# Permittivity of a weakly inhomogeneous plasma

V. S. Beskin, A. V. Gurevich, and Ya. I. Istomin

*P. N. Lebedev Physical Institute, Academy of Sciences of the USSR*
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We find for a weakly inhomogeneous collisionless plasma an expression for the permittivity $\varepsilon^{\rm eff}_{\alpha\beta}$. The anti-Hermitean part of $\varepsilon^{\rm eff}_{\alpha\beta}$ provides us with a correct description of the exchange of energy between a wave and the particles of the plasma under conditions when the nature of the resonance interaction changes strongly due to the presence of inhomogeneities. We consider specific applications of the $\varepsilon^{\rm eff}_{\alpha\beta}$ tensor. We show that the form of the Landau damping of long-wavelength Langmuir oscillations in an inhomogeneous plasma in an electric field changes completely. A beam of fast electrons causes in such a plasma alternating bands of excitation and damping regions for the oscillations. We consider the instability of a stationary flux of an ultra-relativistic plasma moving in a strong curvilinear magnetic field; this is important for the theory of the origin of the radio-emission of pulsars.

The propagation of small amplitude waves in a rarefied plasma is described by a set of linearized kinetic equations for the particles and by the Maxwell equations for the field. In a uniform medium the eigensolutions of these equations are plane waves. Their properties are completely determined by the permittivity tensor $\varepsilon_{\alpha\beta}(\mathbf{k},\omega)$ where $\omega$ is the frequency and $\mathbf{k}$ the wave vector of the wave. Using the $\varepsilon_{\alpha\beta}(\mathbf{k},\omega)$ tensor we find from the dispersion equation the oscillation eigenmodes of the medium and for each mode we determine the way the wave vector $\mathbf{k}$ and the spatial growth rate (or damping rate) $\varkappa$ of the oscillations depend on the wave frequency $\omega$. The quantity $\varkappa$ describes the change in wave amplitude due to an exchange of energy between the wave and the particles of the medium; it is determined by the anti-Hermitean part of $\varepsilon$.[1,2]

In an inhomogeneous medium the picture is significantly different. First of all, plane waves are no longer eigenfunctions of the linearized equations. It is true that in the case of a weakly inhomogeneous medium which we shall consider in what follows,

$$kL \gg 1, \quad \varkappa/k \ll 1 \tag{1}$$

($L$ is a characteristic scale of the inhomogeneity) one should expect that the eigenfunctions can be represented in the form of a packet of plane waves with a structure determined by the geometric-optics formulae. However, to construct the packet it is necessary to know the permittivity tensor $\varepsilon_{\alpha\beta}(\mathbf{r},\mathbf{k},\omega)$ of an inhomogeneous plasma and finding it encounters well defined difficulties.

The main difficulties arise when separating the anti-Hermitean part $\varepsilon^{\rm aH}_{\alpha\beta}(\mathbf{r},\mathbf{k},\omega)$ of the tensor. First of all, the change in the wave amplitude in a non-uniform plasma is due not only to an exchange of wave energy with the medium, but also to a change in its group velocity. It is necessary to find a way to separate these two processes when determining the $\varepsilon_{\alpha\beta}$ tensor in a non-uniform medium. An even more complicated situation arises in respect to the anti-Hermitean part $\varepsilon^{\rm aH}_{\alpha\beta}(\mathbf{r},\mathbf{k},\omega)$ determined by the resonance particles, since the scale $L$ of the inhomogeneity may become comparable with or less than the characteristic scales for the interaction of the particles with a wave (these scales depend strongly on $\omega$ and $\mathbf{k}$). Thanks to the effect of the inhomogeneity, the whole character of the resonance interaction is then transformed, and this must lead to a complete change in the corresponding part of $\varepsilon^{\rm aH}_{\alpha\beta}$.

The study of wave propagation in a plasma in a non-uniform magnetic or electric field or possessing non-uniform parameters—densities, particle temperature—is important for many problems connected with plasma stability, its containment and heating, the generation of radiation, and so on. Many papers (see the monographs and reviews of Refs. 3 to 5) have therefore been devoted to a theoretical study of the problems indicated here, both in the general form and in some specific statements of it. However, its complete solution has not been obtained. In particular, the effect of the non-uniformity of a plasma on the resonance interactions of the particles with a wave has in fact not been considered earlier.

In §1 of the present paper we shall, by a special expansion using the small parameters of (1), construct a solution of the linearized equations under stationary conditions and find a general expression for the permittivity tensor $\varepsilon^{\rm eff}_{\alpha\beta}(\mathbf{r},\mathbf{k},\omega)$ of an inhomogeneous plasma. This expression provides a correct description of the exchange of energy between a wave and the particles of the medium under all conditions, amongst them the case when the inhomogeneity of the plasma significantly affects the resonance interaction. Using a special symmetrization of the Fourier transform, we can prove that the permittivity tensor guarantees equivalence for inversion of the wave ($\omega \to -\omega, \mathbf{k} \to -\mathbf{k}$) and time reversal ($t \to -t$) thanks to which the quantity $\varepsilon^{\rm eff}$ which we have found possesses the same properties as the permittivity of a uniform medium.

In subsequent sections of the paper we give examples of actual applications of the $\varepsilon^{\rm eff}_{\alpha\beta}$ tensor. In §2 we consider cases when the resonance interaction is unimportant and it is sufficient to restrict oneself to a simplified expression for $\varepsilon^{\rm eff}_{\alpha\beta}$. We study the excitation of long-wavelength ion-sound oscillations and consider one example of drift instability (the ion-cyclotron drift instability). We show that using the correct expression for the permittivity tensor may lead to considerable corrections in the theory of the drift instability.

We study in §3 Langmuir oscillations of a plasma in an electric field. We show that the Landau damping changes its

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form for sufficiently long-wavelength oscillations due to a change in the nature of the resonance interaction in an inhomogeneous plasma. When the oscillations are excited in such a plasma by a fast electron beam the picture may differ completely from the classical one—the growth rates change strongly and there arise alternating bands of regions where the oscillations are excited and damped.

In the concluding §4 we consider a plasma which is a stationary beam of ultra-relativistic particles moving along a strong non-uniform (curvilinear) magnetic field. In this case the resonance interaction is especially large as almost all particles turn out to be at resonance. We find an expression for the permittivity tensor. An analysis of the dispersion equation obtained in this case shows that under conditions of a sufficiently dense plasma there appear fast growing hydrodynamic modes. This result is of considerable interest in astrophysics for the theory of the origin of the extraordinary powerful and highly directional radio-emission from pulsars—it has not been possible previously to find a mechanism for generating this radiation.

## §1. RESPONSE OF A NON-UNIFORM PLASMA TO A PLANE WAVE. EFFECTIVE PERMITTIVITY

We consider a stationary non-uniform rarefied plasma. The propagation of small amplitude waves in it is described by a set of linearized collisionless kinetic equations for the plasma particles and the Maxwell equations for the field. The non-uniformity of the plasma appreciably affects then both the motion of the particles which determine the response of the plasma to the action of the wave, and also the process itself of the propagation of the wave field.

An effective method for finding the response of the plasma to the action of an electromagnetic field is the path-integral method suggested by Shafranov.[2] It enables us to obtain expressions for the currents and charges arising in a plasma with arbitrary inhomogeneities and non-stationarities when a small amplitude electromagnetic wave acts on it. Restricting ourselves here to a stationary medium when all perturbed quantities are proportional to $\exp\{-i\omega t\}$, we get from the solution of the collisionless kinetic equation the following expression for the current density in the plasma:

$$j_\alpha(\mathbf{r},\omega)=-e^2\int d\mathbf{k}\exp\{i\mathbf{kr}\}$$
$$\times\int v_\alpha\, d\mathbf{p}\int_{-\infty}^{t}\left\{\mathbf{E}(\mathbf{k})+\frac{1}{c}[\mathbf{v}(t')\mathbf{B}(\mathbf{k})]\right\}$$
$$\times\exp\{i\omega(t-t')-i\mathbf{k}(\mathbf{r}-\mathbf{r}')\}[\partial F(\mathbf{r},\mathbf{p},t)/\partial\mathbf{p}(t')]dt'.$$

Here $F(\mathbf{r},\mathbf{p},t)$ is the unperturbed distribution function of particles with charge $e$; $\mathbf{p}(t')$, $\mathbf{v}(t')$, $\mathbf{r}'=\mathbf{r}(t')$ are the momentum, velocity, and coordinate at time $t'$ of a particle moving along the unperturbed trajectory such that at time $t$ it is at the point $\mathbf{r}$ considered with momentum $\mathbf{p}$ and velocity $\mathbf{v}$. The quantities $\mathbf{E}(\mathbf{k})$ and $\mathbf{B}(\mathbf{k})$ are the Fourier components of the electromagnetic field of the wave:

$$\mathbf{E}(\mathbf{k})=\frac{1}{(2\pi)^3}\int\mathbf{E}(\mathbf{r})e^{-i\mathbf{kr}}\,d\mathbf{r};\qquad \mathbf{B}(\mathbf{k})=\frac{c}{\omega}[\mathbf{kE}(\mathbf{k})].$$

The conductivity found thus as a response of a non-uniform plasma to a plane wave we denote by $\sigma^0_{\alpha\beta}(\mathbf{r},\mathbf{k},\omega)$:

$$\sigma_{\alpha\beta}{}^0=-e^2\int v_\alpha\, d\mathbf{p}\int_{-\infty}^{t}\left[\left(1-\frac{\mathbf{kv}'}{\omega}\right)\delta_{\beta s}+\frac{k_s v_\beta'}{\omega}\right]$$
$$\times\exp\{i\omega(t-t')-i\mathbf{k}(\mathbf{r}-\mathbf{r}')\}\frac{\partial F}{\partial \mathbf{p}_s'}\,dt'. \qquad (2)$$

In a uniform medium when the distribution function $F(\mathbf{p})$ is independent of the coordinate the quantity $\sigma^0_{\alpha\beta}$ is also independent of $\mathbf{r}$ and is indentically the same as the usual complex conductivity of a uniform plasma: $\sigma^0_{\alpha\beta}=\sigma_{\alpha\beta}(\omega,\mathbf{k})$.[1]

Moreover, the total current is

$$j_\alpha(\mathbf{r},\omega)=\int d\mathbf{k}\,e^{i\mathbf{kr}}\sigma_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k},\omega)E_\beta(\mathbf{k}). \qquad (3)$$

Expressing in (3) the Fourier component $E_\beta(\mathbf{k})$ in terms of the field $E_\beta(\mathbf{r})$ and substituting this quantity into the Maxwell equations we get a linear integro-differential equation describing the evolution of the wave field:

$$\Delta\mathbf{E}-\nabla\operatorname{div}\mathbf{E}+\frac{\omega^2}{c^2}\mathbf{E}=-\frac{4\pi i\omega}{c^2}\mathbf{j}(\mathbf{r},\omega),$$
$$j_\alpha=\frac{1}{(2\pi)^3}\int\int d\mathbf{k}\,d\mathbf{r}'\,e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\sigma_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k},\omega)E_\beta(\mathbf{r}'). \qquad (4)$$

In a uniform medium the eigenfunctions of this equation are plane waves and the condition of solvability of this equation has the form of the usual dispersion equation

$$\operatorname{Det}\left|k_\alpha k_\beta-k^2\delta_{\alpha\beta}+\frac{\omega^2}{c^2}\varepsilon_{\alpha\beta}\right|=0,\qquad \varepsilon_{\alpha\beta}=\delta_{\alpha\beta}+\frac{4\pi i}{\omega}\sigma_{\alpha\beta}. \qquad (5)$$

In a non-uniform medium this is, of course, not the case—plane waves are not eigenfunctions of the linear Eq. (4) and the solvability condition does not have the form of the dispersion Eq. (5) neither for the tensor $\varepsilon_{\alpha\beta}(\mathbf{k},\omega)$ nor for

$$\varepsilon_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k},\omega)=\delta_{\alpha\beta}+\frac{4\pi i}{\omega}\sigma_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k},\omega). \qquad (6)$$

However, in the case of a weakly non-uniform medium, when the wavelength is much smaller than the characteristic scale of the inhomogeneity

$$\mu\approx 1/kL\ll 1,\quad L\sim|\sigma^0|/|\partial\sigma^0/\partial\mathbf{r}| \qquad (7)$$

and its damping (or growth rate) relatively small:

$$p=|\varkappa|/|\mathbf{k}|\approx|\varepsilon^{aH}|/|\varepsilon^{H}|\ll 1,\quad \varepsilon_{\alpha\beta}=\varepsilon_{\alpha\beta}{}^{H}+i\varepsilon_{\alpha\beta}{}^{aH} \qquad (8)$$

($\varepsilon^{H}_{\alpha\beta}$ is the Hermitean and $\varepsilon^{aH}_{\alpha\beta}$ the anti-Hermitean part of the $\varepsilon_{\alpha\beta}$ tensor, $\boldsymbol{\mu}=\operatorname{Im}\ \mathbf{k}$), we can construct the eigenfunctions of Eq. (4) and find the condition for its solvability. It will be shown below that this condition can be written in the form of the dispersion Eq. (5) for a function $\varepsilon^{\text{eff}}_{\alpha\beta}(\mathbf{r},\mathbf{k},\omega)$. We shall call this function the effective dielectric permittivity of a non-uniform plasma.

It is natural to look for the eigenfunctions of Eq. (4) in a weakly non-uniform plasma in the form of a wavepacket

$$E_\beta(\mathbf{r})=E_\beta{}^0(\mathbf{r})e^{i\psi(\mathbf{r})}. \qquad (9)$$

We neglect here the reflection of waves; this is valid with exponential accuracy by virtue of condition (7). Substituting expression (9) into (4) we get

$$j_\alpha(\mathbf{r},\omega)=\frac{1}{(2\pi)^3}\iint d\mathbf{k}\,d\mathbf{r}'\exp\{i\psi(\mathbf{r}')+i\mathbf{k}(\mathbf{r}-\mathbf{r}')\}$$
$$\times\sigma_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k})E_\beta{}^0(\mathbf{r}'). \qquad (10)$$

When integrating over $d\mathbf{k}$ in (10) the main contribution to $j_\alpha$ under the conditions (7), (8) comes from values of the wavevector close to $\mathbf{k}_0(r)\equiv\nabla\Psi(\mathbf{r})$. It is therefore convenient to expand the quantity $\sigma^0_{\alpha\beta}(\mathbf{k})$ near $\mathbf{k}_0(r)$ in a power series in $\mathbf{k}-\mathbf{k}_0$ (we assume that $\sigma^0_{\alpha\beta}(\mathbf{k})$ does not have a singularity close to the point $\mathbf{k}_0$):

$$\sigma_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k},\omega)=\sigma_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k}_0)+\frac{\partial\sigma_{\alpha\beta}{}^0(\mathbf{k}_0)}{\partial k_i}(k_i-k_{0i})+\frac{1}{2}\frac{\partial^2\sigma_{\alpha\beta}{}^0(\mathbf{k}_0)}{\partial k_i\,\partial k_k}(k_i-k_{0i})(k_k-k_{0k})+\frac{1}{6}\frac{\partial^3\sigma_{\alpha\beta}{}^0(\mathbf{k}_0)}{\partial k_i\,\partial k_k\,\partial k_l}(k_k-k_{0k})(k_i-k_{0i})(k_l-k_{0l})\quad\ldots\quad(11)$$

We further apply the following method of successive approximations for solving Eqs. (4), (10), (11): we first restrict ourselves to one term in the series (11) and to the zeroth order in $p$ on the left-hand side of (4), afterwards to three terms in (11) and the first-order terms in $p$, next to four terms in the expansion and second-order terms in $p$, and so on. In the first approximation Eq. (4) takes the form

$$(k_{0\alpha}k_{0\beta}-k_0{}^2\delta_{\alpha\beta}+\omega^2c^{-2}\varepsilon_{\alpha\beta}{}^{0\text{э}})E_\beta{}^0=0.$$

Its solution determines the magnitude of the vector $\mathbf{k}_0$ which satisfies the dispersion equation

$$\mathrm{Det}|k_{0\alpha}k_{0\beta}-k_0{}^2\delta_{\alpha\beta}+\omega^2c^{-2}\varepsilon_{\alpha\beta}{}^{0\text{э}}(\mathbf{r}\mathbf{k}_0)|=0,\qquad \mathbf{k}_0=\nabla\psi(\mathbf{r}),\quad(12)$$

the same, as for a uniform medium, (5) with $\varepsilon_{\alpha\beta}=\varepsilon^{0\mathrm{H}}_{\alpha\beta}$. As only the Hermitean part of $\varepsilon^0_{\alpha\beta}$ occurs in the dispersion Eq. (12), $\mathbf{k}_0$ and also $\psi(\mathbf{r})$ are real quantities. The amplitude $\mathbf{E}^0(\mathbf{r})$ then turns out to be a complex quantity.

In the second approximation (first order in the parameter $p$ and with three terms in the series (11)) Eq. (4) takes the form

$$\frac{i}{2}E_\beta{}^0\left[\frac{\partial k_{0\beta}}{\partial r_\alpha}+\frac{\partial k_{0\alpha}}{\partial r_\beta}-2\frac{\partial k_{0i}}{\partial r_i}\delta_{\alpha\beta}+\frac{\omega^2}{2c^2}\frac{\partial^2\varepsilon_{\alpha\beta}{}^{0\text{э}}}{\partial k_i\,\partial k_k}\left(\frac{\partial k_{0k}}{\partial r_i}+\frac{\partial k_{0i}}{\partial r_k}\right)\right]+i\frac{\partial E_\beta{}^0}{\partial r_i}\left(k_{0\beta}\delta_{i\alpha}+k_{0\alpha}\delta_{i\beta}-2k_{0i}\delta_{\alpha\beta}+\frac{\omega^2}{c^2}\frac{\partial\varepsilon_{\alpha\beta}{}^0}{\partial k_i}\right)=i\frac{\omega^2}{c^2}\varepsilon^{0\,\text{аэ}}_{\alpha\beta}E_\beta{}^0.\quad(13)$$

The right-hand side of this equation is proportional to the anti-Hermitean part of the $\varepsilon^0_{\alpha\beta}$ tensor. When there is no dissipation of electromagnetic energy, it vanishes and Eq. (13) must lead to the energy conservation law, which in a non-uniform medium (as in a uniform medium) has the form

$$\mathrm{div}\,\mathbf{S}=-\frac{\omega}{8\pi}\varepsilon_{\alpha\beta}{}^{\text{аэ}}E_\beta{}^0E_\alpha{}^{0*},\quad(14)$$

where $\mathbf{S}$ is the electromagnetic energy flux. To find the quantity $\mathbf{S}$ we multiply, as usual, (13) by $E_\alpha^{(0)*}$ and combine it with the complex-conjugate expression. As a result we get

$$\frac{\partial}{\partial r_i}\left[E_\alpha{}^{0*}E_\beta{}^0\left(k_{0\beta}\delta_{i\alpha}+k_{0\alpha}\delta_{i\beta}-2k_{0i}\delta_{\alpha\beta}+\frac{\omega^2}{c^2}\frac{\partial\varepsilon_{\alpha\beta}{}^{0\text{э}}}{\partial k_i}\right)\right]=2\frac{\omega^2}{c^2}\left(\varepsilon^{0\,\text{аэ}}_{\alpha\beta}+\frac{1}{2}\frac{\partial^2\varepsilon_{\alpha\beta}{}^{0\text{э}}}{\partial k_i\,\partial r_i}\right)E_\beta{}^0E_\alpha{}^{0*}.\quad(15)$$

We see that the left-hand side of Eq. (15) contains, in first approximation in the small parameter $\mu$ from (7), a quantity which is proportional to the well known expression for the divergence of the energy flux $\mathbf{S}$:[2]

$$S_i=-\frac{c^2}{16\pi\omega}\left(k_{0\beta}\delta_{i\alpha}+k_{0\alpha}\delta_{i\beta}-2k_{0i}\delta_{\alpha\beta}+\frac{\omega^2}{c^2}\frac{\partial\varepsilon_{\alpha\beta}{}^{0\text{э}}}{\partial k_i}\right)E_\alpha{}^{0*}E_\beta{}^0.$$

However, the right-hand side of (15), which describes absorption or buildup of the wave energy, is not expressed simply in terms of the anti-Hermitean part of the permittivity tensor, as in a uniform medium, but contains a correction $\frac{1}{2}\partial^2\varepsilon^{0\mathrm{H}}_{\alpha\beta}/\partial k_i\partial r_i$. Therefore, on the right-hand side of Eq. (15) there stands the anti-Hermitean part not of the tensor $\varepsilon^0_{\alpha\beta}$, but of the tensor $\varepsilon^{\mathrm{eff}}_{\alpha\beta}$:

$$\varepsilon_{\alpha\beta}^{\mathrm{eff}}=\varepsilon_{\alpha\beta}{}^0+\frac{i}{2}\frac{\partial^2\varepsilon_{\alpha\beta}{}^0}{\partial k_i\,\partial r_i}.\quad(16)$$

In this case Eq. (15) is equivalent to the relation

$$(k_\alpha k_\beta-k^2\delta_{\alpha\beta}+\omega^2c^{-2}\varepsilon_{\alpha\beta}^{\mathrm{eff}})E_\alpha{}^{0*}E_\beta{}^0=0,\qquad \mathbf{k}=\mathbf{k}_0+i\boldsymbol{\varkappa},\quad(17)$$

where we have performed an expansion in powers of the parameter $p=|\varkappa|/|\mathbf{k}|$ and retained only the first two terms in that expansion. From (17) follows the dispersion Eq. (5) for the tensor $\varepsilon^{\mathrm{eff}}_{\alpha\beta}$. The quantity $\varepsilon^{\mathrm{eff}}_{\alpha\beta}$ is thus, in the second approximation considered here, that permittivity tensor $\varepsilon_{\alpha\beta}$ which we must substitute in the usual dispersion Eq. (5) in order to determine the eigenmodes in a non-uniform medium under the conditions (7) and (8).

The need to transform $\varepsilon$ is connected with the fact that in a non-uniform plasma the wave amplitude changes not only due to the anti-Hermitean part of the tensor, as in a uniform medium, but also due to its Hermitean part (change in the group velocity). This fact is automatically taken into account by the energy transfer equation. The energy conservation law (14) therefore plays the role of an additional condition guaranteeing the correct choice of corrections to the permittivity tensor.

One might conclude from Eq. (15) for the transfer of the energy of the electromagnetic oscillations that there arises a correction only to the anti-Hermitean part of the permittivity tensor. However, the construction of the next, third approximation shows that this is not the case and that Eq. (16) is valid both for the anti-Hermitean part and for the Hermitean part (see the left-hand side of Eq. (19)). Moreover, there appears yet one more additional term in Eq. (16). In this approximation it is necessary for the solution of Eq. (4) to take four terms in the expansion (11) and to retain all terms up to second order in $p$. As a result we have

$$\frac{i}{2}E_\beta{}^0\left[\frac{\partial k_{0\beta}}{\partial r_\alpha}+\frac{\partial k_{0\alpha}}{\partial r_\beta}-2\frac{\partial k_{0i}}{\partial r_i}\delta_{\alpha\beta}+\frac{\omega^2}{2c^2}\frac{\partial^2\varepsilon_{\alpha\beta}{}^0}{\partial k_i\,\partial k_k}\left(\frac{\partial k_{0k}}{\partial r_i}+\frac{\partial k_{0i}}{\partial r_k}\right)-\frac{i}{6}\frac{\omega^2}{c^2}\frac{\partial^3\varepsilon_{\alpha\beta}{}^0}{\partial k_i\,\partial k_k\,\partial k_l}\frac{\partial^2k_{0l}}{\partial r_i\,\partial r_k}\right]$$

$$+i\frac{\partial E_\beta{}^0}{\partial r_i}\Big[k_{0\beta}\delta_{i\alpha}+k_{0\alpha}\delta_{i\beta}-2k_{0i}\delta_{\alpha\beta}$$

$$+\frac{\omega^2}{c^2}\frac{\partial \varepsilon_{\alpha\beta}{}^0}{\partial k_i}-\frac{i}{6}\frac{\partial^3\varepsilon_{\alpha\beta}{}^0}{\partial k_k\,\partial k_l\,\partial k_m}\left(\frac{\partial k_{0l}}{\partial r_k}\delta_{im}+\frac{\partial k_{0m}}{\partial r_l}\delta_{ik}+\frac{\partial k_{0k}}{\partial r_m}\delta_{il}\right)\Bigg]$$

$$+\frac{\partial^2 E_\beta{}^0}{\partial r_i\,\partial r_k}\left(\delta_{i\alpha}\delta_{k\beta}-\delta_{\alpha\beta}\delta_{ik}+\frac{1}{2}\frac{\omega^2}{c^2}\frac{\partial^2\varepsilon_{\alpha\beta}{}^0}{\partial k_i\,\partial k_k}\right)=i\frac{\omega^2}{c^2}\varepsilon_{\alpha\beta}^{0\,\mathrm{aa}}E_\beta{}^0. \qquad (18)$$

Moreover, as before, we multiply (18) by $E_\alpha^{0*}$ and add it to the complex conjugate expression. We transform the expression obtained in such a way that the left-hand side has the form of a divergence. To do this we split the quantity $E_\beta^0$ into a real amplitude and a phase $E_\beta^0 = A_\beta e^{i\varphi}$ and we introduce the wavevector $\mathbf{k} = \mathbf{k}_0 + \nabla\varphi$. As a result we get

$$\frac{\partial}{\partial r_i}\left[E_\alpha{}^{0*}E_\beta{}^0\left(k_\beta\delta_{i\alpha}+k_\alpha\delta_{i\beta}-2k_i\delta_{\alpha\beta}+\frac{\omega^2}{c^2}\frac{\partial\varepsilon_{\alpha\beta}^{\mathrm{eff}\;\mathrm{H}}(\mathbf{r},\mathbf{k})}{\partial k_i}\right)\right]$$

$$+\frac{1}{4}\frac{\omega^2}{c^2}\frac{\partial^2}{\partial r_i\,\partial r_k}\left[E_\alpha{}^{0*}E_\beta{}^0\frac{\partial^2\varepsilon_{\alpha\beta}^{\mathrm{eff}\;\mathrm{aH}}(\mathbf{r},\mathbf{k})}{\partial k_i\,\partial k_k}\right]$$

$$=2\frac{\omega^2}{c^2}E_\alpha{}^{0*}E_\beta{}^0\left[\varepsilon_{\alpha\beta}{}^0+\frac{i}{2}\frac{\partial^2\varepsilon_{\alpha\beta}{}^0}{\partial k_i\,\partial r_i}-\frac{1}{8}\frac{\partial^4\varepsilon_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k})}{\partial k_i\,\partial k_k\,\partial r_i\,\partial r_k}\right]^{\mathrm{aa}}. \qquad (19)$$

One should note that in the expression for the electromagnetic energy flux in (19) there occurs the magnitude of the Hermitean part of the effective permittivity tensor (16). The expression in the right-hand side of (19) determines according to (14) to (16) the effective permittivity. We see that in the third approximation there appears a new correction to the expression for the permittivity tensor (cf. (12), (16)).

We now examine how important these corrections are. At first sight it looks as if the first correction to the tensor $\varepsilon_{\alpha\beta}^0$ is of order $\mu$ and the second of order $\mu^2$. This, however, is valid only for the Hermitean part of the tensor and the contribution of non-resonance particles to it. As to the anti-Hermitean part, the picture is completely different here—it depends on the form of the tensor $\varepsilon_{\alpha\beta}^0$ and the contribution from the corrections may be important and even decisive. First of all, if the tensor $\varepsilon_{\alpha\beta}^0$ is Hermitean, it is just the corrections which give the anti-Hermitean part of the $\varepsilon_{\alpha\beta}^{\mathrm{eff}}$ tensor, i.e., they completely determine the damping and growth processes of waves in a non-uniform plasma. In those cases one can usually restrict oneself to the first correction, i.e., to Eq. (16) for $\varepsilon_{\alpha\beta}^{\mathrm{eff}}$.

A more complicated situation arises when the tensor $\varepsilon_{\alpha\beta}^0$ has an anti-Hermitean part which is determined by the resonant particles. In that case, when the scale of the inhomogeneity of $\varepsilon_{\alpha\beta}^0$ becomes comparable to or less than the characteristic scales of the resonance interaction of particles with the wave (and these scales depend significantly on the frequency $\omega$ and the wavevector $\mathbf{k}$ of the perturbations considered) the whole nature of the resonance interaction is transformed and the anti-Hermitean part of the $\varepsilon_{\alpha\beta}^{\mathrm{eff}}$ tensor is correspondingly completely changed. The violation of the conditions for the resonance interaction of the wave with plasma particles occurs also in an external electric field and also in a non-uniform magnetic field because the particles are accelerated, $\mathbf{v} = \mathbf{v}(\mathbf{r})$. It will become clear in what follows that in that case the contribution from the first and second order to the anti-Hermitean part of the $\varepsilon_{\alpha\beta}^{\mathrm{eff}}$ tensor (19) turns out to be of the same order which is a consequence of the exponential character of the phase synchronism in the resonance interaction. In the general case, therefore, it is necessary for the determination of the effective permittivity to extend the expansion and to find the subsequent corrections (19) to the $\varepsilon_{\alpha\beta}^{\mathrm{eff}}$ tensor. It is natural to assume (and this is confirmed by calculations) that the general form of the expansion of $\varepsilon_{\alpha\beta}^{\mathrm{eff}}$ obtained in successive approximations retains the same form as for its first three terms standing in Eq. (19). Then

$$\varepsilon_{\alpha\beta}^{\mathrm{eff}}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n\frac{\partial^{2n}\varepsilon_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k})}{\partial k_i{}^n\,\partial r_i{}^n}. \qquad (20)$$

Hence it follows that the $\varepsilon_{\alpha\beta}^{\mathrm{eff}}$ tensor can be written in the following convoluted integral form:

$$\varepsilon_{\alpha\beta}^{\mathrm{eff}}(\mathbf{r},\mathbf{k},\omega)=\frac{1}{(2\pi)^3}\iint\varepsilon_{\alpha\beta}{}^0(\mathbf{r}+\boldsymbol{\eta}/2,\mathbf{k}',\omega)\times\exp\{i(\mathbf{k}'-\mathbf{k})\boldsymbol{\eta}\}\,d\mathbf{k}'\,d\boldsymbol{\eta}. \qquad (21)$$

One can easily check that (20) and (21) are identical by expanding $\varepsilon_{\alpha\beta}^0$ under the integral in a Taylor series.

The solution found gives a complete answer to the problem we posed. $\varepsilon_{\alpha\beta}^{\mathrm{eff}}$ tensor is in a weakly non-uniform plasma, i.e., under the conditions (7), (8), given by Eqs. (2), (6), (20). This expression for the $\varepsilon_{\alpha\beta}^{\mathrm{eff}}$ tensor is then used in the usual dispersion Eq. (5) to sort out the eigenmodes and to determine for them the real $\mathbf{k}(\mathbf{r})$ and the imaginary $\varkappa(\mathbf{r})$ parts of the wavevector. The electrical field of the wave is after this determined by the usual geometrical optics formulae.

One can simplify Eq. (21) by integrating in it over $d\boldsymbol{\eta}$ and $d\,\mathbf{k}'$. Indeed, we take into account that the quantity $\mathbf{r}-\mathbf{r}'$ in (2) is a function of the coordinate $\mathbf{r}$, the momentum $\mathbf{p}$ and the time difference $t-t'$:

$$\mathbf{r}-\mathbf{r}'=\lambda(\mathbf{r},\mathbf{p},t-t').$$

Then integrating in (21) we get

$$\sigma_{\alpha\beta}^{\mathrm{eff}}=-e^2\int d\mathbf{p}v_\alpha\int_{-\infty}^{t}dt'\exp\{i\omega(t-t')-i\mathbf{k}\boldsymbol{\eta}^*\}$$

$$\mathrm{Det}^{-1}\left|\delta_{ik}-\frac{1}{2}\frac{\partial\lambda_i(\mathbf{r}+\boldsymbol{\eta}^*/2)}{\partial r_k}\right|\left[\left(1-\frac{\mathbf{k}\mathbf{v}'}{\omega}\right)\delta_{\beta s}+k_s v_\beta'/\omega+\frac{i}{2\omega}\frac{\partial}{\partial r_s}v_\beta'-\frac{i\delta_{\beta s}}{2\omega}\frac{\partial}{\partial r_i}v_i'\right]\frac{\partial F}{\partial p_s'}\bigg|_{\mathbf{r}=\mathbf{r}+\boldsymbol{\eta}^*/2}, \qquad (22)$$

where the vector $\boldsymbol{\eta}^*(\mathbf{r},\mathbf{p},t-t')$ is a solution of the following equation:

$$\boldsymbol{\eta}^*=\lambda(\mathbf{r}+\boldsymbol{\eta}^*/2,\mathbf{p},t-t'). \qquad (23)$$

Formula (22) is the final one in which all possible simplifications have been made.

We now discuss the problem of the relation between the permittivity (21) and the complex tensor $\hat{\varepsilon}_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ which is the kernel of the integral connection between the induction and the electrical field strength vectors:[1]

$$D_\alpha(\mathbf{r})=\int\hat{\varepsilon}_{\alpha\beta}(\mathbf{r},\mathbf{r}')E_\beta(\mathbf{r}')\,d\mathbf{r}'.$$

It is well known that in a uniform medium $\hat{\varepsilon}_{\alpha\beta} = \hat{\varepsilon}_{\alpha\beta}(\mathbf{r} - \mathbf{r}')$. In a weakly non-uniform medium one also distinguishes in $\hat{\varepsilon}_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ a "fast" variable $\mathbf{r} - \mathbf{r}'$. For the "slow" variable one then uses two forms: the simple one[4,6] $\hat{\varepsilon}_{\alpha\beta}(\mathbf{r} - \mathbf{r}',\mathbf{r})$ and the symmetric one $\hat{\varepsilon}^{s}_{\alpha\beta}(\mathbf{r} - \mathbf{r}',(\mathbf{r} + \mathbf{r}')/2)$.[5,7]

In our case, from (3),

$$D_\alpha(\mathbf{r}') = \int \varepsilon_{\alpha\beta}{}^0(\mathbf{r}', k, \omega) \exp(i\mathbf{k}\mathbf{r}') E_\beta(\mathbf{k})\, d\mathbf{k}.$$

Consequently, the tensor $\hat{\varepsilon}^0_{\alpha\beta}$ takes the form

$$\hat{\varepsilon}_{\alpha\beta}{}^0(\mathbf{r}'-\mathbf{r}'', \mathbf{r}') = \int \varepsilon_{\alpha\beta}{}^0(\mathbf{r}', \mathbf{k}) \exp\{i\mathbf{k}(\mathbf{r}'-\mathbf{r}'')\}\, d\mathbf{k}. \qquad (24)$$

We now find the Fourier component of the tensor $\hat{\varepsilon}^0_{\alpha\beta}$, symmetrized with respect to $\mathbf{r}$. To do this, we put $(\mathbf{r}' + \mathbf{r}'')/2 = \mathbf{r}$, $\mathbf{r}' - \mathbf{r}'' = \boldsymbol{\eta}$ and hence $\mathbf{r}' = \mathbf{r} + \boldsymbol{\eta}/2$. Then

$$\varepsilon_{\alpha\beta}(\mathbf{r}, \mathbf{k}) = \frac{1}{(2\pi)^3} \int \hat{\varepsilon}^0_{\alpha\beta}\left(\boldsymbol{\eta}, \mathbf{r} + \frac{\boldsymbol{\eta}}{2}\right) e^{-i\mathbf{k}\boldsymbol{\eta}}\, d\boldsymbol{\eta}. \qquad (25)$$

Now comparing (24) and (25) with (21) we see that

$$\varepsilon_{\alpha\beta}(\mathbf{r}, \mathbf{k}) \equiv \varepsilon^{\text{eff}}_{\alpha\beta}(\mathbf{r}, \mathbf{k}). \qquad (26)$$

One should note also a simple connection between the $\hat{\varepsilon}^0_{\alpha\beta}$ tensor and the symmetrized $\hat{\varepsilon}^s_{\alpha\beta}$ tensor:

$$\hat{\varepsilon}^0(\boldsymbol{\eta}, \mathbf{r}+\boldsymbol{\eta}/2) = \hat{\varepsilon}^s(\boldsymbol{\eta}, \mathbf{r}).$$

The effective permittivity tensor $\varepsilon^{\text{eff}}_{\alpha\beta}(\mathbf{r},\mathbf{k},\omega)$ is thus the simply symmetrized with respect to $\mathbf{r}$ Fourier component of the tensor $\hat{\varepsilon}^0_{\alpha\beta}(\mathbf{r}' - \mathbf{r}'',\mathbf{r}')$ which determines the relation between the induction and electrical field strength in a non-uniform plasma, (24), (3), (6). The physical meaning of the symmetrization with respect to $\mathbf{r}$ of the Fourier transformation in the fast variable $\mathbf{r}' - \mathbf{r}''$ is elucidated in Fig. 1. The vectors $\mathbf{r}'$ and $\mathbf{r}''$ are on both sides of the given vector $\mathbf{r}$ such that their half-sum is always constant and equal to $\mathbf{r}$. For such a symmetrization the two vectors $\mathbf{r}'$ and $\mathbf{r}''$ turn out to be equivalent: interchanging $\mathbf{r}'$ and $\mathbf{r}''$ only changes the sign of $\boldsymbol{\eta}$ without changing the second argument of $\hat{\varepsilon}^0$. Thanks to this the Fourier transformation of the tensor $\hat{\varepsilon}^0$ in which the effect of the inhomogeneity on the particle motion is taken into account proceeds as in the uniform case, and $\mathbf{r}$ plays simply the role of a parameter. The $\varepsilon^{\text{eff}}_{\alpha\beta}$ tensor, therefore, possesses, as in the uniform case, the following symmetry properties: under time reversal $(t \to -t)$ the wave with an inverted front $(\omega \to -\omega, \mathbf{k} \to -\mathbf{k})$ i.e., moving in the opposite direction, gives the same response in the medium as the initial one (apart from a permutation of the indices $\varepsilon^{\text{eff}}_{\alpha\beta}(t \to -t, -\omega, -\mathbf{k}) = \varepsilon^{\text{eff}}_{\alpha\beta}(\omega,\mathbf{k})$). This leads to a correct formulation of the energy conservation law. One verifies easily that the initial tensor $\varepsilon^0_{\alpha\beta}$ of (2), (6) does not possess this symmetry. One can show that the symmetrization performed here is the only way to construct in a non-uniform medium the $\varepsilon^{\text{eff}}_{\alpha\beta}(\mathbf{r},\mathbf{k},\omega)$ tensor from the $\varepsilon^0_{\alpha\beta}(\mathbf{r},\mathbf{k},\omega)$ tensor. Thus, the construction of the symmetrized $\varepsilon_{\alpha\beta}(\mathbf{r},\mathbf{k})$ tensor (25) which is the same as $\varepsilon^{\text{eff}}_{\alpha\beta}(\mathbf{r},\mathbf{k})$ confirms the correctness of Eq. (22) which we have obtained. Yet another proof will be demonstrated in §4 where we show that only the tensor $\varepsilon^{\text{eff}}_{\alpha\beta}$ guarantees complete satisfaction of the Einstein relations which determine the connection between the stimulated and spontaneous radiations.

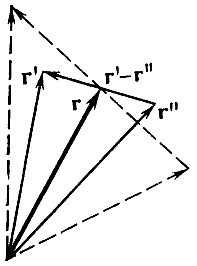


FIG. 1. Position of the vectors $\mathbf{r}'$ and $\mathbf{r}''$ for the symmetrization of the tensor $\hat{\varepsilon}^0_{\alpha\beta}(\mathbf{r}' - \mathbf{r}'',\mathbf{r}')$ of (24) with respect to the radius vector $\mathbf{r}$. The vectors $\mathbf{r}'$ and $\mathbf{r}''$ are positioned always such that their half-sum equals $\mathbf{r}$.

In conclusion we compare the solution obtained with the results of earlier papers. Pitaevskiĭ[8] was the first to obtain a correction of the form (16) for the permittivity in a medium which varied slowly in time and Kadomtsev[7] the first to get it for a medium weakly non-uniform in space. At the same time in papers on the theory of a non-uniform plasma (see the monograph, Ref. 3) basically either the quantity $\varepsilon^0(\mathbf{r},\mathbf{k},\omega)$ was used—the response of a non-uniform medium to a plane wave—or the so-called local permittivity tensor $\varepsilon^{\text{loc}}$. The $\varepsilon^{\text{loc}}(\mathbf{r},\mathbf{k},\omega)$ tensor is obtained from the uniform tensor $\varepsilon(\mathbf{k},\omega)$ by a direct replacement of the uniform plasma parameters—the density $N$, the electron and ion temperatures, $T_e$ and $T_i$, and the magnetic field $\mathbf{B}_0$ by the non-uniform $N(\mathbf{r})$, $T_e(\mathbf{r})$, $T_i(\mathbf{r})$, and $\mathbf{B}_0(\mathbf{r})$. Comparing $\varepsilon^{\text{loc}}$ and $\varepsilon^0$ we find that in the conditions (7), (8) the tensors $\varepsilon^{\text{loc}}$ and $\varepsilon^0$ guarantee the correct transition to the limit of a uniform medium and give the correct main term in the Hermitean part of the tensor. The non-Hermitean part of the permittivity tensor like the corrections to the Hermitean part, under conditions when the non-uniformity is important, may turn out to be incorrect. This means that under well defined conditions the use of the tensors $\varepsilon^{\text{loc}}$ or $\varepsilon^0$ may lead to incorrect expressions for the growth and damping rates of the waves, to non-observance of the energy conservation law, and to the appearance of false instabilities. The use of the $\varepsilon^{\text{eff}}_{\alpha\beta}$ tensor with a correction of the form (16) may also lead to the same errors under circumstances when the effect of the non-uniformity on the resonance interaction between the waves and the plasma particles is important. This will be shown in more detail using the examples which we consider in the following sections.

We considered here for the sake of simplicity only one species of particles. When there are several species present it is necessary to simply sum over them when determining the quantity $\sigma^0_{\alpha\beta}$. The results obtained can also be generalized to the case when there is a weak non-stationarity present or to the case of a few collisions.

## §2. ISOTROPIC PLASMA. DRIFT OSCILLATIONS

In the present section we consider two simple examples when the anti-Hermitean part of the permittivity does not have a resonant character or when the non-uniformity does not affect the resonant interaction between particles and waves. In that case we can restrict ourselves to Eq. (16) for $\varepsilon^{\text{eff}}$.

### Long-wavelength ion-sound oscillations

One-dimensional long-wavelength ion-sound oscillations of a non-isothermal plasma are described by Korteweg–de Vries equations:[9]

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\beta\frac{\partial^3 v}{\partial x^3}=0. \qquad (27)$$

Equation (27) is written down in a system of coordinates moving with the sound velocity $c_s=(T_e/M)^{1/2}$ ($T_e$ is the electron temperature, $M$ the ion mass, $v$ the hydrodynamic ion velocity, $\beta=c_s\mathscr{D}^2/2$ is a small parameter which determines the contribution from the dispersion terms, and $\mathscr{D}$ is the Debye radius).

In a uniform medium ($v_0=\text{const}$) Eq. (27) corresponds to a permittivity

$$\varepsilon=1-(v_0k-\beta k^3)/\omega. \qquad (28)$$

The dispersion relation for longitudinal ion-sound oscillations then takes the form

$$\omega=v_0k-\beta k^3.$$

The permittivity $\varepsilon$ does not contain an anti-Hermitean part—the oscillations in a uniform plasma are neither damped nor do they grow.

We now consider a large-scale non-uniform flow, i.e., we assume that $v_0=v_0(x)$ where according to (7) the quantity $v_0$ changes appreciably only on a scale $L\gg\mathscr{D}/(v_0/v_s)^{1/2}$. From (27) we find

$$\varepsilon^0=1-\frac{v_0k-\beta k^3}{\omega}+\frac{i}{\omega}\frac{\partial v_0}{\partial x}. \qquad (29)$$

Using then Eq. (16) we get

$$\varepsilon^{\text{eff}}=1-\frac{v_0k-\beta k^3}{\omega}+\frac{i}{2\omega}\frac{\partial v_0}{\partial x}. \qquad (30)$$

The presence of an anti-Hermitean part in $\varepsilon^{\text{eff}}$ leads to the appearance of a damping of the oscillations ($\gamma$ the temporal and $\varkappa$ the spatial damping):

$$\gamma=-\frac{1}{2}\frac{\partial v_0}{\partial x},\qquad \varkappa=\frac{1}{2}\frac{\partial v_0}{\partial x}\Big/(v_0-3\beta k^2). \qquad (31)$$

Hence it follows that in the region of a rarefaction wave, where $\partial v_0/\partial x>0$ the flow is stable, but in the region of a compression wave ($\partial v_0/\partial x<0$) it is unstable. It is clear that the growth rate of the instability is completely determined by the inhomogeneity.

The equation for the wave energy transfer in the case (27) which we consider has the form

$$\frac{\partial a^2}{\partial t}+\frac{\partial}{\partial x}(Va^2)+\frac{\partial v_0}{\partial x}a^2=0,\qquad V=v_0-3\beta k^2. \qquad (32)$$

Here $a$ is the amplitude of the oscillations and $V$ is the group velocity. The change in the wave energy described by the last term in (32) completely corresponds to the growth rate (31). The same term but with the opposite sign appears also in the equation for the energy of the large-scale motion when we take into account its effect on the wave. The growth rate (31) and hence also the permittivity (30) thus correctly describe the process of the exchange of energy between the large-scale motion and the wave.

We note also that using the local approximation $\varepsilon=\varepsilon^{\text{loc}}$ (Eq. (28) with $v_0=v_0(x)$) does not give an instability as there is no anti-Hermitean part in $\varepsilon^{\text{loc}}$. Using, however the quantity $\varepsilon^0$ of (29) which is the response to a plane wave leads to a change in the wave amplitude which is faster than what follows from (31). It is clear from (32) that the change in amplitude consists of two parts: one part is connected with a change in the group velocity $V$ and the other with the damping (or growth) rate of the oscillations. In the case considered here both these parts turn out to be the same.

### Drift oscillations in a uniform magnetic field

We now consider the example of an anisotropic medium—a collisionless plasma in a uniform magnetic field $\mathbf{B}_0$ with non-uniform parameters $N$ and $T$. We direct the field $\mathbf{B}_0$ along the $z$-axis and assume, as usual, that the plasma particle distribution function depends on a single coordinate $x$ in an orthogonal direction. We shall assume that the size of the non-uniformity $L$ is much larger than either the wavelength $k^{-1}$ of the oscillations considered or the particle Larmor radius $\rho_B$.

In that case Eq. (22) for the effective permittivity can be strongly simplified. Indeed, in a uniform magnetic field the particle trajectory does not depend on the coordinates so that the quantity $\boldsymbol{\eta}^*$ in (23) is the same as $\mathbf{r}-\mathbf{r}'$. This means that the inhomogeneity does not affect the resonance interaction of the particles with the oscillations and we can restrict ourselves in Eq. (20) for $\varepsilon^{\text{eff}}_{\alpha\beta}$ to a finite number of terms. To first order in the quantity $L^{-1}$ we have (16):

$$\varepsilon^{\text{eff}}_{\alpha\beta}=\varepsilon_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k})+\frac{i}{2}\frac{\partial^2\varepsilon_{\alpha\beta}{}^0}{\partial k_x\,\partial x}.$$

The tensor $\varepsilon^0_{\alpha\beta}(\mathbf{r},\mathbf{k})$ is in our case well known[10,11] and is used to consider drift oscillations in a non-uniform plasma. It contains both the magnitude of the local permittivity and corrections taking into account drift effects which corresponds to an expansion in the ratio of the particle Larmor radius to the inhomogeneity length, $\rho_B/L$. In the same order in $L^{-1}$ we can in (16) replace the term $(i/2)\partial^2\varepsilon^0_{\alpha\beta}/\partial k_x\partial x$ by $(i/2)\partial^2\varepsilon^{\text{loc}}_{\alpha\beta}/\partial k_x\partial x$ where $\varepsilon^{\text{loc}}_{\alpha\beta}(\mathbf{r},\mathbf{k})$ is the local permittivity tensor which is well known from the theory of oscillations in a uniform plasma.

We show that taking the correction $(i/2)\partial^2\varepsilon^{\text{loc}}_{\alpha\beta}/\partial k_x\partial x$ into account is important. We confine ourselves here to solely the quantity

$$\delta\varepsilon=\frac{i}{2}\frac{k_\alpha k_\beta}{k^2}\frac{\partial^2\varepsilon_{\alpha\beta}{}^{\text{loc}}}{\partial k_x\,\partial x},$$

which is necessary to obtain the dispersion equation for potential oscillations

$$\delta\varepsilon=\frac{4\pi ie^2}{k^2\omega}\frac{k_x}{k_\perp{}^2}\frac{\partial}{\partial x}\Bigg[\int d\mathbf{p}\,(\omega-k_zv_z)\frac{1}{v_z}\frac{\partial F}{\partial p_z}-\sum_{n=-\infty}^{\infty}\int\frac{J_n{}^2(k_\perp v_\perp/\omega_c)}{\omega-k_zv_z-n\omega_c}\times\left(2\omega-k_zv_z+\omega\varepsilon_\perp\frac{\partial}{\partial\varepsilon_\perp}\right)\left[k_z\frac{\partial F}{\partial p_z}+(\omega-k_zv_z)\frac{\partial F}{\partial\varepsilon_\perp}\right]d\mathbf{p}\Bigg], \qquad (33)$$

$J_n(\xi)$ is Bessel function, $\varepsilon_\perp$ is the energy of the transverse motion of the particles in the magnetic field, $k_\perp^2=k_x^2+k_y^2$. We see that the magnitude of the correction (33) is proportional to the wave vector component along the inhomogeneity $x$-axis as should be the case as a wave propagating at right angles to the inhomogeneity gradient does not feel it. For a Maxwellian plasma with a temperature $T(x)$ Eq. (33) takes the form

$$\delta\varepsilon = i\frac{T}{k^2\mathscr{D}^2}\frac{k_x}{k_\perp^2}\left[\frac{\partial \ln N}{\partial x}+\frac{\partial T}{\partial x}\frac{\partial}{\partial T}\right]\frac{\omega}{|k_z|v_T} \times \sum_{n=-\infty}^{\infty} Z\left(\frac{\omega-n\omega_c}{|k_z|v_T}\right)\frac{\partial}{\partial T}e^{-\xi}I_n(\xi); \quad (34)$$

here

$$\mathscr{D}^2=T/4\pi Ne^2, \quad v_T=(2T/m)^{1/2}, \quad \xi=k_\perp^2T/M\omega_c^2;$$

$$Z(\xi)=i\pi^{1/2}e^{-\xi^2}\left(1+\frac{2i}{\pi^{1/2}}\int_0^\xi e^{t^2}dt\right),$$

$Z(\xi)$ is a Kramp function, $I_n(\xi)$ a modified Bessel function. The correction to the permittivity (34) has a structure similar to the usual drift correction[10] with the only difference that the drift frequency $\omega^*$ proportional, say, to the gradient of the particle density, depends only on the quantity $k_y$:

$$\omega^*=\frac{k_yT}{M\omega_c}\frac{\partial \ln N}{\partial x},$$

whereas in (34) there appears a frequency proportional to $k_x$:

$$\tilde{\omega}^*=\frac{k_xT}{M\omega_c}\frac{\partial \ln N}{\partial x}.$$

As a result the dispersion equation describing drift oscillations in a non-uniform plasma depends not only on the magnitude of the wavevector $k_y$ at right angles to the inhomogeneity gradient, but also on the component of the wave vector along the gradient itself, $k_x$. This can led to an appreciable change in the drift instability.

As an example we consider the so-called drift-cyclotron instability.[11] It is an instability of oscillations close to the harmonics of the ion cyclotron frequency $\omega\approx n\omega_{ci}$ when this frequency approaches the ion drift frequency: $\omega^*\approx n\omega_{ci}$. This is possible under the conditions $(k_\perp\rho_{Bi})^2\approx nLk_\perp\gg 1$. We put $k_z=0$ and $\nabla T=0$ and we get, using (34) the following dispersion equation:

$$\left(1-\frac{\omega^*}{\omega}\right)\left[1-\frac{\omega}{(\omega-n\omega_{ci})\alpha}\right]+i\pi\frac{\tilde{\omega}^*\omega}{\omega_{ci}(\omega-n\omega_{ci})\alpha^3} = -k_\perp^2\rho_{Bi}^2\frac{m}{M}\left(1+\frac{\omega_{ce}^2}{\omega_{pe}^2}\right), \quad (35)$$

$\alpha=(2\pi k_\perp^2T/M\omega_{ci}^2)^{1/2}=(2\pi)^{1/2}k_\perp\rho_{Bi}$, $m$ is the electron mass, $M$ the ion mass, and $\omega_{pe}$ the electron Langmuir frequency. When $k_x=0$, i.e., when $\tilde{\omega}^*=0$, the maximum growth rate of the instability is reached when

$$\frac{\omega^*-n\omega_{ci}}{n\omega_{ci}}=k_\perp^2\rho_{Bi}^2\frac{m}{M}\left(1+\frac{\omega_{ce}^2}{\omega_{pe}^2}\right)+\frac{1}{\alpha}$$

and equal to

$$\frac{\gamma}{n\omega_{ci}}=(2\pi)^{-1/4}\left[k_\perp\rho_{Bi}\frac{m}{M}\left(1+\frac{\omega_{ce}^2}{\omega_{pe}^2}\right)\right]^{1/2}.$$

When one does not take the correction (34) into account this result for constant $k_\perp$ is independent of the magnitude of $k_x$. Taking this correction into account it is clear from (35) that the magnitude of the growth rate decreases with increasing $k_x$ and for a sufficiently large ratio $k_x/k_y$ the instability disappears:

$$\left|\frac{k_x}{k_y}\right|\geqslant 2(2\pi)^{1/2}\frac{1}{n}(k_\perp\rho_{Bi})^{1/2}\left[\frac{m}{M}\left(1+\frac{\omega_{ce}^2}{\omega_{pe}^2}\right)\right]^{1/2}.$$

We see thus that the drift instability is, when the correction (34) is taken into account, not indifferent to the direction of the wavevector **k** whether it is directed along the gradient of the inhomogeneity or at right angles to it. In the given case propagation strictly along the gradient led to the vanishing of the instability, although the range of angles $k_y/k_x$ where correction (34) is important is small. It is, however, clear that each concrete example needs its own considerations.

## §3. LANGMUIR OSCILLATONS

We now consider an isotropic plasma in which there is no external magnetic field. We shall assume that the nonuniformity of the plasma is sustained by a large-scale electrical field with a potential $\varphi(\mathbf{r})$ which changes along a length of the order $L$ which is much longer than the wavelength of the perturbations considered, (7). We assume that the particle distribution function $F(\mathbf{p},\mathbf{r})$ depends only on their energy $\varepsilon=p^2/2m+e\varphi(\mathbf{r})$.

We determine the magnitude of the effective permittivity

$$\varepsilon_{\alpha\beta}^{\text{eff}}=\delta_{\alpha\beta}+\frac{4\pi i}{\omega}\sigma_{\alpha\beta}^{\text{eff}}.$$

It is convenient in this case to split off in Eq. (22) the main part $\varepsilon_{0\alpha\beta}^{\text{eff}}$ and the correction of order $\mu$:

$$\varepsilon_{\alpha\beta}^{\text{eff}}=\varepsilon_{0\alpha\beta}^{\text{eff}}+\varepsilon_{1\alpha\beta}^{\text{eff}},$$

$$\varepsilon_{0\alpha\beta}^{\text{eff}}=\delta_{\alpha\beta}+\frac{4\pi e^2}{\omega}\int v_\alpha v_\beta\Phi(\mathbf{r},\mathbf{v},\mathbf{k},\omega)\frac{\partial F(\mathbf{r}+\boldsymbol{\eta}^*/2)}{\partial\varepsilon}d\mathbf{p},$$

$$\varepsilon_{1\alpha\beta}^{\text{eff}}=\frac{4\pi e^2}{\omega}\int(v_\alpha'-v_\alpha)v_\beta\Phi\frac{\partial F}{\partial\varepsilon}d\mathbf{p}. \quad (36)$$

Here $\Phi$ is a function describing the main dispersion properties of the non-uniform plasma:

$$\Phi(\mathbf{r},\mathbf{v},\mathbf{k},\omega)=-i\int_{-\infty}^{t}dt'\exp\{i\omega(t-t')-i\mathbf{k}\boldsymbol{\eta}^*\}. \quad (37)$$

The quantity $\boldsymbol{\eta}^*=\boldsymbol{\eta}^*(\mathbf{r},\mathbf{v},t-t')$ is given by Eq. (23). To find it we must consider the particle trajectories. We consider here particles of charge $e$ and mass $m$ moving in the potential field $\varphi(\mathbf{r})$. Hence

$$\mathbf{r}-\mathbf{r}'=\mathbf{v}(t-t')-\frac{\mathbf{a}}{2}(t-t')^2+\frac{\dot{\mathbf{a}}}{6}(t-t')^3-\ldots,$$

$$\mathbf{a}=-\frac{e}{m}\frac{\partial\varphi}{\partial\mathbf{r}}, \quad \dot{\mathbf{a}}=-\frac{e}{m}\frac{\partial^2\varphi}{\partial\mathbf{r}\,\partial r_k}v_k. \quad (38)$$

As the value of the distribution function in the point $\mathbf{r}+\boldsymbol{\eta}^*/2$ occurs in Eq. (36) it is convenient for us to change from the momentum **p** and the velocity **v** to the momentum and velocity which a particle has also in the point $\mathbf{r}+\boldsymbol{\eta}^*/2$, i.e., $\mathbf{w}=\mathbf{v}(\mathbf{r}+\boldsymbol{\eta}^*/2)$. Substituting next the expansion (38) into (23) we find

$$\eta^{*}=\mathbf{v}(t-t')+\frac{1}{24}\dot{\mathbf{a}}(t-t')^3-\frac{1}{8}\mathbf{a}\frac{(\mathbf{av})}{v^2}(t-t')^3+\dots \quad (39)$$

We have renamed $\mathbf{w}$ by $\mathbf{v}$ since

$$F(\mathbf{w},\ \mathbf{r}+\eta^{*}/2)=F(\mathbf{v},\ \mathbf{r}).$$

It is clear from (39) that $\eta^*$ does not contain a term proportional to $(t-t')^2$ and moreover is generally, since one can show, an odd function of $t-t'$. This means that the quantity $\omega(t-t')-\mathbf{k}\eta^*$ occurring in (37) will also be an odd function of $t-t'$ which as already noted in §1 is a consequence of the symmetrization of the effective permittivity tensor (25): under time reversal $t\to -t$ the dispersion function (37) does not change for the inverted wave ($\omega\to-\omega,\mathbf{k}\to-\mathbf{k}$). It is also important that in $\eta^*$ occur only terms which are proportional to $\dot{\mathbf{a}}$ or $a^2$—this corresponds directly to the well known fact that the emission of an accelerated particle is proportional either to the derivative $\dot{\mathbf{a}}$ or to the square of the acceleration.[12]

Substituting (39) into (37) and integrating we find

$$\Phi(\mathbf{r},\ \mathbf{v},\ \mathbf{k},\ \omega)=2\pi|\xi|[\mathrm{Gi}(z)-i\,\mathrm{Ai}(z)],\quad z=2(\omega-\mathbf{kv})\xi, \quad (40)$$

where

$$\xi=\left(3\mathbf{ka}\frac{(\mathbf{av})}{v^2}-\mathbf{k}\dot{\mathbf{a}}\right)^{-1/3},$$

$\mathrm{Ai}(z)$ is the Airy function and $\mathrm{Gi}(z)$ a function related to it:

$$\mathrm{Ai}(z)+i\,\mathrm{Gi}(z)=\frac{1}{\pi}\int_0^\infty \exp\left\{izt+\frac{it^3}{3}\right\}dt.$$

We give the asymptotic expansions of these functions[13] which are important for what follows. When $|z|\gg 1$

$$\mathrm{Ai}+i\,\mathrm{Gi}=\begin{cases}\dfrac{i}{\pi z}\left(1+\dfrac{1\cdot 2}{z^3}+\dfrac{1\cdot2\cdot4\cdot5}{z^6}+\dots\right), & \pi>\arg z>-\dfrac{\pi}{3}\\ \pi^{-1/2}z^{-1/4}\exp\left(-\dfrac{2}{3}z^{3/2}\right)+\dfrac{i}{\pi z}\left(1+\dfrac{1\cdot2}{z^3}+\dfrac{1\cdot2\cdot4\cdot5}{z^6}+\dots\right), & -\dfrac{\pi}{3}>\arg z\geqslant-\pi\end{cases}. \quad (41)$$

Moreover, on the real axis when $z>0$ the function $\mathrm{Ai}(z)$ is exponentially small:

$$\mathrm{Ai}(z)={}^1/_2\pi^{-1/2}z^{-1/4}\exp(-{}^2/_3z^{3/2}),\quad \mathrm{Im}\,z=0,\ \mathrm{Re}\,z>0. \quad (42)$$

In a uniform medium, i.e., when $\dot{\mathbf{a}}$, $\mathbf{a}\to 0$, the quantity $z\to\infty$ and Eq. (40) goes over into the well known dispersion function of a uniform isotropic plasma:[6]

$$\Phi(\mathrm{v},\ \mathrm{k},\ \omega)=\mathrm{P}/(\omega-\mathbf{kv})-i\pi\delta(\omega-\mathbf{kv}); \quad (43)$$

$P$ denotes that the integral over the velocities is taken in the sense of a principal value integral. The imaginary part of $\Phi$ is connected with Cherenkov emission of plasma waves by resonance particles $\omega=\mathbf{kv}$.

In a non-uniform plasma the form of the dispersion function (40) depends significantly on the particle velocity. For non-resonant particles in the region $(\omega-\mathbf{k}\cdot\mathbf{v})\gg\xi^{-1}$ the parameter $z\gg 1$. Accordingly the real part of $\Phi$ is here according to (41) practically the same as in a uniform plasma, (43). On the other hand, in the imaginary part there appears a finite term (42) although the Cherenkov emission must be absent when $\omega>\mathbf{k}\cdot\mathbf{v}$. The physical emergence of the imaginary part (42) is connected with the fact that particles move in the inhomogeneous field $\varphi(\mathbf{r})$ with an acceleration which must lead to the generation of bremsstrahlung which is not present in a uniform collisionless plasma. Thus, in a non-uniform medium there appears apart from Cherenkov radiation also bremsstrahlung with an intensity which in the non-resonance region is exponentially small:

$$\mathrm{Im}\,\Phi\sim\exp\left\{-8\omega^{3/2}/3\left|6\mathbf{ka}\frac{\mathbf{av}}{v^2}-2\mathbf{k}\dot{\mathbf{a}}\right|^{1/2}\right\},$$

which is in accordance with Ref. 14.

In the region $\omega<\mathbf{k}\cdot\mathbf{v}$ the function $\mathrm{Gi}(z)$ oscillates near the same asymptotic value $1/\pi z$ and the function $\mathrm{Ai}(z)$ oscillates, changing sign which corresponds to an alternation of bands in which there is emission and absorption of waves. These oscillations are caused by the phase desynchronization of the Cherenkov radiation connected with the inhomogeneity of the medium and leading to the formation of interference bands. The dispersion function in a non-uniform plasma is thus significantly different from that of a uniform plasma which is connected both with the appearance of a new kind of radiation (in our case bremsstrahlung) and with a change in the phase structure of the radiation existing in the uniform medium.

We now consider high-frequency Langmuir oscillations with a dispersion which is determined by the electron component of the plasma. The magnitude of the longitudinal permittivity is given by Eq. (36)

$$\varepsilon_0^{\mathrm{eff}}=\frac{k_\alpha k_\beta\varepsilon_{0\alpha\beta}^{\mathrm{eff}}}{k^2}=1+\frac{4\pi e^2}{\omega}\int(\mathbf{kv})^2\Phi(\mathbf{r},\mathbf{v},\mathbf{k},\omega)\frac{\partial F}{\partial\varepsilon}d\mathbf{p}, \quad (44)$$

in which, as usual, the dispersion function (40) is averaged over the electron velocities. We first consider an equilibrium Maxwell distribution (for constant temperature $T_e$)

$$F(\varepsilon)=\frac{N_0}{(2\pi T_e/m)^{3/2}}\exp\left(-\frac{\varepsilon}{T_e}\right).$$

As the phase velocity of the oscillations $v_{\mathrm{ph}}=\omega/k$ is much higher than the thermal velocity $v_T=(2T_e/m)^{1/2}$ the main contribution to the real part of $\varepsilon_0^{\mathrm{eff}}$ comes from particles for which the quantity $z\gg 1$. The function $\mathrm{Gi}(z)$ determining the real part of the dispersion equation is then equal to $1/\pi z$. Thus, $\mathrm{Re}\,\varepsilon_0^{\mathrm{eff}}$ is the same as the expression determining $\varepsilon$ in a uniform plasma with the local value of the density:

$$\mathrm{Re}\,\varepsilon_0^{\mathrm{eff}}=\varepsilon^\pi=1-\frac{4\pi e^2N(\mathbf{r})}{m\omega^2}(1+3k^2\mathscr{D}^2).$$

As to the imaginary part of $\varepsilon_0^{\mathrm{eff}}$ determining the damping of the Langmuir waves the main contribution to it comes from resonance particles $v\approx v_{\mathrm{ph}}$ for which $z\approx 1$ and the non-uniformity may turn out to be important:

$$\text{Im}\,\varepsilon_0^{\text{eff}}=2\pi^{1/2}\frac{\omega_p^2}{\omega}v_T^{-3}|\xi|\int_{-\infty}^{\infty}v^2\exp\left(-\frac{v^2}{v_T^2}\right)\times\text{Ai}[2(\omega-\mathbf{kv})\xi]dv. \quad (45)$$

The quantity $\xi$ in (45) equals ($v\gg v_T$)

$$\xi=\left(-\frac{e}{m}(\mathbf{kv})\frac{\partial^2\varphi}{\partial\mathbf{r}\,\partial\mathbf{r}}\right)^{-1/3}=\left[-\frac{v_T^2}{2}k\frac{\partial}{\partial\mathbf{r}}\frac{\partial N/\partial\mathbf{r}}{N}v\right]^{-1/3}.$$

We distinguish here two cases: when the width of the oscillations of the Airy function occurring in (45) $\Delta v\approx 1/k|\xi|$ is much smaller than the thermal velocity and conversely. In the first case

$$v_{\text{ph}}/v_T<(\omega|\xi_{\text{ph}}|)^{3/4}\approx(L/\mathscr{D})^{1/2}(\delta N/N)^{-1/4}$$

($\xi_{\text{ph}}=\xi(v=v_{\text{ph}})$, $L$ is the characteristic scale of the inhomogeneity, $\delta N$ is the magnitude of the inhomogeneity) the imaginary part of $\varepsilon_0^{\text{eff}}$ is practically the same as in a uniform plasma:

$$\text{Im}\,\varepsilon_0=2\pi^{1/2}\frac{\omega_p^2}{\omega^2}\left(\frac{v_{\text{ph}}}{v_T}\right)^3\exp\left(-\frac{v_{\text{ph}}^2}{v_T^2}\right),$$

i.e., the damping of the Langmuir waves is the usual Landau damping, $\gamma=-(\omega/2)\text{Im}\,\varepsilon_0$. In the opposite case $v_{\text{ph}}/v_T>(\omega|\xi_{\text{ph}}|)^{3/4}$, i.e., for sufficiently long-wavelength oscillations $k<(L\mathscr{D})^{-1/2}(\delta N/N)^{1/4}$, the Landau damping is significantly changed in a non-uniform plasma. Calculating the integral in (45) by the steepest descent method shows that the main contribution to the Landau damping in this case comes not from locally resonant particles $v\approx v_{\text{ph}}$ but (owing to the smearing out of the resonance by the inhomogeneity) from particles with appreciably smaller velocities $v_T\ll v_0<v_{\text{ph}}$:

$$v_0=5^{1/2}\cdot 3^{-2/5}\cdot 2^{1/5}v_T(\omega|\xi_{\text{ph}}|)^{2/5}\left(\frac{v_{\text{ph}}}{v_T}\right)^{1/5}\approx v_T\left(\frac{\mathscr{D}^2}{L^2}\frac{\delta N}{N}\right)^{-1/5}\left(\frac{v_{\text{ph}}}{v_T}\right)^{1/5}.$$

In this case

$$\text{Im}\,\varepsilon_0^{\text{eff}}=2\pi^{1/2}\cdot 3\cdot 10^{-1/2}\frac{\omega_p^2}{\omega^2}\left(\frac{v_0}{v_T}\right)^3\exp\left(-\frac{v_0^2}{v_T^2}\right)$$

and the Landau damping in a non-uniform medium becomes for sufficiently long-wavelength oscillations significantly larger than in a uniform plasma and it also changes its dependence on the wavevector:

$$\gamma\sim\exp[-k^{-2/5}\mathscr{D}^{-6/5}L^{4/5}(\delta N/N)^{-2/5}].$$

The inhomogeneity of the plasma also changes the growth rate of the kinetic beam instability. The result depends greatly in that case on the parameter

$$a=2|\xi_b|\omega_p v_{Tb}/v_b \quad (46)$$

($\xi_b=\xi(v=v_b)$, $v_b$ is the beam velocity, $v_{Tb}$ the thermal spread). When $a\gg 1$, i.e., for a sufficiently wide beam

$$v_{Tb}\gg v_b(L/\mathscr{D})^{-2/3}(\delta N/N)^{1/3},$$

the non-uniformity is unimportant and the growth rate of the beam instability is determined in the same way as in a uniform medium. In the opposite limiting case, $a\ll 1$, the smearing out of the Cherenkov resonance becomes larger than the beam width. The growth rate then takes on an oscillating nature:

$$\text{Im}\,\varepsilon=2\pi\frac{N_b}{N}\left(\frac{v_b}{v_{Tb}}\right)^2a^2\frac{kv_b}{\omega_p}\text{Ai}'[2(\omega-kv_b)\xi_b], \quad (47)$$

periodically changing sign and considerably decreasing in magnitude:

$$|\gamma_b|/\gamma_{b\,max}\approx a^2\ll 1;$$

here

$$\gamma_{b\,max}=\frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}\omega_p\frac{N_b}{N}e^{-1/2}\left(\frac{v_b}{v_{Tb}}\right)^2$$

is the maximum value of the beam instability growth rate in a uniform plasma. For a value of the parameter $a\sim 1$ the way the beam instability growth rate depends on the magnitude of the wave vector of the oscillations considered $\delta k=(kv_b/\omega_p-1)v_b/v_{Tb}$ is shown in Fig. 2. We see that not only the structure of the unstable regions in the wave-vector space is changed, but also the magnitude of the growth rate which tends to zero when the inhomogeneity of the plasma is sufficiently large.

In concluding this section we give the expression for the correction to the permittivity tensor of an isotropic non-uniform plasma which is proportional to $\mu$:[7]

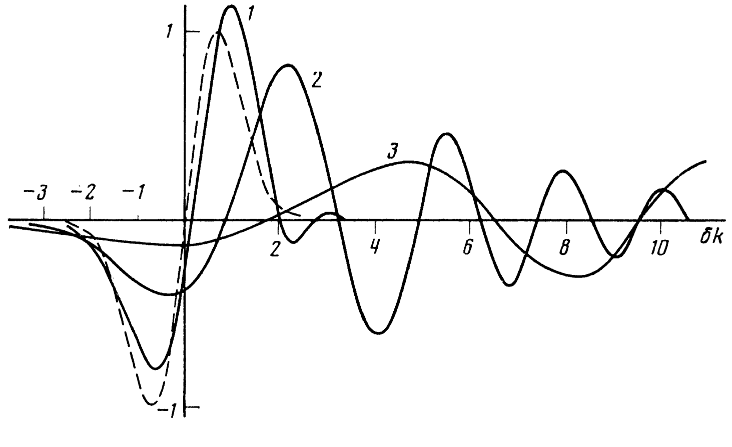


FIG. 2. The growth rate of the kinetic beam instability $\gamma_b$ as function of the wave number $k$. Along the horizontal axis is plotted the quantity $\delta k=(kv_b/\omega_p-1)v_b/v_{Tb}$, along the vertical axis $\gamma_b/\gamma_{b\,\max}$; $\gamma_{b\,\max}$ is the maximum value of the growth rate of the beam instability in a uniform plasma. Curve 1 corresponds to the value of the parameter $a$ of (46) equal to 2; curve 2: $a=1$; curve 3: $a=0.5$. The dashed curve depicts the same function in a uniform plasma ($a\gg 1$). The change in sign of $\xi$ of (40) corresponds to a charge in the signs of $\delta k$ and $\gamma_b$ ($\xi\to-\xi$, $\delta k\to-\delta k$, $\gamma_b\to-\gamma_b$).

$$\varepsilon_{1\alpha\beta}^{\text{eff}}=\frac{2\pi ie^2}{m\omega}\left(\frac{\partial}{\partial k_\alpha}\frac{\partial}{\partial r_\beta}-\frac{\partial}{\partial k_\beta}\frac{\partial}{\partial r_\alpha}\right)\int F(\varepsilon)\Phi\,d\mathbf{p}. \quad (48)$$

From (48) there follows an important fact: if the quantity $\Phi(\mathbf{r},\mathbf{v},\mathbf{k},\omega)$ is real, the correction $\varepsilon$ is Hermitean. This means that taking the inhomogeneities correctly into account does not lead to an incorrect instability either in the next order in the parameter $\mu$. The instability is determined only by the real processes of wave emission by a moving charged particle which in our case are Cherenkov radiation and bremsstrahlung.

## §4. RELATIVISTIC PLASMA IN A NON-UNIFORM MAGNETIC FIELD

We consider a medium which is a stationary beam of relativistic charged particles with a Lorentz factor $\gamma\gg1$ moving along a very strong curvilinear magnetic field. In such a medium each separate charged particle emits electromagnetic waves as they move along a curved trajectory. This so-called curvature (or magneto-drift) radiation is completely analogous to the usual synchrotron radiation:[12] it has the same characteristic frequency $\omega\approx c\gamma^3/\rho$ (where $\rho$ is the radius of curvature of the magnetic field line) and is directed into a narrow cone of opening $\theta\sim1/\gamma\ll1$ along the direction of motion of the particle, i.e., along the magnetic field.

The study of this radiation under conditions of a rather dense relativistic plasma when the distance between the particles is less than the wavelength of the emitted wave is of great interest in astrophysics, in particular, for an understanding of the origin of the extraordinary powerful and strongly directed radio-emission flux from pulsars. This problem has not been solved before.

We emphasize that there is no emission from the plasma as a whole in contrast to emission by a separate particle under stationary conditions as a constant current does not radiate. The curvilinearity, i.e., the non-uniformity of the magnetic field, plays a decisive role in the generation of the radiation considered here. Yet another important feature of it is that since the velocity of all particles in the plasma is close to the light velocity $c$, Cherenkov interactions between the radiation and the plasma become possible for an oscillation mode with a refractive index barely larger than unity. In that case all or almost all particles in the plasma turn out to be at resonance. The simultaneous coexistence and the interaction of curvature and Cherenkov radiation must lead to the appearance of new oscillation modes which are completely different from the oscillations of a uniform plasma.

The radiation of the plasma considered is determined by the permittivity tensor (22). It is convenient for its evaluation to introduce at each point $\mathbf{r}$ three unit vectors: $\mathbf{h}$ along the direction of the magnetic field, $\mathbf{n}$ the vector of the normal, and $\mathbf{l}$ the binormal vector. As the particles move along the magnetic field their distribution function has the form

$$F(\mathbf{p})=f_\parallel(p_\parallel)\delta(\mathbf{p}_\perp),$$

$p_\parallel$ is the momentum component along $\mathbf{h}$, $\mathbf{p}_\perp$ that at right angles to $\mathbf{h}$. We used here the fact that in a very strong magnetic field $\omega_c\to\infty$ the Larmor rotation is instantaneously stopped—de-excited so that all particles are in the zero Landau level. Transverse drift motion with $\mathbf{p}_\perp=\mathbf{p}_{10}(\mathbf{r})$ is in principle possible but we consider here conditions when it is unimportant. Since possible transverse rotations are instantaneously "forgotten" as $\omega_c\to\infty$, we have

$$\frac{\partial F}{\partial p_s'}=h_s(\mathbf{r}')\frac{\partial f_\parallel}{\partial p_\parallel}\delta(\mathbf{p}_\perp)=\frac{v_s(t')}{v_\parallel}\frac{\partial f_\parallel}{\partial p_\parallel}\delta(\mathbf{p}_\perp).$$

It thus follows from (22) that

$$\sigma_{\alpha\beta}^{\text{eff}}(\mathbf{r},\mathbf{k})=-e^2\int v_\alpha\,d\mathbf{p}\int_{-\infty}^{t}dt'\exp\{i\omega(t-t')-i\mathbf{k}\boldsymbol{\eta}^*\}$$

$$\times\text{Det}^{-1}\cdot v_\beta(t')\frac{1}{v_\parallel}\frac{\partial f_\parallel}{\partial p_\parallel}\delta\left(\mathbf{p}_\perp\left(\mathbf{r}+\frac{\boldsymbol{\eta}^*}{2}\right)\right). \quad (49)$$

The function $\boldsymbol{\eta}^*(\mathbf{r},\mathbf{p},t-t')$ occurring in (49) can be found from the particle equation of motion (23) which in our case takes the form

$$\mathbf{r}-\mathbf{r}'=\mathbf{v}(t-t')-\frac{\mathbf{a}}{2}(t-t')^2+\frac{\dot{\mathbf{a}}}{6}(t-t')^3-\ldots,$$

$$\mathbf{a}=\frac{v_\parallel^2}{\rho}\mathbf{n},\quad \dot{\mathbf{a}}=-\frac{v_\parallel^3}{\rho^2}\left(\mathbf{h}+\mathbf{n}\left(\mathbf{h}\frac{d\rho}{d\mathbf{r}}\right)+\mathbf{l}\frac{\rho}{\rho_\tau}\right). \quad (50)$$

Here $\rho(\mathbf{r})$ is the radius of curvature of the magnetic field line, and $\rho_\tau$ its torsion radius. We shall in what follows assume that $\rho_\tau\to\infty$, i.e., that the particle trajectory is planar.[1)] We have limited ourselves in (50) to the cubic terms in $t-t'$ for the same reasons as in (38); each higher term makes a smaller contribution and the quadratic term in fact vanishes. Indeed, substituting (50) into (23) we find

$$\boldsymbol{\eta}^*=\mathbf{v}(t-t')-\frac{v_\parallel^2}{2\rho}\mathbf{n}(t-t')^2-\frac{v_\parallel^3}{6\rho^2}(t-t')^3\left[\mathbf{h}-\frac{1}{2}\mathbf{n}\left(\mathbf{h}\frac{d\rho}{d\mathbf{r}}\right)\right]. \quad (51)$$

In deriving (51) we must bear in mind that the vectors $\mathbf{h}$ and $\mathbf{n}$ are functions of the velocity $\mathbf{v}$ or of the momentum $\mathbf{p}$ which is the independent integration variable:

$$v_\parallel\mathbf{h}=\mathbf{v}-\mathbf{l}(\mathbf{vl}),\quad v_\parallel\mathbf{n}=[\mathbf{lv}],\quad \mathbf{l}=\text{const}.$$

We get a similar expression for $\mathbf{v}(t')$:

$$\mathbf{v}(t')=\mathbf{v}-\frac{v_\parallel^2}{\rho}(t-t')\mathbf{n}-\frac{v_\parallel^3}{2\rho^2}(t-t')^2\left[\mathbf{h}+\mathbf{n}\left(\mathbf{h}\frac{d\rho}{d\mathbf{r}}\right)\right].$$

When integrating in (49) there occurs in all expressions the quantity $\mathbf{w}$—the value of the particle velocity at the point $\mathbf{r}+\boldsymbol{\eta}^*/2$:

$$\mathbf{w}=v_\parallel\mathbf{h}\left(\mathbf{r}+\frac{\boldsymbol{\eta}^*}{2}\right)$$

$$=v_\parallel\mathbf{h}+\frac{v_\parallel^2}{2\rho}(t-t')\mathbf{n}-\frac{1}{8}\frac{v_\parallel^3}{\rho^2}(t-t')^2\left[\mathbf{h}+\mathbf{n}\left(\mathbf{h}\frac{d\rho}{d\mathbf{r}}\right)\right].$$

Using this the functions of $t-t'$ occurring in (49) equal (cf. (39))

$$\boldsymbol{\eta}^*(\mathbf{w})=v_\parallel(t-t')\mathbf{h}-\frac{1}{24}\frac{v_\parallel^3}{\rho^2}(t-t')^3\left[\mathbf{h}+\mathbf{n}\left(\mathbf{h}\frac{d\rho}{d\mathbf{r}}\right)\right],$$

$$\mathbf{v}'(\mathbf{w})=v_\parallel\mathbf{h}-\frac{v_\parallel^2}{2\rho}(t-t')\mathbf{n}-\frac{1}{8}\frac{v_\parallel^3}{\rho^2}(t-t')^2\left[\mathbf{h}+\mathbf{n}\left(\mathbf{h}\frac{d\rho}{d\mathbf{r}}\right)\right].$$

The term quadratic in $t-t'$ disappeared from the expression for $\boldsymbol{\eta}^*$ as should be the case.

Finally we have the following expression for the conductivity tensor:

$$\sigma_{\alpha\beta}^{\text{eff}}(\mathbf{r},\mathbf{k},\omega)=-e^2\int v_\parallel\frac{\partial f_\parallel}{\partial p_\parallel}dp_\parallel\int_0^\infty E(\omega,\mathbf{k},p_\parallel,\tau)$$

$$\times\Big[h_\alpha h_\beta\Big(1-\frac{v_\parallel^2\tau^2}{4\rho^2}\Big)+(n_\alpha h_\beta-h_\alpha n_\beta)\frac{v_\parallel\tau}{2\rho}-(n_\alpha h_\beta+h_\alpha n_\beta)\times\frac{v_\parallel^2\tau^2}{8\rho^2}\Big(\mathbf{h}\frac{d\rho}{d\mathbf{r}}\Big)-n_\alpha n_\beta\frac{v_\parallel^2\tau^2}{4\rho^2}\Big]d\tau,$$

$$E(\omega,\mathbf{k},p_\parallel,\tau)=\exp\Big[i(\omega-k_\parallel v_\parallel)\tau+\frac{i}{24}\frac{v_\parallel^3\tau^3}{\rho^2}\Big(k_\parallel+(\mathbf{kn})\Big(\mathbf{h}\frac{d\rho}{d\mathbf{r}}\Big)\Big)\Big].\qquad(52)$$

Considering a mode propagating at a small angle to the direction of the magnetic field $\theta\lesssim 1/\gamma$ (as only they can turn out to be unstable) we can neglect the last term in the exponent (when $\mathbf{h}d\rho/d\mathbf{r}<\gamma$). Integrating in (52) over $\tau$ we finally get

$$\sigma_\parallel=-2\pi e^2\int\frac{\partial f_\parallel}{\partial p_\parallel}\frac{\rho^{2/3}}{k_\parallel^{1/3}}\Big[\mathrm{Ai}(\xi)\Big(1+\frac{\xi}{(k_\parallel\rho)^{2/3}}\Big)+i\,\mathrm{Gi}(\xi)\Big(1+\frac{\xi}{(k_\parallel\rho)^{2/3}}\Big)-\frac{i}{\pi(k_\parallel\rho)^{2/3}}\Big]dp_\parallel,$$

$$\sigma_\perp=-2\pi e^2\int\frac{\partial f_\parallel}{\partial p_\parallel}\frac{1}{k_\parallel}\Big[\xi\,\mathrm{Ai}(\xi)+i\xi\,\mathrm{Gi}(\xi)-\frac{i}{\pi}\Big]dp_\parallel,$$

$$\sigma_{\perp,\parallel}=-2\pi e^2\int\frac{\partial f_\parallel}{\partial p_\parallel}\Big\{-\frac{\rho^{1/3}}{k_\parallel^{2/3}}i[\mathrm{Ai}'(\xi)+i\,\mathrm{Gi}'(\xi)]+\frac{1}{2k_\parallel}\times\Big(\mathbf{h}\frac{d\rho}{d\mathbf{r}}\Big)\Big[\xi\,\mathrm{Ai}(\xi)+i\xi\,\mathrm{Gi}(\xi)-\frac{i}{\pi}\Big]\Big\}dp_\parallel,\qquad(53)$$

$$\sigma_{\parallel,\perp}=-2\pi e^2\int\frac{\partial f_\parallel}{\partial p_\parallel}\Big\{\frac{\rho^{1/3}}{k_\parallel^{2/3}}i[\mathrm{Ai}'(\xi)+i\,\mathrm{Gi}'(\xi)]+\frac{1}{2k_\parallel}\times\Big(\mathbf{h}\frac{d\rho}{d\mathbf{r}}\Big)\Big[\xi\,\mathrm{Ai}(\xi)+i\xi\,\mathrm{Gi}(\xi)-\frac{i}{\pi}\Big]\Big\}dp_\parallel.$$

Here $\xi=2(\omega-k_\parallel v_\parallel)_\rho{}^{2/3}/v_\parallel k_\parallel^{1/3}$. In its structure Eq. (53) is close to (36).

As $\rho\to\infty$ (i.e., $\xi\to\infty$) the expression for $\sigma_{\alpha\beta}^{\text{eff}}$ changes into the corresponding expression for a uniform plasma (cf. (43)):

$$\sigma_\parallel=-ie^2\int\frac{v_\parallel}{\omega-k_\parallel v_\parallel}\frac{\partial f_\parallel}{\partial p_\parallel}dp_\parallel.$$

However, for finite values of $\rho$ and $\omega\approx k_\parallel v_\parallel\approx k_\parallel c$ the conductivity of such a non-uniform plasma has practically nothing in common with the local conductivity of a uniform medium. This is also understandable: first of all there is no curvature radiation in a uniform medium—it is completely connected with the inhomogeneity. Moreover, the whole plasma consists here of resonance particles and, as we showed in §3, the strongest effect of the inhomogeneity just turns to be on the contribution from the resonance particles to the dispersion properties of the medium. When $\xi\lesssim 1$ the real and imaginary parts of the conductivity are of the same order. As in (36), in the region $\omega<k_\parallel v_\parallel$ there occur oscillations of the Airy function describing the imaginary part of the permittivity. These oscillations appear as the result of the interaction between the curvature and the Cherenkov radiations and with them the complete transformation of the radiation properties of the plasma may be connected.

One must use Eq. (53) to find the dispersion equation and to determine, using it, the oscillation eigenmodes of the plasma considered. An analysis shows that in that case in a wide range of frequencies there arises a set of hydrodynamic modes part of which possess large growth rates. However, a study of this problem is not the subject of the present paper. We consider here only the limiting case of a low density plasma in order to have the possibility to make a comparison with the theory of synchrotron emission. First of all, in a low density plasma the anti-Hermitean part of the permittivity tensor is small and the refractive index $n$ of electromagnetic waves is close to unity, i.e., $k\approx\omega/c$. Using this we can write the dispersion equation in the form

$$(1-n^2)+(1-n^2\sin^2\theta\cos^2\varphi)\delta\varepsilon_\perp+(1-n^2\cos^2\theta)\delta\varepsilon_\parallel+n^2\cos\theta\sin\theta\cos\varphi(\delta\varepsilon_{\parallel,\perp}+\delta\varepsilon_{\perp,\parallel})=0.$$

Here $\delta\varepsilon=4\pi i\sigma/\omega$. Hence it follows that the imaginary part of the wavevector, i.e., the spatial growth rate of the wave $\varkappa=\operatorname{Im}k$ in the case considered is

$$2\varkappa=\frac{4\pi}{c}\operatorname{Re}[\sigma_\parallel\sin^2\theta+\sigma_\perp(1-\sin^2\theta\cos^2\varphi)].\qquad(54)$$

Here $\theta$ is the angle between the wave vector $\mathbf{k}$ and the magnetic field $\mathbf{B}_0$, $\varphi$ is the angle in the $\mathbf{n}l$ plane reckoned from the direction of the normal $\mathbf{n}$. We assumed, moreover, that $\rho=\rho_0=$ constant.

We take further into account that the growth rate of the waves $\varkappa$ describes a stimulated radiation process. In a plasma with a sufficiently small density it is through the Einstein relation connected with the spontaneous emission process of a single particle:[15]

$$2\varkappa=\frac{(2\pi)^3c^2}{\omega^3}\int P_\Omega(\omega,\mathbf{p})\frac{\partial f}{\partial\mathbf{p}}\mathbf{k}\,d\mathbf{p}.\qquad(55)$$

Here $P_\Omega(\omega,\mathbf{p})$ is the spectral density of the emission power of a single particle into a solid angle $d\Omega$. Comparing (55) and (53),(54) we find the quantity $P_\Omega(\omega,p_\parallel)$:

$$P_\Omega(\omega,p_\parallel)=\frac{e^2}{3^{1/2}\pi^2}\frac{\rho_0\omega^2}{c^2}(\gamma^{-2}+\theta^2)^{1/2}[(1+\theta^2\sin^2\varphi)\times(\gamma^{-2}+\theta^2)+\theta^2]K_{1/3}\Big[\frac{2\rho_0\omega}{3c\gamma^3}(1+\theta^2\gamma^2)^{3/2}\Big],$$

$$p_\parallel=mc(\gamma^2-1)^{1/2}.\qquad(56)$$

We have used here the fact that in actual fact the radiation is concentrated in a narrow cone of angles $\theta\sim 1/\gamma$ and replaced $\sin\theta$ by $\theta$; $K_{1/3}(z)$ is a Macdonald function. Integrating (56) over the solid angle

$$P(\omega)=\int P_\Omega(\omega)d\Omega=\pi\int_0^\infty P_\Omega(\omega)d\theta^2,$$

we get the magnitude of the spectral power of the particle emission:

$$P(\omega)=\frac{e^2}{3^{1/2}\pi^2}\frac{\omega}{c\gamma^2}\int_{\omega/\omega_c}^\infty K_{5/3}(\eta)d\eta,\qquad\omega_c=\frac{3}{2}c\gamma^3/\rho_0.$$

This quantity is exactly the same as the well known expression from synchrotron radiation theory.[12] This should be the case as synchrotron radiation arising when a relativistic particle performs cyclotron rotation along an orbit with a constant radius of curvature $\rho_0$ is identically the same as its curvature radiation when it is moving along a curvilinear magnetic field with the same value of $\rho = \rho_0$.

The obtained agreement with the synchrotron radiation theory confirms the correctness of Eqs. (21), (22) for the permittivity tensor of the plasma. The use of other expressions for $\varepsilon$ does not lead to the same result. For example, using (16) we get the formula

$$P(\omega)=\frac{e^2\omega}{c\gamma^2}\frac{1}{\xi}\int_{\xi}^{\infty}(z-\xi)\,\mathrm{Ai}(z)\,dz, \qquad \xi=\left(\frac{\rho\omega}{2c\gamma^3}\right)^{2/3},$$

which does not correctly go over into the frequency dependence of the synchrotron radiation power.

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[1)] In the pulsar magnetosphere in the region $r \ll R_c$ where the radio emission is generated, both this condition and the condition $\mathbf{p}_{\perp 0}(r) \approx 0$ are well satisfied. Here $r$ is the distance from the surface of the neutron star and $R_c = c/\Omega$ the light cylinder radius ($\Omega$ is the angular rotation frequency of the star).

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