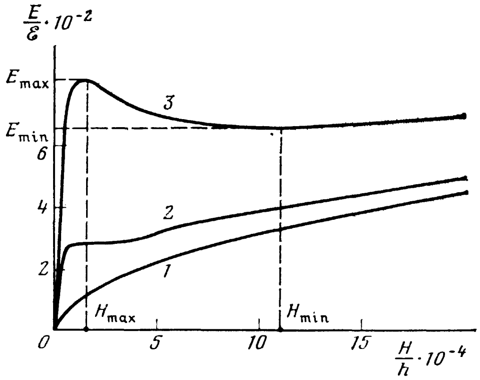

FIG. 3. Current-voltage characteristic (dependence of E on H) of a metal plate at various values of the external magnetic field h_0: curves 1, 2, and 3 correspond respectively to $h_0 = 0$, $h_0 = h_{cr}$, and $h_0 = 5h_{cr}$. The notation is explained in the text.

Figure 3 shows the CVC calculated from Eq. (3.8) for $h_0 = 0$, $h_0 = h_{cr}$, and $h_0 = 5h_{cr}$, with $l/d = 3 \cdot 10^3$. At this ratio of l and d a numerical solution of Eq. (3.9) leads to the following values of the critical quantities:

$$h_{cr} = 2{,}3\cdot 10^4 h, \quad H_{cr} = 10^4 h, \quad E_{cr} = 2{,}9\cdot 10^2 \mathscr{E}. \tag{3.12}$$

4. INSTABILITY OF DROPPING SEGMENT OF THE CVC. STRATIFICATION OF CURRENT

The CVC investigated in the preceding section was obtained assuming the current to be uniformly distributed over the width of the plate (along the z axis). On the basis of the analogy with the theory of semiconductors having S-shaped CVC (see, e.g., Ref. 6), it is natural to raise the questions of the stability of such a current distribution to perturbations that depend on the coordinate z.

Nonuniformity of the current along the z axis means the appearance of an x component $H_x(x, z)$ of the magnetic field. Then $H_z \equiv H(x, z)$ also begins to depend on z (since $\operatorname{div} \mathbf{H} = 0$). If the characteristic scale of this nonuniformity $\Delta z \gtrsim d$ (so that $H_x \lesssim H_z$), the effective conductivity of the metal is determined as before by Eq. (3.5). It must be borne in mind, however, that now the y component of the vector potential also depends on z. We introduce the dimensionless vector potential

$$a = \bar{A}/h_0 d, \tag{4.1}$$

for which the magnetostatics equation can be written in the form

$$\frac{d^2 a}{dz^2} = \frac{2a}{d^2} - \frac{4\pi E}{ch_0 d}\sigma(a). \tag{4.2}$$

The homogeneous (indepedent of z) solution of this equation

$$E = \frac{ch_0}{2\pi d}\frac{a}{\sigma(a)} \tag{4.3}$$

specifies the CVC (3.6). Let us investigate its stability to small perturbations of the form

$$\delta a, \ \delta E \propto \exp(ikz - i\omega t). \tag{4.4}$$

Recognizing that

$$\delta E=-\frac{1}{c}\frac{\partial A_y}{\partial t}=i\omega\,\frac{h_0 d}{c}\,\delta a, \tag{4.5}$$

we obtain from (4.2) the following dispersion equation:

$$\omega=-i\frac{c}{h_0 d}\left[\frac{E}{2a}(kd)^2+\frac{\partial E}{\partial a}\right]. \tag{4.6}$$

It is obvious from (4.6) that on the dropping segment of the CVC, where the derivative $\partial E/\partial a$ is negative, the uniform distribution is unstable to long-wave perturbations. Assuming $|\partial E/\partial a|\sim E/a$, we find that the growing perturbations are those with wave numbers $k<k_0\approx d^{-1}$.

The instability established here is absolute and aperiodic. It is natural therefore to expect the new system state that results from the development of this instability to be static. We must therefore consider inhomogeneous solutions for Eq. (4), which we now find it convenient to write in the form

$$\frac{\partial^2 a}{\partial z^2}+\frac{\partial W}{\partial a}=0,\qquad W=\int_0^a da'\left[\frac{4\pi E}{ch_0 d}\sigma(a')-\frac{2a'}{d^2}\right]. \tag{4.7}$$

We shall analyze Eq. (4.7) with the aid of a method widely used to investigate, e.g., superheat nonlinearity in semiconductors.[6] Equation (4.7) coincides with the equation of motion of a particle in a field with a potential $W(a)$. In this case z plays the role of the time, and a that of the coordinate. The potential $W(a)$ depends on the electric field E as an external parameter. In the interval

$$E_{min}<E<E_{max}, \tag{4.8}$$

where a dropping segment of the CVC exists (see curve 3 of Fig. 3), the derivative $\partial W/\partial a$, i.e., the force acting on the particle, is zero at three points $a_1(E)<a_2(E)<a_3(E)$. These points are obtained from Eq. (4.3) for a homogeneous CVC. Analysis shows that the function $W(a)$ has maxima at the points a_1 and a_3 and a minimum at a_2. It can be easily seen from this that in the interval (4.8) there exist bounded solutions $a(z)$ that are oscillating functions of the "time" z and correspond to motion of the particle in a potential well $W(a)$. It is very important that at some value of the parameter $E=E_0$ the maxima $W(a_1)$ and $W_3(a)$ are equal, and a unique (existing only in this situation) monotonic solution $a_0(z)$ of Eq. (4.7) appears. Without writing the explicit form of $a_0(z)$, we mentioned that as z varies from $-\infty$ to $+\infty$ the function $a_0(z)$ varies monotonically in the range

$$a_1(E_0)<a_0(z)<a_3(E_0). \tag{4.9}$$

This solution decribes the "coordinate" of a particle that executes an infinitely "slow" aperiodic motion with a total energy equal to $W(a_1)=W(a_3)$.

For the vector potential

$$\bar{A}(z)=h_0 d a_0(z) \tag{4.10}$$

the existence of such an inhomogeneous solution means stratification of the current and the appearance of a kind ("domain wall") in the plate. On one side of the kink $A(z)$ reaches the value $h_0 d a_1(E_0)$, and on the other $h_0 d a_3(E_0)$. The characteristic interval Δz over which the function $a_0(z)$ varies, i.e., the width of the kink, is of the order of the thickness of the plate d. The electric field E_0 at which the current stratification takes place is obtained from the equation

$$\int_{a_1(E_0)}^{a_3(E_0)} da\left[\frac{4\pi E_0}{ch_0 d}\sigma(a)-\frac{2a}{d^2}\right]=0. \tag{4.11}$$

Equation (4.11) has a unique solution in the interval (4.8). In an external magnetic field $h_0\gg h_{cr}$ we have

$$\frac{E_0}{\mathscr{E}}=2{,}1\left(\frac{h_0}{h}\right)^{1/2},\qquad a_1(E_0)=2{,}1\left(\frac{h_0}{h}\right)^{-1/2}\ln\frac{R_0}{d},$$

$$a_3(E_0)=1{,}70. \tag{4.12}$$

We emphasize that the electric field E_0 is close to E_{min} [see (3.11)], and the current densities on the two sides of the kink differ by a factor $a_3/a_1\gg 1$.

The special role of the monotonic solution $a_0(z)$ becomes clear in investigations of the stability of inhomogeneous current distributions to perturbations of the type

$$\delta a(z,t)=\delta a(z)\exp(-\lambda t). \tag{4.13}$$

According to (4.7) and (4.5) we obtain for $\delta a(z)$

$$\hat{\Lambda}\delta a\equiv-\frac{c^2}{4\pi\sigma(a)}\left(\frac{d^2}{dz^2}+\frac{\partial^2 W}{\partial a^2}\right)\delta a=\lambda\delta a. \tag{4.14}$$

It follows hence that for the solution $a(z)$ of Eq. (4.7) to be stable it is necessary that all the eigenvalues λ of the operator $\hat{\Lambda}$ be non-negative. By direct differentiation of (4.7) with respect to z it is easy to verify that the function $\delta a(z)=da/dz$ is an eigenfunction of the operator $\hat{\Lambda}$ with an eigenvalue $\lambda=0$. If $a(z)$ is a monotonic function $[a=a_0(z)]$, then da/dz has no zeros, i.e., it corresponds to the "ground state" of the problem (4.14). In this case $\lambda=0$ is the minimum eigenvalue of the operator $\hat{\Lambda}$. Consequently, the monotonic solution $a_0(z)$ of Eq. (4.7) is stable to the perturbation (4.13) If, however, da/dz has zeros then, by virtue of the oscillation theorem, $\lambda=0$ is not the minimum eigenvalue of the problem (4.14). Nonmonotonic (oscillating) solutions $a(z)$ are therefore unstable.

Figure 4 shows schematically the CVC at $h_0>h_{cr}$, when

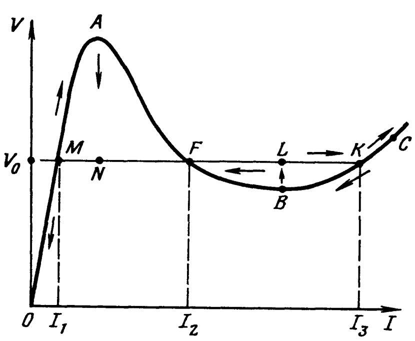

FIG. 4

the current should become stratified over the width of the sample. In the given-current regime the voltage $V \propto E$ first follows the ohmic law (2.12), increasing to the value at the point A. This is followed (with further increase of the current I) by a discontinuity of the voltage to a value $V_0 \propto E_0$ corresponding to inhomogeneous distribution of the current with respect to the coordinate z. This inhomogeneous current states exists right up to $I_3 = a_3 h_0 Dc/2\pi$, while the CVC is the horizontal segment NK. With further increase of the current, at $I > I_3$, the voltage varies along the section KC. In the reverse direction, the CVC follows the "route" $CKBLFNMO$; a jumplike increase of V takes place at the point B, and the distribution becomes homogeneous at the point M for $I = I_1 = a_1 h_0 Dc/2\pi$.

On the horizontal branch of the CVC, the widths $D_1 = D - D_3$ and D_3 of the regions through which currents $I_1 D_1/D$ and $I_3 D_3/D$ flow respectively are governed by the total current I:

$$D_3/D = (I - I_1)/(I_3 - I_1) \approx (I - I_1)/I_3. \tag{4.15}$$

The growth of the total current I is accompanied by expansion of the region with the large current. The plate's lateral face at which the kink appears is determined by extraneous factors that make the two faces nonequivalent.

We present numerical estimates for a plate of thickness $= 10^{-1}$ cm having an electron bulk mean free path $l = 0.3$ cm and a Fermi momentum $p_F = 10^{-19}$ g · cm/s. The characteristic scales of h and $\mathscr{E}$ are, according to (3.7) and (A.12),

$$h = 5 \cdot 10^{-3} \text{ Oe} \quad \mathscr{E} = 3 \cdot 10^{-5} \text{ V/cm}. \tag{4.16}$$

The parameters of the critical point (3.12) are

$$h_{cr} = 110 \text{ Oe} \quad H_{cr} = 50 \text{ Oe} \quad E_{cr} = 9 \cdot 10^{-3} \text{ V/cm}.$$

These values correspond to a current density $j_{cr} = 8 \cdot 10^5$ A/cm^2, while the specific power release per unit surface is 0.7 W/cm^2. The characteristic CVC "switching" time from one branch to the other is of the order of the reciprocal growth rate (4.6) and turns out to be about 10^{-8} s.

We note that from the viewpoint of the thermodynamics of irreversible processes the current stratification in a conductor is the simplest example of the so-called dissipative structures, which have been attracting interest of late in various branches of physics (see, e.g., Ref. 7).

We point out in conclusion that it is of interest to analyze the stabilities of segments MA and BK on the CVC to finite perturbations. Another no less important study is that of magnetodynamic nonlinearity in thin-wall hollow cylinders. The latter case corresponds to "short-circuiting" the lateral faces of the plate, i.e., to boundary conditions periodic in z for Eq. (4.7). Since (4.7) has no stable static and periodic solution, a moving current domain (soliton) whose velocity is regulated by the total current should appear in place of the static kink.

APPENDIX

We present the results of an asymptotically exact solution of the magnetostatics problem (3.1), 3.1) for individual CVC segments on which the current density is specified by one of the expressions (2.10), (2.12), (2.17), and (2.18).

We begin with the case when the metal conductivity is determined by the group of untrapped electrons. In a weaks combined magnetic field

$$H + h_0 \ll h \tag{A.1}$$

the inequalities (2.9) are satisfied and the current density takes the form (2.10). Owing to the spatial homogeneity of the current, the magnetic field $H(x)$ it produces is linear in the coordinate x and Eq. (3.1) with boundary conditions (3.2) can be easily solved. The corresponding ohmic segment of the CVC is described by the expression

$$\frac{E}{\mathscr{E}} = \frac{H}{h} \ln^{-1}\left(\frac{l}{d}\right)^2. \tag{A.2}$$

It is just as easy to find the CVC in the region

$$\frac{H + h_0}{h} \gg 1, \quad \left(\frac{H}{h}\right)^{1/2} \ln^{-1}\left(\frac{R_+}{d}\right), \tag{A.3}$$

where the current density of the untrapped electrons is determined by (2.12)

$$\frac{E}{\mathscr{E}} = \frac{H}{h} \ln^{-1}\left(\frac{R_+}{d}\right). \tag{A.4}$$

We consider now the more complicated situation when the current density is due to trapped electrons, and the conductivity of the untrapped ones can be neglected. If the inequality

$$H \gg h_0, \; h \ln^2\left(\frac{R}{d}\right) \tag{A.5}$$

holds, the current density takes the form (2.18). The magnetostatics problem (3.1), (3.2) with a current (2.18) was solved by us in Ref. 2. The current-voltage characteristic is here parabolic:

$$E/\mathscr{E} = (H/h)^{1/2}. \tag{A.6}$$

A special study is required for the case

$$\left(\frac{h_0}{h} \ln \frac{R_0}{d}\right)^{2/3} \ll \frac{H}{h} \ll \frac{h_0}{h}, \tag{A.7}$$

which corresponds to the dropping section of the CVC. The trapped-electrons current density is given in this region by (2.17). We introduce the dimensionless vector potential $f(\xi)$ and the coordinate ξ:

$$f(\xi) = 32 \frac{\sigma_0^2 d^2 E^2}{R_0 h_0^3 c^2} A_y \left(\xi \frac{d}{2}\right), \quad \xi = \frac{2x}{d}. \tag{A.8}$$

In terms of these variables, Eq. (3.1) with current density (2.17) takes the form

$$f''(\xi) + f^{3/2}/(1 - \xi^2) = 0. \tag{A.9}$$

The prime denotes the derivative with respect to ξ. Equation (A.9) must be solved in the region $0 \leqslant \xi \leqslant 1$ with boundary conditions

$$f'(0)=0, \quad f(1)=0. \tag{A.10}$$

It can be seen from (A.8) that to find the CVC we need calculate only the quantity $f'(1)$ that determines the numerical factor C in the expression

$$\frac{E}{\mathscr{E}} = C\frac{h_0}{h}\left(\frac{H}{h}\right)^{-1/2}. \tag{A.11}$$

In this case

$$C = \frac{3\pi}{16\gamma^{1/2}}\,|f'(1)|^{1/2}.$$

Numerical calculations yield for the sought constants the values $f'(1) = -10$ and $C = 2$.

It is easy to verify that in all the segments considered the CVC (3.8) obtained in the effective-conductivities model agrees with the results of the asymptotically exact calculations. The only difference is that in (3.8) it is assumed that $C = 1$. In addition, the numerical coefficient γ that determines the scale of the magnetic field [see (3.7)] turns out to be

$$\gamma = \frac{25\pi^{1/2}\Gamma^4(5/4)\Gamma^3(5/3)}{32\Gamma^3(7/6)} \approx 0.86. \tag{A.12}$$

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