Propagation of heat pulses generating quantized vortices in He II

S. K. Nemirovskiĭ

"Energokhimmash" Special Design Bureau

(Submitted 1 April 1986)
Zh. Eksp. Teor. Fiz. **91**, 1363–1375 (October 1986)

The problem of the propagation of intense heat pulses, generating quantized vortices and interacting with these "proper" vortices, is solved. The limiting cases of short and long pulses are considered. A relation is obtained for the velocity of propagation of short pulses, and the evolution of their shape is described. It is demonstrated that a vapor film can form near the surface of the heater, and the time of formation of this film is estimated. In the case of long pulses of low intensity the dynamics of the temperature field and normal velocity when the heat flux is switched on in a stepwise manner is investigated. The results are compared with data from a number of papers and it is shown that there is good qualitative and quantitative agreement with the experimental results.

1. INTRODUCTION AND BASIC EQUATIONS

In the flow of helium in capillaries the critical flow velocities at which quantized vortex lines appear are of the order of fractions of a centimeter per second. At the same time, in experiments to investigate nonlinear second-sound pulses, local values of the relative velocity $v_n - v_s$ reach magnitudes of the order of 1–2 m/sec (see, e.g., Refs. 1–3), while in Ref. 2 it was shown that the evolution of such pulses is well described by the Burgers equation, which follows directly from the ordinary (vortex-free) hydrodynamics of He II. It appears to us that the reason for the contradiction lies in the fact that, in the cited experiments, the wave pulses were too short to allow time for the development in the helium of a vortex structure capable of having an appreciable influence on the acoustic characteristics of the system. This point of view has been confirmed to a certain extent in Ref. 4, in which it was shown that in the wake of an intense and sufficiently long heat pulse (to be called henceforth the reference pulse) one observes strong damping of the second-sound probe wave as a result of friction with the vortex lines that are generated by the reference pulse. An analogous conclusion about the role of the pulse length was also reached by the authors of Refs. 5 and 6, who investigated the propagation of paired heat pulses (with the same input parameters) and observed strong attenuation of the trailing pulse as a result of damping by vortices generated by the leading signal.

Recently, a number of papers on the evolution of very intense (and long) heat pulses in He II have appeared. The authors of these papers point out the considerable discrepancies between the effects they observe and the predictions of the nonlinear acoustics of superfluid helium. Among such discrepancies are, e.g., the fact that the asymptotic shape of the signal differs from the Burgers triangle,[2,5–7] the quantitative disagreement of the data on the nonlinear sound velocity,[2,8] and a number of others. In contradiction with the classical acoustics of He II is the fact, noted by several authors, of the boiling up of helium and the formation of a vapor film near the surface of the heater.[9,10] In fact, from the formulas of acoustics (see, e.g., Ref. 11), even in intense pulses of 50 W/cm^2, the amplitude of the temperature disturbance in them does not exceed 0.05 K, which is frequently not enough even to reach the He-II-vapor equilibrium curve (in p-T coordinates).[1)] At the same time, it is known that considerable superheating, by up to 0.4 K, is required for the formation of a vapor film in He II.[12,13] It appears that the discrepancies described are due to the circumstance that quantized vortices capable of altering the laws of the dynamics of such pulses to a considerable degree are created in intense and long wave pulses.

In the present paper we give an account of the solution of the problem of the evolution of heat pulses generating quantized vortex lines and interacting with these "proper" vortices. The investigation is carried out in the framework of the hydrodynamics of He II, in which randomly oriented vortex lines, or a vortex tangle, appear and develop. Following Feynman,[14] we shall call such a state of He II superfluid trubulence. The equations of the hydrodynamics of superfluid turbulence were obtained earlier by the author and Lebedev in Ref. 15.

In the case of second sound propagating along the x axis, the equations of motion of turbulent He II reduce to the following system of relations for the quantities $v(x,t)$—the dimensionless velocity of the normal component, $\theta(x,t)$—the dimensionless temperature disturbance, and $L(x,t)$—the dimensionless density of the vortex tangle (the total length of lines per unit volume)[15]:

$$\frac{\partial \Theta}{\partial t}+\frac{\partial}{\partial x}[v+A_{11}v\Theta]=A_{12}Lv^2+A_{13}L^2, \quad (1)$$

$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}[\Theta+A_{21}\Theta^2+A_{22}v^2]=-A_{23}Lv-A_{24}L^{3/2}, \quad (2)$$

$$\frac{\partial L}{\partial t}=A_{31}(vL^{3/2}-L^2)+A_{32}v^{5/2}. \quad (3)$$

The third relation is the well known Vinen-Schwarz (VS) equation (see Refs. 15, 16, and 17). It is the balance relation between the growth of the tangle on account of the Magnus force [the first term in the parentheses in (3)] and the decrease of L as a consequence of the intersection and

 0038-5646/86/100803-08$04.00

breaking down of the vortex lines (the second term). The tangle draws the energy to increase its length from the main flux, and, as a result of breaking down of vortex rings, returns it to the system in the form of heat. The second relation is the equation for the velocity of the normal component. Without the right-hand side, it coincides with the equations of the nonlinear acoustics of He II (compare with Ref. 18). The right-hand side contains terms connected with the friction with the "frozen" system of vortices and with slowing down on account of transfer of part of the energy of the growing tangle. The relation (1) is, in essence, the entropy-transport equation. The expression on the right-hand side is the dissipative function associated with the work of the forces of friction and with the energy liberated in the process of breaking down of the vortex rings.

The quantities in Eqs. (1)-(3) were made dimensionless by means of the following relations (the index p marks the dimensional quantities, and the basic notation corresponds to Refs. 11 and 15):

$$v_n{}^p=v_{n0}v,\quad x^p=c_2t_sx,\quad \delta T^p=(c_2\rho_sv_{n0}/\sigma\rho_n)\Theta,\quad t^p=t_st,$$
$$\mathscr{L}^p=(\alpha\rho v_{n0}/\beta\rho_s)^2L. \tag{4}$$

Here v_{n0} (cm/sec) and t_s (sec) are the characteristic velocity of the normal component and the duration, respectively, in the initial second-sound pulse. The quantities are made dimensionless in such a way that the dimensionless amplitudes of the velocity and temperature in the intial pulse, and also the velocity of second sound and the equilibrium density of the tangle (see below) are equal to unity. The coefficients A in the relations (1)-(3) are functions of the temperature, and also of the quantities v_{n0} and t_s. For illustration, we shall write out the values of the coefficients A calculated for the temperature $T=1.75$ K. In the calculation the data of Refs. 16 and 17 were used:

$$A_{11}=2\cdot10^{-4}v_{n0},\quad A_{12}=4.7\cdot10^{-4}v_{n0}{}^3t_s,\quad A_{13}=6\cdot10^{-5}v_{n0}{}^3t_s,$$
$$A_{21}=8.5\cdot10^{-5}v_{n0},\quad A_{22}=4\cdot10^{-4}v_{n0},\quad A_{23}=20.1v_{n0}{}^2t_s, \tag{5}$$
$$A_{24}=2.3\cdot v_{n0}{}^2t_s,\quad A_{31}=15.6v_{n0}{}^2t_s,\quad A_{32}=0.37v_{n0}{}^{1/2}t_s.$$

In contrast to Ref. 15, in the right-hand side of Eq. (3) we have introduced a small term $A_{32}v^{5/2}$, corresponding to the spontaneous formation of vortices in the helium flow. In this form, this source term was proposed by Vinen.[16] Without the source term, as is easily seen by integration of Eq. (3) (for a certain fixed v, switched on at time $t=0$), the time t_v of development of the vortex tangle to half its equilibrium value ($L_{eq}=v^2$) becomes infinite. In fact, in accordance with Eq. (3) the time of development of a tangle in a stationary flux of intensity W is equal to (for $A_{32}v^{5/2}=0$)

$$\tau_v=\int_0^{v^2/2}\frac{dL}{A_{31}(vL^{3/2}-L^2)}, \tag{6}$$

and it can be seen that the integral diverges at the lower limit $L\rightarrow0$. The origin of this discrepancy is connected with the fact that the VS equation (3) (without the source term) is the balance equation between the growth and disappearance of vortex lines. The mechanism of spontaneous appearance of vortices in the helium flow has not been built into this equation. Naturally, Vinen actually observed finite times of development of a vortex tangle. Analyzing his experiments on the probing of the critical counterflow by second sound, he obtained the following empirical relation for the quantity $\tau_v(W)$ (here W is the intensity of the heat flux switched on at time $t=0$):

$$\tau_v(W)=a(T)W^{-3/2}. \tag{7}$$

Here $a(T)$ is a quantity that depends on the temperature and (weakly) on the geometry, and has a characteristic value of the order of $a(T)\cong0.05$ sec·cm^3/W$^{3/2}$. It is not difficult to verify that the small source term $A_{32}v^{5/2}$ in Eq. (3) leads [upon calculation of the integral of the type (6)] to formula (7).

The form of the source term is explained by Vinen in the following way.[16] If we assume that the initial appearance of vortex nuclei is such that $\mathscr{L}_{nuc}\propto|v_n-v_s|$, and assume further that the vortex nuclei that appear develop in accordance with the VS equation (3), we can see that the term $(\partial\mathscr{L}/\partial t)_{nuc}$, having the meaning of the source term, is related as follows to the velocity of the normal component:

$$(\partial\mathscr{L}/\partial t)_{nuc}\propto|\mathbf{v}_n-\mathbf{v}_s|\mathscr{L}_{nuc}^{3/2}\propto|\mathbf{v}_n-\mathbf{v}_s|^{5/2}\propto v_n^{5/2}. \tag{8}$$

The latter relation follows from the fact that, in second sound, $\rho_nv_n+\rho_s\mathbf{v}_s=0$. As regards the coefficient of the quantity $v^{5/2}$, here there are no theoretical arguments available, and the quantity A_{32} can be determined only experimentally. Vinen obtained his results (we have in mind the dependence (7)) for heat fluxes W of magnitude not exceeding 1 W/cm^2, and, generally speaking, we have no grounds to carry these results over into the region of the heat fluxes (up to tens of watts per square centimeter) with which nonlinear acoustics is concerned. On the other hand, in Ref. 4 there are certain quantitative results that make it possible to determine the development of a vortex tangle, and, consequently, to calculate the coefficient A_{32}.

In Ref. 4 a second-sound signal with large ($\sim$40W/cm^2) amplitude (i.e., a nonlinear wave) was used to probe the wake of the reference pulse. It is well known that the velocity of a nonlinear wave depends on its amplitude, and the latter, because of itneraction with the vortices, decreases during propagation of the wave through a vortex tangle. Thus, there is the possibility of relating the time of flight t_f of the probing pulse to characteristics of the superfluid turbulence. Corresponding calculational formulas were obtained in Ref. 15, and experimental results for t_f, for different intensities and durations of the reference pulse, are given in Ref. 4 in the form of tables. Calculations performed in accordance with the above ideas lead to the value of A_{32} given in the relations (5). We note immediately that for a number of reasons, associated both with experimental complexities and with difficulties arising in the theoretical interpretation of specific experiments, the coefficient A_{32} has been calculated rather roughly—in practice, in order of magnitude. Henceforth, therefore, we shall regard the quantity A_{32} as a parameter of the theory, a provisional value of which is given in the relations (5).

In the system of equations (1)-(3) we have omitted small (in the conditions of the cited experiments) terms associated with the contribution of vortices to the fluxes of energy and entropy, to the momentum-flux tensor, to the chemical potential, etc. (see Ref. 15). In addition, in the left-hand side of the VS equation (3) we have omitted the term div $(\mathbf{v}_L L)$ describing the drift of the tangle. The point is that the drift velocity $\mathbf{v}_L$ is connected with the velocity $\mathbf{v}_n$ of the normal component by the relation $\mathbf{v}_L = b(T)\mathbf{v}_n$, and the quantity $b(T)$ is small in the temperature range 1.8-2.1 K in which most of the experiments have been performed (see, e.g., Ref. 17).

The equations (1)–(3), as can be seen, are very cumbersome, and in the general case their complete solution (for certain boundary and initial conditions) can be obtained only numerically. It has turned out to be possible, however, to perform an analytical investigation in the case of very short and very long pulses. By short pulses we mean heat pulses of intensity W whose duration t_s is much shorter than the quantity τ_v. The evolution of pulses of short duration will be described in the next section of the article. The opposite limiting case, of long pulses with $\tau_v \ll t_s$, henceforth called the adiabatic case, will be considered in Sec. 3 of the article.

2. PROPAGATION OF SHORT PULSES

In pulses of short duration the growth of the tangle is determined by the source term in the VS equation (3), which, in the present case, has an obvious solution:

$$L(x,t)=A_{32}\int_{-\infty}^{t} v^{5/2}(t')\,dt', \tag{9}$$

i.e., the relation between the value of the velocity $v(x,t)$ and the tangle density $L(x,t)$ is nonlocal in time, and the length of the vortex lines at a certain point x increases as the pulse passes through. By means of (9), we can eliminate the quantity L from Eqs. (1)–(2) and thus obtain a closed system of equations for $v(x,t)$ and $\theta(x,t)$ that contains, however, nonlocal terms of the type (9).

Simple estimates show that in the case under consideration the right-hand sides of Eqs. (1) and (2) are small in comparison with both the linear terms and the nonlinear terms in the left-hand sides.[2] This gives the possibility of solving the equations by the methods of perturbation theory, in which, as the zeroth iteration, one must select the ordinary nonlinear second-sound wave described in Ref. 2. First, in Eqs. (1) and (2) we perform a certain transformation, connected with the following circumstance. If in these equations the right-hand side were absent, we would have a system of two homogeneous quasilinear equations. It is known that in such a formulation of the problem, when the wave is propagating from a wall into the undisturbed liquid, the solution of the homogeneous system is realized in the form of a so-called simple wave or Riemann invariant (see, e.g., Ref. 19). In a simple wave there is a single-valued functional relationship $\theta = \theta(v)$ between the (generally speaking) independent variables v and θ, and the evolution of the wave is determined by one equation, which in our case has the form

$$\partial v/\partial t+[1+\alpha v]\partial v/\partial x=0. \tag{10}$$

Here $\alpha = \alpha_{Khal} v_{n0}/c_2$, where α_{Khal} is the nonlinearity coefficient of the second sound, calculated earlier by Khalatnikov.[20] For $T = 1.75$ K the quantity $\alpha_{Khal} \cong 0.5$. The form of the function $\theta = \theta(v)$ is given (in dimensional quantities) in Ref. 18. In the presence of vortices, i.e., with allowance for the right-hand sides of Eqs. (1) and (2), the above statements are incorrect, but one can assume that the corrections to the function $\theta = \theta(v)$ and to Eq. (10) are small to the extent that the terms associated with the vortex tangle are small, i.e., one can set

$$\Theta=\Theta(v)+\psi(x,t), \tag{11}$$

with $\psi \ll \theta$. A solution of the type (11) is called a quasisimple wave.[18,21] The function $\psi(x,t)$ can be determined from the conditions for compatibility of Eqs. (1) and (2), which [after substitution of (11)] can be considered as a system of algebraic equations for $\partial v/\partial t$ and $\partial v/\partial x$. Computations performed in the spirit of Ref. 18 lead to the following evolution equation for the quantity v (in the coordinate frame moving with the velocity of the second sound)[3]:

$$\partial v/\partial t+\alpha v\partial v/\partial x=-{}^1/{}_2A_{23}vL, \tag{12}$$

and the connection between the temperature $\theta(x,t)$ and the velocity $v(x,t)$ is expressed by the relation

$$\Theta=\Theta(v)+\frac{A_{23}}{2}\int_{-\infty}^{t} Lv\,dt'. \tag{13}$$

We shall solve Eq. (12) by the method of successive approximations. The zeroth iteration $v^{(0)}$ is described by an equation of the type (12) without the right-hand side. As is well known (see, e.g., Ref. 2), the single-pulse evolves in accordance with this equation via a series of stages, such as steepening of the wave profile, formation of a shock front, and transformation of the wave into a "Burgers" triangle. The asymptotic form of any single pulse is a triangle with decreasing amplitude. Therefore, in order not to encumber the text with long calculations, we shall carry through the solution for a signal that is triangular from the outset.

The analytical expression for the evolution of a triangular pulse having (at time $t = 0$) amplitude v_{n0} at the shock front and duration t_s is described by the following formula (in the moving coordinate frame):

$$v^{(0)}(x,t)=x/(1+\alpha t),\quad 0\leqslant x\leqslant(1+\alpha t)^{1/2}. \tag{14}$$

During the evolution the spatial length l and the amplitude $v_d^{(0)}$ of the discontinuity behave as follows:

$$l=(1+\alpha t)^{1/2},\qquad v_d^{(0)}=(1+\alpha t)^{-1/2}. \tag{15}$$

We note that $l v_d^{(0)} = 1$, corresponding to the conservation law

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} v^{(0)}(x,t)\,dx=0,$$

which follows from Eq. (12) (without the right-hand side).

The next step in the calculations is to take Eq. (12) and determine its right-hand side, in which for $L(x,t)$ and $v(x,t)$ we must use the relations (9) and (14). Taking into account that in reality the integration over the time in formula (9) is restricted to the interval $\Delta t' \cong 1$ during which the zeroth interation changes only slightly (formally, because $\alpha\Delta t' \ll 1$), we can calculate the quantity $L(x,t)$ as follows:

$$L(x,t) = \frac{A_{32}}{(1+\alpha t)^{5/2}} \int_{x}^{(1+\alpha t)^{1/2}} x^{5/2} dx$$

$$= \frac{2A_{32}}{7} [(1+\alpha t)^{-3/4} - x^{7/2} (1+\alpha t)^{-5/2}]. \qquad (16)$$

By substituting (16) into Eq. (12) and then linearizing Eq. (12) about $v^{(0)}$, we obtain the following equation for the first iteration $v^{(1)}(x,t)$:

$$\partial v^{(1)}/dt + \alpha x/(1+\alpha t) = \widetilde{B} x[(1+\alpha t)^{-7/4} - x^{7/2}(1+\alpha t)^{-7/2}] - \alpha v^{(1)}/(1+\alpha t), \qquad (17)$$

where $\widetilde{B} = -A_{23}A_{32}/7$. We shall solve Eq. (17) by the method of characteristics. The characteristics in $x-$, $t-$, and $v^{(0)}$-space are described by the following system of ordinary differential equations:

$$dx/dt = \alpha x/(1+\alpha t), \qquad (18)$$

$$dv^{(1)}/dt + \alpha v^{(1)}/(1+\alpha t) = \widetilde{B}[x(1+\alpha t)^{-7/4} - x^{9/2}(1+\alpha t)^{-7/2}]. \qquad (19)$$

The system (18), (19) has the following solution, which depends on two arbitrary constants C_2 and C_2:

$$x = C_1(1+\alpha t),$$

$$v^{(1)} = \frac{4\widetilde{B}C_1}{5\alpha}(1+\alpha t)^{1/4} - \frac{C_1^{9/2}\widetilde{B}}{3\alpha}(1+\alpha t)^{2} + C_2(1+\alpha t)^{-1}. \qquad (20)$$

We shall select the constants C_1 and C_2 in such a way that the characteristics pass through the point $t = 0$, $x = x_0$, $v^{(1)} = 0$, corresponding to the zeroth initial condition for the first iteration, i.e., we shall set $v^{(1)}(x,t) = 0$. By next eliminating the parameter x_0 from the relations obtained, we find that $v^{(1)}(x,t)$ is equal to

$$v^{(1)} = \frac{4\widetilde{B}x}{5\alpha}[(1+\alpha t)^{-3/4} + (1+\alpha t)^{-2}]$$

$$- \frac{\widetilde{B}x^{-9/2}}{3\alpha}[(1+\alpha t)^{-5/2} - (1+\alpha t)^{-11/2}]. \qquad (21)$$

The sum $v^{(0)} + v^{(1)} = v$ (the sum of the expressions (14) and (21)) gives the solution of the problem in the approximation of interest to us. In the laboratory frame the wave profile is described by the function $v(x-t,t)$.

In Fig. 1 we depict schematically the function $v(x_i - t,t)$ at different distances x_i from the position $x = 0$ of the emitter; the function $v^{(0)}(x-t,t)$ is depicted by a dashed line. The bending in the function $v(x_i - t,t)$ is connected with the fact that the first iteration $v^{(1)}(x_i - t,t)$ has an intermediate minimum ($v^{(1)}$ is negative), the presence of which is easily understood from the following qualitative considerations. In the leading parts of the heat pulses, vortices do not have time to be generated, and so the right-hand side of Eq. (12) vanishes. At the end of the pulse the right-hand side is equal to zero since the zeroth-iteration velocity vanishes. In other words, there is no mutual-friction force proportional to $v^{(0)}$. It is interesting to note that the first iteration is not equal to zero at the shock front of the wave pulse. This may appear strange at first sight, since in the leading parts of the pulse there are no vortices. However, by virtue of the fact that the original equation (12) contains in the left-hand side a nonlinear term of the form $\alpha v\partial v/\partial x$, the correction $v^{(1)}$ is carried over from the central parts toward the leading edge, and the velocity deficit $v_d^{(1)}(t)$ at the shock front increases monotonically. Since, furthermore, the velocity U_{fr} of propagation of the shock front, as follows from the Hugoniot relations for He II, is equal to $U_{fr} = \alpha v_d/2$ (see, e.g., Ref. 11), the velocity deficit $v_d^{(1)}$ at the shock front leads to the result that the pulse moves more slowly than it would without the vortices. We shall return somewhat later to the question of the change of the velocity of propagation of a pulse that is generating vortices.

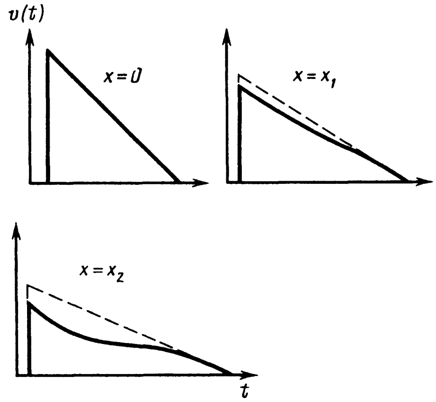

FIG. 1.

The above-described behavior of intense heat pulses has been observed in many experiments. In Fig. 2 we show for comparison a wave-profile oscillogram taken from Ref. 22, for $x = 0.2$ cm, $W = 73.5$ W/cm^2, and $t_s = 70$ μsec.

The evolution of a pulse having an arbitrary initial initial shape can be calculated in a similar manner. It is possible also to calculate the evolution of a group of pulses, and this is important in the discussion of experiments in which pulses are triggered in the stroboscopic regime. The corresponding calculations are very cumbersome, and we shall not give them here. We note only that after the formation of triangles the evolution of the pulses coincides qualitatively with that described.

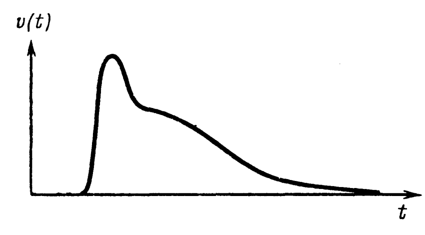

FIG. 2.

We shall discuss the behavior of the temperature perturbation $\theta(x,t)$. The temperature field can be calculated from formula (13). It follows from this formula that the temperature disturbance is somewhat larger than it would be in the vortex-free regime. The temperature rise is due to dissipation of the wave energy in friction with the vortices, and can turn out to be extremely important and lead to superheating of the helium relative to the saturation line; this can become the cause of the formation of a vapor film. We shall calculate the time t_{boil} of formation of a vapor film during pumping of a constant heat flux of intensity $W = STv_n$, switched on at time $t = 0$. The quantity t_{boil} can be obtained from the relation (13), if we are given the value of ΔT_{cr}—the superheating critical temperature at which the film appears. In formula (13) we neglect the first term $\theta(v)$, i.e., we neglect the acoustic temperature disturbance in the wave pulse (as was noted earlier, this quantity is not large and cannot be the cause of boiling up of the helium).[4] We also neglect the shift from the saturation line caused by the excess pressure of the column of helium above the heater. Then, for a given superheating critical temperature (dimensionless) $\Delta\theta_{\text{cr}}$, the time t_{boil} of onset of boiling can be obtained from the following relations:

$$\Delta\Theta_{\text{cr}} = \frac{A_{23}}{2}\int_0^{t_{\text{boil}}} vL(t')\,dt' = \frac{A_{23}A_{32}}{2}\int_0^{t_{\text{boil}}} v\,dt'\int_0^{t'} v^{5/2}dt'' = \frac{A_{23}A_{32}}{4}\,v^{7/2}t^2_{\text{boil}}. \qquad (22)$$

In dimensional form, formula (22) is equivalent to the following:

$$Wt^{4/7} = \left(\frac{4\Delta T_{\text{cr}}}{A_{23}A_{32}}\frac{\sigma\rho_s}{c_2\rho_n}\right)^{2/7} ST = F(T). \qquad (23)$$

By substituting into (23) data taken from Refs. 12 and 13 for the superheating critical temperature ΔT_{cr}, we find that for $T = 1.75$ K the quantity $F(T)$ is equal to 0.5–1 (here the heat-flux intensity W is expressed in W/cm^2, and the time t_{boil} in seconds).

The formation of a vapor film during pumping of intense heat pulses in He II was observed in Refs. 9 and 10. The authors of these papers proposed the following dependence for the time of formation of the film:

$$Wt_{\text{boil}}^{0.5\pm0.1} = \begin{cases} 0.5 & [\text{Ref. } 9] \\ 0.04 & [\text{Ref. } 10] \end{cases}. \qquad (24)$$

We return to the question of the variation of nonlinear extra term that appear in the velocity of sound on account of the interaction with vortices generated by this sound. Generally speaking, the concept of the velocity of propagation of a pulse is not adequate in the sense that this quantity is affected by a large number of parameters, including the initial shape of the pulse. Moreover, this quantity is variable. Therefore, to speak of the nonlinear velocity of propagation of a pulse as, e.g., a function of the pulse intensity W has meaning only if we have stipulated all the other conditions of the experiment. In Ref. 8 measurements were made of the quantity $c_2(W)$ (which the authors called the nonlinear signal velocity), defined as $c_2(W) = d/t_{\text{fl}}$, where d is the distance between two second-sound detectors positioned at a certain distance from the emitter, and t_{fl} is the time of flight of the signal between these detectors. This quantity, or, more correctly, the difference $c_2(W) - c_2$ (c_2 is the velocity of linear second sound), is depicted as a function of the intensity by the solid line in Fig. 3. Using the conditions of the experiment of Ref. 8, the authors of Ref. 2 calculated $c_2(W) - c_2$ from the Burgers equation [in essence, from Eq. (10)]. The result of these calculations is represented in Fig. 3 by the dashed line. As can be seen, the discrepancy is very considerable, and cannot be attributed to experimental error.

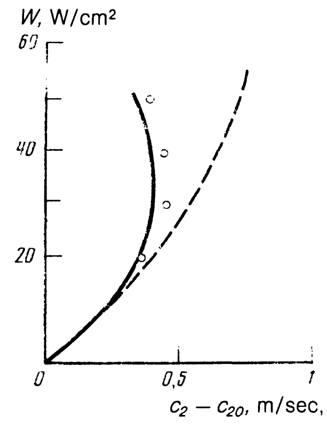

FIG. 3.

As follows from what was said earlier, the velocity of the shock front in a pulse that creates vortices differs from the velocity of an ordinary (vortex-free) wave. In principle, solving Eq. (12) with the initial conditions that follow from the experiment of Ref. 8, one can calculate the quantity $c_2(W) - c_2$. However, quite apart from the fact that here we would necessarily encounter difficulties associated with taking all the stages of the evolution of the nonlinear signal into account, an exact calculation is also impossible for the following reason. The point is that the main effect (see Fig. 3) is observed at large values of W, for which the condition $t_s \ll \tau_v$ is violated. Therefore, as in the discussion of the question of the boiling of helium, we shall confine ourselves to order-of-magnitude relations. In accordance with formula (21), the change ΔU_{fr} in the velocity of the shock front on account of interaction with vortices is equal to

$$\Delta U_{\text{fr}} = (av_d^{(1)}/2)c_2 f(t) = (A_{23}A_{32}/14)\,c_2 f(t). \qquad (25)$$

Here $v_d^{(1)}$ is the vlaue of the first iteration at the shock front [see (21)], and $f(t)$ is a function that can be obtained from the exact solution and has a characteristic value close to unity. By substituting into (25) the values of the coefficients A_{23} and A_{32}, we find that $\Delta U_{\text{fr}} \cong 10^{-5} v_{n0}^{5/2}$ (cm/sec). The results of calculations using this relation are represented in Fig. 3 by points. Thus, despite the order-of-magnitude character of the calculations, here there is good quantitative agreement with experiment.

3. THE ADIABATIC APPROXIMATION

In this section we shall consider the case when the characteristic duration t_s of the disturbances is much longer than the time of development of a vortex tangle, and the quantity L takes its equilibrium (for the velocity v) value. It is clear that in this case we are certainly dealing with pulses of low intensity, since in powerful pulses the level of the vortices is high, the dissipation is correspondingly high, and the helium boils before the tangle is saturated. As a result, the adiabatic approximation is valid only for pulses of low (but supercritical) intensity and large duration. (For steplike pulses, i.e., constant heat fluxes switched on at time $t = 0$, instead of the duration t_s we must speak of the observation time after the beginning of the experiment.) Such conditions were maintained in the experiment of Refs. 13, 23, and 24. In these experiments investigations were made of the dynamics of the temperature field $T(x,t)$ in long (up to 8 m) tubes filled with superfluid helium, to one end of which a constant heat flux, switched on at time $t = 0$, was supplied. The resulting dynamics of the temperature field, like that for short pulses, cannot be described in the framework of classical two-fluid hydrodynamics.

With the assumption that $\tau_v \ll t_s$ it may be supposed that the tangle density L has time to adjust to the change of the parameters of the wave and, as follows from the VS equation (3), is equal to $L = v^2$ (the source term is small and acts only at the very beginning, when L is small). Substituting the value $L = v^2$ into Eqs. (1)–(2), we obtain the following system of equations, which, unlike the equations in the preceding sections, we write in dimensional form:

$$\frac{\partial v_n}{\partial t}+\frac{\rho_s\sigma}{\rho_n}\frac{\partial T'}{\partial x}=-av_n^3, \tag{26}$$

$$\frac{\partial T'}{\partial t}+\frac{\sigma}{\sigma_T}\frac{\partial v_n}{\partial x}=bv_n^4. \tag{27}$$

Here T' is the temperature disturbance, $a = A\rho^3/\rho_s^2$, and $b = A\rho_n\rho^3/C\rho_s^3$, where A is the Gorter-Mellink constant (see, e.g., Refs. 16 and 17). We note that, in contrast to the case of short pulses, in the adiabatic approximation the source term (the determination of which involves the familiar difficulties—see Sec. 1) is absent, and we may expect results of higher accuracy. In addition, in the left-hand sides of the equations of the system (26), (27) we have omitted the usual nonlinear terms, which are unimportant in the case under consideration.

The relations (26) and (27) are the equations for the propagation of second sound with allowance for damping by vortices generated by this sound (the term $-av_n^3$) and for dissipation associated with this damping (the term bv_n^4). We shall supplement the system (26), (27) with boundary conditions corresponding to stepwise liberation of heat $W = STv_{n0}$ into undisturbed He II:

$$v_n(0,\ t)=v_{n0}e(t),\quad v_n(x,\ 0)=0,\quad T'(x,\ 0)=0. \tag{28}$$

Here $e(t)$ denotes the unit step function: $e(t) = 0$ for $t<0$, and $e(t) = 1$ for $t \geqslant 0$.

Despite the outwardly simple form of Eqs. (26) and (27), there are at present no general methods for solving them analytically. However, it has turned out to be possible to separate out a particular integral in the form of a traveling wave in which the variables v_n and T' are functions of the quantity $\tau = x - Ut$, i.e., $v_n(x,t) = v_n(\tau)$ and $T'(x,t) = T'(\tau)$. For the homogeneous (in the derivatives) systems of equations, traveling waves are complete solutions of the boundary-value problems, i.e., problems in which the waves propagate from a wall into the undisturbed liquid; here the boundary condition $v_n(0,t)$ can be arbitrary. The equations, so to speak, can be functionally arbitrary (see Ref. 19). In contrast to this, the inhomogeneous equations [i.e., the relations (26) and (27)] admit solutions in the form of traveling waves only for a certain class of boundary conditions in which, however, a certain constant (here, U) can be varied. The system admits a constant leeway (see Ref. 19). To determine the class of admissible boundary conditions, we substitute $v_n(\tau)$ and $T'(\tau)$ into the system (26), (27) and solve the system for the derivatives $v_{n\tau}$ and T'_τ. As a result of straightforward calculations we obtain the following system of ordinary differential equations:

$$v_{n\tau}=\frac{1}{U^2-c_2^2}\left(aUv_n^3-\frac{b\sigma\rho_s}{\rho_n}v_n^4\right), \tag{29}$$

$$T_\tau'=\frac{1}{U^2-c_2^2}\left(-bUv_n^4+\frac{a\sigma}{\sigma_T}v_n^3\right). \tag{30}$$

By integrating Eq. (29) and setting $x = 0$, we obtain the class of admissible boundary conditions $\tilde{v}_n(0,t)$ that depend on the parameter U. It is easy to see that for a value of U equal to $U = b\rho_s\sigma v_{n0}/a\rho_n = \sigma v_{n0}/C$ the function $\tilde{v}_n(0,t)$ is a diffuse step $\tilde{v}_n(0,t) = v_{n0}e(t\,|\,\Delta t)$ with diffuseness $\Delta t = C^2c_2^2\rho_s^2/\sigma^2 A\rho^3 v_{n0}^4$ and amplitude v_{n0}.

Next we describe the exact solution of the system (26), (27) with the boundary condition $\tilde{v}_n(0,t)$, and show that the time $t_{\rm init}$ after which the evolution of the rectangular step of interest [see (28)] reaches the regime of the exact solution is much shorter than the observation time in the experiments of Refs. 13, 23, and 24.

In accordance with what has been said, the velocity field $\tilde{v}_n(x,t)$ is described by the relation $\tilde{v}_n(x,t) = v_{n0}e(t - x/U\,|\,\Delta t)$, i.e., the diffuse step moves as a whole in space with velocity U. We now consider the evolution of the temperature field $\tilde{T}'(x,t)$. Substituting the value found for U into Eq. (30), we obtain the relation

$$T_\tau'\sim -a\rho_n\tilde{v}_n^3(\tau)/\rho_s\sigma$$

(in the derivation of this relation it has been taken into account that the third and fourth powers of the function $e(t\,|\,\Delta t)$ are also in the form of a diffuse unit step, i.e., $e^3(t\,|\,\Delta t) \cong e(t\,|\,\Delta t)$), from which, since we know the function $\tilde{v}_n(\tau)$, it is easy to determine the function $\tilde{T}'(x,t) = \tilde{T}'(\tau)$. In particular, at the wall, by setting $x = 0$ and taking into account that $\partial/\partial t = -U\partial/\partial x$ we obtain

$$T'(0,t)\sim\int_{-\infty}^{t} b\tilde{v}_n^4(0,t')\,dt'.$$

If for a rectangular (not diffuse) step we neglect the change of temperature in the initial time interval $\sim t_{\rm init}$, it is possible to show that the temperature $T'(0,t)$ of the helium near the wall increase linearly:

$$T'(0,\ t)=bv_{n0}{}^4t. \tag{31}$$

The region with the linear increase of temperature propagates next in the liquid with velocity U. The rise of the temperature near the wall can lead to superheating of the helium relative to the saturation line and (as in the case of short pulses, too) can lead to the formation of a vapor film. However, the dependence of the time t_{boil} of onset of boiling on the intensity of the heat flux will now differ from that in the short-pulse case considered in the preceding section. The time t_{boil} can be calculated from the relation (31) if we are given the limiting superheating temperature ΔT_{cr}, and is found to be equal to

$$t_{\text{boil}}=\Delta T_{\text{cr}}(ST)^4/bW^4=BW^{-4}. \tag{32}$$

A dependence of the form (32) was observed in the experiment of Ref. 13. The value of the constant B agrees well with that obtained in Ref. 13. For example, for $T=1.8$ K, for B_{theor} we have (with allowance for the spread in the data) $B_{\text{theor}}=90\text{–}150\ \text{W}^4\cdot\text{sec/cm}^8$, whereas the experimental value is $B_{\text{exp}}=110\ \text{W}^4\cdot\text{sec/cm}^8$. At higher temperatures the agreement is worse; e.g., for $T=2$ K, $B_{\text{exp}}=17$ while $B_{\text{theor}}=20\text{–}40\ \text{W}^4\cdot\text{sec/cm}^8$.

The described evolution of the temperature field was observed in Refs. 23 and 24 (see, e.g., Fig. 3 in Ref. 23). The following values were given for the rate of growth $\partial T'/\partial t$ of the temperature and for the velocity U_{exp} of the front (for $W=4.4\cdot10^{-2}$ W/cm^2): $\partial T'/\partial t=8.3\cdot10^{-7}$ K/sec, $U_{\text{exp}}=0.27$ cm/sec. Our calculations give for these quantities the values $\partial T'/\partial t=8.6\cdot10^{-7}$ K/sec and $U_{\text{theor}}=0.32$ cm/sec.

We now discuss the physical meaning of the exact solution obtained. The term in the right-hand side of Eq. (26) corresponds to the mutual friction and should lead to a decrease of the quantity $v_n(x,t)$. At the same time, because of the dissipation associated with this friction, the temperature near the wall increases. But, as is well known, in He II a temperature drop is a driving force for the velocity $\mathbf{v}_n$ (for $j=0$) and can compensate the decrease of the quantity v_n. The interplay of these two effects gives rise to the possibility of the existence of a solution in the form of a traveling wave of unchanging profile. It is necessary to note that this mechanism is specific for a superfluid liquid, since in ordinary media the effect of the dissipative function on the flow is much weaker. In fact, in classical media a temperature gradient due to dissipation does not lead directly to macroscopic convective flows, and the pressure drop associated with it (through the expansion coefficient $\beta_T=-\rho^{-1}(\partial\rho/\partial T)$), is small by virtue of the smallness of β_T.

Starting from the physical meaning of the solution, we can estimate the time t_{init} after which the evolution of a strictly rectangular step (specified by the boundary conditions (28)) coincides with the solution $\tilde{v}_n(x,t)$ that we have described. This solution, as already stated, is a step whose front is smeared out in space over a length $\tilde{l}$ equal to $\tilde{l}\sim(\partial\tilde{v}_n/\partial\tau)^{-1}v_{n0}$. A rectangular step acquires the same diffuseness after a time $t_{\text{init}}=\tilde{l}/c_2$, since the leading edge of the step propagates with velocity c_2 while the value of the velocity v_n at the leading edge falls practically to zero because of the friction. After the rectangular step has acquired spatial diffuseness comparable with the quantity $\tilde{l}$, the problems of the propagation of an exact step and a diffuse step become identical. In other words, the evolution of an exact step is described with good accuracy by the relations obtained earlier. By comparing $\tilde{l}=(\partial\tilde{v}_{n0}/\partial\tau)^{-1}v_{n0}$ and $\tilde{l}=t_{\text{init}}c_2$, we find that $t_{\text{init}}\sim Cc_2\rho_s/A\rho^3\sigma v_{n0}^3$. It is not difficult to convince oneself that the initial time interval t_{init} is much shorter than the times of observation in Refs. 13, 23, and 24. For example, in the experiment of Ref. 23 the quantity $t_{\text{init}}\sim10$ sec, whereas the experiment was performed for an hour, i.e., 10^3–10^4 sec. We note that the time of vortex formation in the cited experiment, as calculated from formula (7), was equal to $\tau_v\sim0.5$ sec.

4. CONCLUSION

We shall summarize briefly the content of the article and make some comments. In the paper it has been shown that the vortices that are formed in He II upon the passage of intense second-sound pulses can alter to a considerable degree (in comparison with ordinary two-fluid hydrodynamics) the laws of the evolution of such pulses. The results obtained agree qualitatively and quantitatively with the experimental data from a number of papers in which different aspects of the dynamics of heat pulses in a large range of their parameters were investigated.

In Sec. 1 of the article, in a discussion of the discrepancies between the results of experiments in the study of heat pulses and the nonlinear theory of waves in He II, we put forward the hypothesis that these discrepancies are due to interaction with vortices generated by the pulses. It seems to us that the good agreement between the conclusions of the present article and the results of the cited experiments confirms this hypothesis. In fact, the experiments with which the comparisons have been made are rather diverse (namely, investigation of the evolution of signals, measurement of the nonlinear velocity of their propagation, measurement of the time of formation of a vapor film, description of the dynamics of the temperature field, and investigation of the motion of trubulence fronts), but nevertheless they all fit well into the framework of the model developed here.

We have remarked on the order-of-magnitude character of the relations (23) and (25), which is associated with the approximate solution of Eqs. (1)–(3). There is also another, "extrinsic" reason why the given relations can describe a real experiment only approximately and in order of magnitude. The point is that, in the interpretation of experiments associated with the excitation of heat pulses in He II by a heated wall, a number of problems linked with the presence of the He-II-solid boundary arise. Among such problems are, e.g., the as-yet uninvestigated question of the stationary Kapitz discontinuity and the associated relaxation effect at the interface, or, e.g., the problem of the extraction of part of the liberated heat into the solid substrate. Also fairly serious is the question of the formation of a heat counterflow (in which the relation $W = STv_n$ is fulfilled) near the wall, where viscosity and ordinary thermal conduction are important. The solution of this problem described in e.g., Ref. 25 cannot be accepted as satisfactory, if only because the thickness of the layer in which the ordinary mechanism of thermal conduction is replaced by convection is found to be of the order of 10^{-7} cm (for $T = 2$ K), and hydrodynamic treatment is clearly inapplicable. In addition, here there are uncertainties in the choice of boundary conditions. The questions enumerated above indicate, on the one hand, that when solving problems connected with the pumping of heat into helium one must analyze carefully the phenomena occurring at the boundary, and, on the other hand, that quantitative results in which the dependence $W = STv_n$ is used have an approximate character. Here it is also appropriate to add that most of the experiments with short pulses are carried out in the stroboscopic regime of triggering of the signals, and in helium there is a certain vortex background which, of course, should affect the evolution of the waves. We note that the above is practically irrelevant to Sec. 3 of the paper.

The results in this article were reported at the Twentieth Bakuriani School on the Physics of Quantum Liquids (1986). The author thanks V. L. Pokrovskiĭ and L. P. Mezhov-Deglin, who took an active part in the discussion of the work.

[1)]As a rule, the source of thermal waves (the emitter) is under the action of hydraulic pressure produced by the column of liquid above it, and, as a consequence, is at a distance (along the temperature axis) $\Delta T = (\partial T/\partial p)_0 \rho g h$ from the equilibrium curve. Here $(\partial T/\partial p)_0$ is the derivative of the temperature with respect to the pressure along the saturation line and h is the height of the column of helium. We note that $(\partial T/\partial p)_0 \rho g \cong 5 \cdot 10^{-3}$ K/cm.

[2)]It is easy to verify this by making use of the fact that in the dimensionless form we have $v \cong 1$, $\theta \cong 1$, $\partial/\partial x \cong 1$, and $\partial/\partial t \cong 1$, and using the values given in (5) for the coefficients A.

[3)]In the right-hand sides of the relations (12) and (13) we have kept only the term describing the force of the friction of the normal component against the "frozen" system of vortex lines. All the other terms appearing in Eqs. (1), (2) and associated with the presence of a vortex tangle give, in the present case, a much smaller contribution.

[4)]Strictly speaking, for vlaues of $\psi(x,t)$ comparable to $\theta(v)$, the quasisimple-wave approximation, i.e., the representation of the solution in the form (11), is inapplicable. In addition, in experiments in which boiling of the helium occurs, the time of formation of the vapor film is comparable to the time of formation of the tangle. Therefore, the relations that follow below must be regarded as estimates, and an exact calculation can be carried out only on the basis of a complete solution of the original system of equations (1)–(3), which, we repeat, is possible only by numerical methods.

[1]D. V. Osborne, Proc. Phys. Soc. London Sec. A **64**, 114 (1951).
[2]M. O. Lutset, S. K. Nemirovskiĭ, and A. N. Tsoĭ, Zh. Eksp. Teor. Fiz. **81**, 249 (1981) [Sov. Phys. JETP **54**, 127 (1981)].
[3]A. Yu. Iznankin and L. P. Mezhov-Deglin, Pis'ma Zh. Eksp. Teor. Fiz. **29**, 755 (1979) [JETP Lett. **29**, 693 (1979)].
[4]S. K. Nemirovskiĭ and A. N. Tsoĭ, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 229 (1982) [JETP Lett. **35**, 286 (1982)].
[5]J. R. Torczynski, Phys. Fluids **27**, 2636 (1984).
[6]A. Yu. Iznankin and L. P. Mezhov-Deglin, Zh. Eksp. Teor. Fiz. **84**, 1378 (1983) [Sov. Phys. JETP **57**, 801 (1983)].
[7]T. N. Turner, Phys. Fluids **26**, 3227 (1983).
[8]J. C. Cummings, D. W. Schmidt, and W. J. Wagner, Phys. Fluids **21**, 713 (1978).
[9]A. N. Tsoĭ and M. O. Lutset, Inzh.-Fiz. Zh. **51**, 5 (1986).
[10]A. Yu. Iznankin and L. P. Mezhov-Deglin, in: Proceedings of the International Conference LT-18, Vol. 1, Elsevier Scientific Publishers (1984), p. 71.
[11]I. M. Khalatnikov, Teoriya sverkhtekuchesti (Theory of Superfluidity), Nauka, Moscow (1971) [English translation (Introduction to the Theory of Superfluidity) published by Benjamin, New York (1965)].
[12]G. Krafft, J. Low Temp. Phys. **31**, 441 (1978).
[13]S. W. Van Sciver, Cryogenics **19**, 385 (1979).
[14]R. P. Feynman, Prog. Low Temp. Phys. **1**, 17 (1957).
[15]S. K. Nemirovskiĭ and V. V. Lebedev, Zh. Eksp. Teor. Fiz. **84**, 1729 (1983) [Sov. Phys. JETP **57**, 1009 (1983)].
[16]W. E. Vinen, Prog. Low Temp. Phys. **3**, 1 (1961).
[17]K. W. Schwarz, Phys. Rev. B **18**, 245 (1978).
[18]S. K. Nemirovskiĭ, Preprint No. 08-76, Institute of Theoretical Physics, Novosibirsk (1976).
[19]B. L. Rozhdestvenskiĭ and N. N. Yanenko, Sistemy kvasilineĭnykh uravneniĭ (System of Quasilinear Equations), Nauka, Moscow (1968).
[20]I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **23**, 253 (1952).
[21]V. I. Karpman, Nelineĭyne volny v dispergiruyushchikh sredakh (Nonlinear Waves in Dispersive Media), Nauka, Moscow (1973).
[22]A. N. Tsoĭ, Kvantovannye vikhri v slede udarnoĭ volny vtorogo zvuka (Quantized Vortices in the Wake of a Second-Sound Shock Wave), in the book: Gidrodinamika odno- i dvukhfaznykh sistem (Hydrodynamics of One- and Two-Phase Systems), Novosibirsk, (1982).
[23]V. P. Peshkov and V. K. Tkachenko, Zh. Eksp. Teor. Fiz. **41**, 1427 (1961) [Sov. Phys. JETP **14**, 1019 (1962)].
[24]S. M. Bhagat, P. R. Critchlow, and K. Mendelssohn, Cryogenics **4**, 166 (1964).
[25]S. J. Putterman, *Superfluid Hydrodynamics*, North Holland, Amsterdam (1974) [Russ. transl., Mir, Moscow (1978)], Sec. 28.

Translated by P. J. Shepherd